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Homogeneous contact metric structures on five-dimensional generalized symmetric spaces

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Abstract. We obtain the full classification of invariant contact metric structures on five-dimensional Riemannian generalized symmetric spaces. Different classes of examples of these spaces show different behaviours. In fact, while some of these spaces do not admit any invariant contact metric structure, we find and describe four new families of homogeneous structures. Investigating their geometric properties, we find that these new examples are not Sasakian (not even K-contact), but they all belong to the wider class of H-contact manifolds. On the other hand, we also obtain a rigidity result, proving that invariant contact metric structures on five-dimensional Riemannian generalized symmetric spaces which are naturally reductive, are exactly the ones giving to them the structure of globally φ-symmetric spaces, already classified in [10].

1. Introduction

A contact manifold (M,η) is said to be *homogeneous* if there exists a connected Lie group G of diffeomorphisms acting transitively on M and leaving η invariant. If g is a Riemannian metric associated to η and G is a group of isometries, then (M,η,g) is said to be a *homogeneous contact metric manifold*. In this case, the whole contact metric structure (η,φ,ξ,g) is invariant.

Three-dimensional homogeneous contact metric manifolds are well understood. If (M,η,g) is a simply connected three-dimensional homogeneous contact metric manifold, then $M=G$ is a Lie group and the contact metric structure

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(η, g, ξ, φ) is left-invariant. This result and a complete classification were obtained in [13]. It is then a natural problem to study five-dimensional homogeneous contact metric manifolds.

The five-dimensional case appears much broader and it allows several different interesting behaviours. The simply connected covering of a five-dimensional contact metric (locally) symmetric space is either $\mathbb{S}^5(1)$ or $\mathbb{E}^3 \times \mathbb{S}^2(4)$ [12]. Five-dimensional φ-symmetric spaces were classified in [10], clarifying their relationship with naturally reductive spaces (we shall provide more details in Section 2). Rigidity results on compact five-dimensional homogeneous contact metric manifolds were given in [15]. More recently, five-dimensional Lie algebras carrying an invariant Sasakian and K-contact structure were completely classified in [1] and [4], respectively.

A *generalized symmetric space* is a connected Riemannian manifold (M, g) admitting a *regular s-structure*, that is, a family $\{s_x : x \in M\}$ of symmetries on M, such that

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y),$$

for every points $x, y, \in M$ [9]. As it is well-known, every generalized symmetric space is a homogeneous Riemannian space [8]. An s-structure $\{s_x : x \in M\}$ is said to be *of order* $k \geq 2$ if $(s_x)^k = \mathrm{id}$ for all $x \in M$ and $(s_x)^i \neq \mathrm{id}$ for $i < k$. A Riemannian manifold (M, g) is said to be *k-symmetric* if it admits a regular s-structure of order k. Each generalized symmetric space is k-symmetric for some k [8]. The *order* of a generalized symmetric space is the least integer k such that (M, g) is k-symmetric.

Low-dimensional generalized symmetric spaces were completely classified in [8] (see also [9]). In particular, five-dimensional Riemannian generalized symmetric spaces are classified into 12 classes of homogeneous manifolds. Comparing this classification list with the classification of five-dimensional naturally reductive spaces [11], it is easily seen that the generalized symmetric spaces which are not naturally reductive are the ones of type $2, 3, 4, 7, 8a, 8b$ (all of order 4) and 9 (of order 6).

The results of [10] on φ-symmetric and naturally reductive spaces lead to the following

Question 1. *Do there exist invariant contact metric structures on five-dimensional generalized symmetric spaces which are not naturally reductive?*

Question 2. *Besides the structures of globally φ-symmetric spaces, do there exist other invariant contact metric structures on five-dimensional generalized symmetric spaces which are naturally reductive?*

In this paper, we shall provide a complete answer to Questions 1 and 2, classifying all invariant contact metric structures on five-dimensional generalized symmetric spaces. With regard to the examples which are not naturally reductive, while several of them do not carry any invariant contact metric structure, we find and explicitly describe four new families of homogeneous contact metric structures, on five-dimensional generalized symmetric spaces of type $3, 8a, 8b$ and 9. These homogeneous contact metric manifolds are not Sasakian (not even K-contact), but belong to the wider class of *H-contact manifolds*, that is, their Reeb vector field ξ is a critical point for the energy functional restricted to the space $\mathfrak{X}^1(M)$ of all unit vector fields [14]. Einstein and η-Einstein invariant contact metric structures will also be pointed out. On the other hand, a rigidity result is obtained for the naturally reductive cases, as it turns out that the only invariant contact metric structures on the naturally reductive examples, are the ones corresponding to globally φ-symmetric spaces.

The paper is organized in the following way. In Section 2 we report some basic information on contact metric structures and the classification of five-dimensional generalized symmetric spaces. In Section 3 we classify invariant contact metric structures on five-dimensional generalized symmetric spaces which are not naturally reductive. The geometry of these examples will be studied Section 4. A negative answer to Question 2 will be obtained in Section 5.

2. Preliminaries

We briefly report some basic information on contact metric structures, referring to [2] for further information. An *almost contact structure* on a $(2n+1)$-dimensional manifold M is triple (φ, η, ξ), where ξ is a nowhere vanishing vector field, η a 1-form and φ a $(1,1)-$tensor, such that

$$\eta(\xi) = 1, \quad \varphi^2 = -\mathrm{I} + \xi \otimes \eta. \tag{2.1}$$

As it is well known, conditions (2.1) imply

$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0. \tag{2.2}$$

The vector field ξ defines the characteristic foliation $\mathcal{F}$ with one-dimensional leaves, and the kernel of η defines the codimension one sub-bundle $\mathcal{D} = \ker \eta$. Then, the tangent bundle TM of M admits the canonical splitting

$$TM = \mathcal{D} \oplus \mathbb{R}\xi.$$

If the 1-form η satisfies the condition

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0,$$

then the subbundle $\mathcal{D}$ defines a *contact structure* on M. In this case, η is called a *contact form* and the vector field ξ is called the *Reeb vector field*. If η is a contact form, then $\mathrm{d}\eta(\xi, X) = 0$, for every vector field X on M.

Considering the product manifold $M \times \mathbb{R}$, denoted by $(X, f\frac{d}{dt})$ an arbitrary vector field on $M \times \mathbb{R}$, one can introduce the almost complex structure

$$J\left(X, f\frac{\partial}{\partial t}\right) = \left(\varphi X - f\xi, \eta(X)\frac{\partial}{\partial t}\right). \tag{2.3}$$

Then, (φ, η, ξ) is said to be *normal* if J is integrable. This is equivalent to requiring that the Nijenhuis tensor N_φ associated to the tensor φ satisfies the condition $N_\varphi = -\mathrm{d}\eta \otimes \xi$.

A Riemannian metric g on an almost contact manifold (M, φ, η, ξ) is *compatible* with the almost contact structure if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for every vector fields X, Y. In particular, $\eta(X) = g(X, \xi)$ for any tangent vector field X. The structure (φ, η, ξ, g) is called an *almost contact metric* structure. Any almost contact structure on a paracompact manifold admits a compatible metric.

The *fundamental form* Φ associated to an almost contact metric structure (φ, η, ξ, g) is given by

$$\Phi(X, Y) = g(X, \varphi Y).$$

An almost contact metric structure (φ, η, ξ, g) is said to be *contact metric* if $2\Phi = \mathrm{d}\eta$. In this case, η is a contact form. We shall denote by (M, η, g) (or $(M, \varphi, \eta, \xi, g)$) a *contact metric manifold*, that is, an odd-dimensional manifold equipped with a contact metric structure. A *Sasakian manifold* is a normal contact metric manifold.

A contact metric manifold $(M, \varphi, \eta, \xi, g)$ is said to be *K-contact* if the tensor $h = \frac{1}{2}\mathcal{L}_\xi\varphi$ vanishes (equivalently, if ξ is a Killing vector field). Any Sasakian manifold is K-contact, but the converse only holds in dimension three.

On any K-contact manifold (M, η, g), the Ricci tensor Ric_g of the contact metric g satisfies $\mathrm{Ric}_g(\xi, X) = 2n\,\eta(X)$, for any vector field X on M, where $\dim M = 2n + 1$. Thus, ξ is a Ricci eigenvector. An *H-contact manifold* is a contact metric manifold whose Reeb vector field ξ is a critical point for the energy functional restricted to the space $\mathfrak{X}^1(M)$ of all unit vector fields on (M, g) (considered as smooth maps from (M, g) into its unit tangent sphere bundle T_1M, equipped with the Sasaki metric). As proved in [14], $(M, \varphi, \xi, \eta, g)$ is H-contact if and only if ξ is an eigenvector of the Ricci operator.

A contact metric manifold (M, η, g) is said to be *η-Einstein* if the Ricci tensor Ric_g of the Riemannian metric g satisfies

$$\mathrm{Ric}_g = \lambda g + \nu\, \eta \otimes \eta,$$

for some smooth functions λ, ν, that is,

$$Q = \lambda \mathrm{I} + \nu\, \eta \otimes \xi, \tag{2.4}$$

where Q is the *Ricci operator*, defined by $g(QX, Y) = \mathrm{Ric}(X, Y)$.

A *φ-symmetric space* may be considered as the odd-dimensional analogue of a Hermitian symmetric space. In fact, it is a Sasakian manifold $(M, \varphi, \eta, \xi, g)$, such that the geodesic reflections with respect to the integral curves of ξ (φ-geodesic symmetries) extend to define global automorphisms of the entire structure. The existence of φ-geodesic symmetries yields that the manifold fibers over a Hermitian symmetric space.

Following [16], on a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ one defines the connection

$$\bar{\nabla}_X Y = \nabla_X Y + d\eta(X, Y)\xi - \eta(X)\phi Y + \eta(Y)\phi X, \tag{2.5}$$

where ∇ denotes the Levi–Civita connection of (M, g). As proved in [5], $\bar{\nabla}$ is the unique metric connection connection with skew-symmetric torsion preserving the Sasakian structure. In particular, (M, η, g) is φ-symmetric if and only if $\bar{\nabla}\bar{R} = 0$, that is, when $\bar{\nabla}$ has symmetric curvature.

A simply connected and complete locally φ-symmetric space is naturally reductive [3]. Conversely, five-dimensional naturally reductive spaces carrying a structure of φ-symmetric space were completely classified in [10]. A classification result in arbitrary dimension was obtained in [6].

We end this section reporting the classification of five-dimensional generalized symmetric spaces.

Theorem 2.1 ([8]). *All non-symmetric five-dimensional generalized symmetric spaces are of order* 4 *and* 6 *and of the following* 12 *types:*

Type 1) *As a homogeneous space, M is the matrix group*

$$\begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ u & v & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, M coincides with $\mathbb{R}^5(x,y,z,u,v)$, equipped with the Riemannian metric

$$g = dx^2 + dy^2 + du^2 + dv^2 + \varrho^2(xdu - ydv + dz)^2,$$

with $\varrho > 0$. The linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ admits an orthogonal basis $\{X_1, Y_1, X_2, Y_2, W\}$, with $\langle X_i, X_i\rangle = \langle Y_i, Y_i\rangle = 1$, $i = 1,2$ and $\langle W, W\rangle = \varrho^2$, such that the Lie brackets can be described as follows:

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W
X_1	0	0	$-W$	0	0
X_2	0	0	0	W	0
Y_1	W	0	0	0	0
Y_2	0	$-W$	0	0	0
W	0	0	0	0	0

(2.6)

Type 2) *As a homogeneous space, M is the matrix group*

$$\begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda_1 t} & 0 & 0 & y \\ 0 & 0 & e^{\lambda_2 t} & 0 & z \\ 0 & 0 & 0 & e^{-\lambda_2 t} & w \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, M coincides with $\mathbb{R}^5(x,y,z,w,t)$, equipped with the Riemannian metric

$$\begin{aligned} g = {} & e^{-2\lambda_1 t}dx^2 + e^{2\lambda_1 t}dy^2 + e^{-2\lambda_2 t}dz^2 + e^{2\lambda_2 t}dw^2 + dt^2 \\ & + 2\alpha[e^{-(\lambda_1+\lambda_2)t}dxdz + e^{(\lambda_1+\lambda_2)t}dydw] + 2\beta[e^{(\lambda_1-\lambda_2)t}dydz - e^{(\lambda_2-\lambda_1)t}dxdw], \end{aligned}$$

where either (i) $\lambda_1 > \lambda_2 > 0$, $\alpha^2 + \beta^2 < 1$, (ii) $\lambda_1 = \lambda_2 > 0$, $\alpha = 0$ *and* $0 \le \beta < 1$, *or* (iii) $\lambda_1 < 0$, $\lambda_2 = 0$, $\alpha = 0$ *and* $0 < \beta < 1$. *The linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ admits a basis $\{X_1, Y_1, X_2, Y_2, W\}$ such that the invariant Riemannian metric and Lie brackets can be described as follows:*

$\langle\,,\,\rangle$	X_1	X_2	Y_1	Y_2	W
X_1	1	α	0	$-\beta$	0
X_2	α	1	β	0	0
Y_1	0	β	1	α	0
Y_2	$-\beta$	0	α	1	0
W	0	0	0	0	1

and

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W	A_1	A_2
X_1	0	0	0	0	$-\lambda_1 X_1$	$-X_2$	$-X_2$
X_2	0	0	0	0	$-\lambda_2 X_2$	X_1	X_1
Y_1	0	0	0	0	$\lambda_1 Y_1$	$-Y_2$	Y_2
Y_2	0	0	0	0	$\lambda_2 Y_2$	Y_1	$-Y_1$
W	$\lambda_1 X_1$	$\lambda_2 X_2$	$-\lambda_1 Y_1$	$-\lambda_2 Y_2$	0	0	0
A_1	X_2	$-X_1$	Y_2	$-Y_1$	0	0	0
A_2	X_2	$-X_1$	$-Y_2$	Y_1	0	0	0

(2.7)

The isotropy subalgebra is given by either $\mathfrak{h} = 0, \mathrm{span}(A_1)$ *or* $\mathrm{span}(A_1, A_2)$, *according to whether conditions* (i), (ii) *or* (iii) *hold, respectively.*

Type 3) *M is the homogeneous space $M = SO(3,\mathbb{C})/SO(2)$, where $SO(3,\mathbb{C})$ is the special complex orthogonal group and the Riemannian metric of M is induced by a real invariant positive semi-definite form of $GL(3,\mathbb{C})$. The subalgebra is $\mathfrak{h} = so(2) = \mathrm{span}(A)$, where, with respect to a basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\mathfrak{m}$, the Riemannian metric and the Lie brackets are respectively given by*

$\langle\,,\,\rangle$	X_1	X_2	Y_1	Y_2	W
X_1	a^2	0	0	$-\gamma$	0
X_2	0	a^2	γ	0	0
Y_1	0	γ	a^2	0	0
Y_2	$-\gamma$	0	0	a^2	0
W	0	0	0	0	b^2

(2.8)

where $a, b > 0$, γ are real numbers, $a^2 > |\gamma|$, and

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W	A
X_1	0	0	0	$-W$	$-X_1$	$-X_2$
X_2	0	0	W	0	$-X_2$	X_1
Y_1	0	$-W$	0	0	Y_1	$-Y_2$
Y_2	W	0	0	0	Y_2	Y_1
W	X_1	X_2	$-Y_1$	$-Y_2$	0	0
A	X_2	$-X_1$	Y_2	$-Y_1$	0	0

(2.9)

Type 4) *M is the complex matrix group*

$$\begin{pmatrix} e^{\lambda t} & 0 & z \\ 0 & e^{-\lambda t} & w \\ 0 & 0 & 1 \end{pmatrix}$$

where $z, w \in \mathbb{C}$ and $t \in \mathbb{R}$. M also coincides with the space $\mathbb{C}^2(z,w) \times \mathbb{R}(t)$, equipped with a Riemannian metric

$$g = e^{-(\lambda+\bar{\lambda})t} dz d\bar{z} + e^{(\lambda+\bar{\lambda})t} dw d\bar{w} + dt^2 + 2c[e^{(\bar{\lambda}-\lambda)t} dz d\bar{w} + e^{(\lambda-\bar{\lambda})t} d\bar{z} dw] + \\ + \gamma e^{-2\lambda t} dz^2 + \bar{\gamma} e^{-2\bar{\lambda} t} d\bar{z}^2 - \gamma e^{2\lambda t} dw^2 - \bar{\gamma} e^{2\bar{\lambda} t} d\bar{w}^2,$$

with $\lambda, \gamma \in \mathbb{C}$, $c \in \mathbb{R}$, $\gamma\bar{\gamma} + c^2 < 1/4$. Putting $\nu = (1+b^2)\gamma$, where $c = \frac{1-b^2}{2(1+b^2)}$, condition $\gamma\bar{\gamma} + c^2 < 1/4$ is equivalent to $\nu\bar{\nu} < b^2$. Put $\lambda = \delta + i\mu, \nu = \alpha + i\beta$. Then, the vector subspace $\mathfrak{m}$ of $\mathfrak{g}$ admits a basis $\{X_1, X_2, Y_1, Y_2, W\}$, such that

$\langle\ ,\ \rangle$	X_1	X_2	Y_1	Y_2	W
X_1	1	α	0	$-\beta$	0
X_2	α	b^2	β	0	0
Y_1	0	β	1	α	0
Y_2	$-\beta$	0	α	b^2	0
W	0	0	0	0	1

and

$[\ ,\ ]$	X_1	X_2	Y_1	Y_2	W	A_1	A_2
X_1	0	0	0	0	$-\delta X_2 - \mu Y_2$	Y_1	$-X_2$
X_2	0	0	0	0	$\delta X_1 - \mu Y_1$	$-Y_2$	X_1
Y_1	0	0	0	0	$-\mu X_2 + \delta Y_2$	$-X_1$	Y_2
Y_2	0	0	0	0	$-\mu X_1 - \delta Y_1$	X_2	$-Y_1$
W	$\delta X_2 + \mu Y_2$	$-\delta X_1 + \mu Y_1$	$\mu X_2 - \delta Y_2$	$\mu X_1 + \delta Y_1$	0	0	0
A_1	$-Y_1$	Y_2	X_1	$-X_2$	0	0	0
A_2	X_2	$-X_1$	$-Y_2$	Y_1	0	0	0

The possible cases are either (i) *$\lambda + \bar{\lambda} \neq 0$ and $\nu \neq 0$,* (ii) *$\lambda + \bar{\lambda} = 0$, $\nu = 0$ and $b^2 \neq 1$, or* (iii) *$\lambda + \bar{\lambda} \neq 0$, $\nu = 0$ and $b^2 = 1$. The isotropy subalgebra is respectively given by $\mathfrak{h} = 0$, $\mathfrak{h} = \mathrm{span}(A_1)$ and $\mathfrak{h} = \mathrm{span}(A_1, A_2)$.*

Types 5a,5b) *M is the homogeneous space $(SO(3) \times SO(3))/SO(2)$ or $(SO(2,1) \times SO(2,1))/SO(2)$. The Riemannian metric is the one induced on M by the real invariant positively semi-definite form*

$$\tilde{g} = a^2[(\omega_1 + \tilde{\omega}_2)^2 + (\tilde{\omega}_1 + \omega_2)^2] + b^2[(\omega_1 - \tilde{\omega}_2)^2 + (\tilde{\omega}_1 - \omega_2)^2] + c^2(\omega_3 + \tilde{\omega}_3)^2,$$

with $a \geq b$ and c three positive real parameters, on the group $GL(3,\mathbb{R}) \times GL(3,\mathbb{R})$ of pairs of regular matrices

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \times \begin{pmatrix} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{pmatrix},$$

where $\omega_1 = a_2 da_3 + b_2 db_3 \pm c_2 dc_3$, $\omega_2 = a_3 da_1 + b_3 db_1 \pm c_3 dc_1$, $\omega_3 = a_1 da_2 + b_1 db_2 \pm c_1 dc_2$, and $\tilde{\omega}_i$ are given by corresponding expressions in $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i$. Cases 5a) and 5b) correspond to the $(+)$ and $(-)$ signs respectively.

For case 5a (case 5b can be described in a similar way), there exists a basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\mathfrak{m}$, such that

$\langle\, , \rangle$	X_1	X_2	Y_1	Y_2	W
X_1	$\frac{a^2+b^2}{4}$	0	0	$\frac{a^2-b^2}{4}$	0
X_2	0	$\frac{a^2+b^2}{4}$	$\frac{a^2-b^2}{4}$	0	0
Y_1	0	$\frac{a^2-b^2}{4}$	$\frac{a^2+b^2}{4}$	0	0
Y_2	$\frac{a^2-b^2}{4}$	0	0	$\frac{a^2+b^2}{4}$	0
W	0	0	0	0	c^2

and

$[\, ,]$	X_1	X_2	Y_1	Y_2	W	A
X_1	0	0	$\frac{1}{2}(W-A)$	0	$-Y_1$	Y_1
X_2	0	0	0	$\frac{1}{2}(W+A)$	Y_2	Y_2
Y_1	$-\frac{1}{2}(W-A)$	0	0	0	X_1	$-X_1$
Y_2	0	$-\frac{1}{2}(W+A)$	0	0	$-X_2$	$-X_2$
W	Y_1	$-Y_2$	$-X_1$	X_2	0	0
A	$-Y_1$	$-Y_2$	X_1	X_2	0	0

(2.10)

The isotropy subalgebra is $\mathfrak{h} = \mathrm{span}(A)$.

Types 6a,6b) *M is the homogeneous space* $SU(3)/SU(2)$ *or* $SU(2{,}1)/SU(2)$, *and coincides with the submanifold of* $\mathbb{C}^3(z_1, z_2, z_3)$ *given by relation* $z_1\bar{z}_1 + z_2\bar{z}_2 \pm z_3\bar{z}_3 = \pm 1$. *The Riemannian metric is the one induced on M by the Hermitian metric on* $\mathbb{C}^3$, *given by*

$$\tilde{g} = \lambda(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 \pm dz_3 d\bar{z}_3) + \mu(z_1 d\bar{z}_1 + z_2 d\bar{z}_2 \pm z_3 d\bar{z}_3)(\bar{z}_1 dz_1 + \bar{z}_2 dz_2 \pm \bar{z}_3 dz_3),$$

where $\lambda > 0, \mu \neq 0$ *are real parameters satisfying* $\mu \pm \lambda > 0$. *Cases 6a) and 6b) correspond to the* $(+)$ *and* $(-)$ *signs respectively.*

For case 6a (case 6b can be described in a similar way), there exists an orthogonal basis $\{X_1, X_2, Y_1, Y_2, W\}$ *of* $\mathfrak{m}$, *with* $\langle X_i, X_i\rangle = \langle Y_i, Y_i\rangle = 3a^2$, $i = 1, 2$, $\langle W, W\rangle = c^2$, *such that*

$[\ ,\ ]$	X_1	Y_1	X_2	Y_2	W	A_1	A_2	A_3
X_1	0	$-A_1$	$3W + A_2$	A_3	$-X_2$	Y_1	$-X_2$	Y_2
X_2	A_1	0	A_3	$-3W + A_2$	Y_2	$-X_1$	$-Y_2$	$-X_2$
Y_1	$-3W - A_2$	$-A_3$	0	A_1	X_1	$-Y_2$	X_1	Y_1
Y_2	$-A_3$	$3W - A_2$	$-A_1$	0	$-Y_1$	X_2	Y_1	$-X_1$
W	X_2	$-Y_2$	$-X_1$	Y_1	0	0	0	0
A_1	$-Y_1$	X_1	Y_2	$-X_2$	0	0	0	0
A_2	X_2	Y_2	$-X_1$	$-Y_1$	0	0	0	0
A_3	$-Y_2$	X_2	$-Y_1$	X_1	0	0	0	0

(2.11)

The isotropy subalgebra is $\mathfrak{h} = \mathrm{span}(A_1, A_2, A_3)$.

Type 7) *M is* $\mathbb{R}^5(x, y, u, v, t)$, *equipped with a Riemannian metric*

$$\begin{aligned} g = {} & dt^2 + e^{-2\lambda t}(tdx - du)^2 + e^{2\lambda t}(tdy + dv)^2 + a^2(e^{-2\lambda t}dx^2 + e^{2\lambda t}dy^2) \\ & + 2\gamma(dydu - dxdv), \end{aligned}$$

where $\lambda, a, \gamma \in \mathbb{R}$, $\lambda \geq 0$, $a > 0$ *and* $\gamma^2 < a^2$. *There exists a basis* $\{X_1, X_2, Y_1, Y_2, W\}$ *of* $\mathfrak{m}$, *such that*

$\langle\ ,\ \rangle$	X_1	X_2	Y_1	Y_2	W
X_1	a^2	0	0	$-\gamma$	0
X_2	0	1	γ	0	0
Y_1	0	γ	a^2	0	0
Y_2	$-\gamma$	0	0	1	0
W	0	0	0	0	1

and

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W	A
X_1	0	0	0	0	$-\lambda X_1 - X_2$	Y_1
X_2	0	0	0	0	$-\lambda X_2$	$-Y_2$
Y_1	0	0	0	0	$\lambda Y_1 + Y_2$	$-X_1$
Y_2	0	0	0	0	λY_2	X_2
W	$\lambda X_1 + X_2$	λX_2	$-\lambda Y_1 - Y_2$	$-\lambda Y_2$	0	0
A	$-Y_1$	Y_2	X_1	$-X_2$	0	0

The possible cases are either (i) $\lambda \neq 0$, *or* (ii) $\lambda = \gamma = 0$. *Respectively, the isotropy subalgebra is either* $\mathfrak{h} = 0$, *or* $\mathfrak{h} = \mathrm{span}(A)$.

Types 8a,8b) *As homogeneous space, M is $I^e(\mathbb{R}^3)/SO(2)$ or $I^h(\mathbb{R}^3)/SO(2)$, where I^e (respectively, I^h) denotes the group of all positive affine transformations of $\mathbb{R}^3$ that preserve $dx^2+dy^2+dz^2$ (respectively, $dx^2+dy^2-dz^2$). M also coincides with the submanifold of $\mathbb{R}^6(x,y,z,\alpha,\beta,\gamma)$, such that $\alpha^2+\beta^2\pm\gamma^2=\pm 1$. The Riemannian metric of M is induced by the regular invariant quadratic form*

$$\bar{g} = dx^2 + dy^2 \pm dz^2 + \lambda^2(d\alpha^2 + d\beta^2 \pm d\gamma^2) + [\mu \pm (-1)](\alpha dx + \beta dy \pm \gamma dz)^2,$$

where $\lambda, \mu > 0$. The five-dimensional generalized symmetric spaces of type 8a (respectively, 8b) are obtained when the sign (+) (respectively, (−)) holds in the previous formulas.

In the case 8a (the case 8b can be described in a similar way), the vector subspace $\mathfrak{m}$ admits an orthogonal basis $\{X_1, X_2, Y_1, Y_2, W\}$, with $\langle X_1, X_1\rangle = \langle Y_1, Y_1\rangle = b^2$, $\langle X_2, X_2\rangle = \langle Y_2, Y_2\rangle = 1$, $\langle W, W\rangle = c^2$, where $b, c > 0$, such that the Lie brackets are given by

$[\,,\,]$	X_1	X_2	Y_1	Y_2	W	A
X_1	0	W	0	0	$-X_2$	Y_1
X_2	$-W$	0	0	0	0	$-Y_2$
Y_1	0	0	0	$-W$	Y_2	$-X_1$
Y_2	0	0	W	0	0	X_2
W	X_2	0	$-Y_2$	0	0	0
A	$-Y_1$	Y_2	X_1	$-X_2$	0	0

(2.12)

and $\mathfrak{h} = \mathrm{span}(A)$.

Type 9) *M is the matrix group*

$$\begin{pmatrix} e^{-(u+v)} & 0 & 0 & x \\ 0 & e^u & 0 & y \\ 0 & 0 & e^v & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Moreover, M coincides with $\mathbb{R}^5(x,y,z,u,v)$, equipped with the Riemannian metric

$$g = \frac{2}{3}a^2(du^2 + dudv + dv^2) + (2b^2+1)(e^{2(u+v)}dx^2 + e^{-2u}dy^2 + e^{-2v}dz^2) + 2(b^2-1)(e^v dxdy + e^u dxdz - e^{-(u+v)}dydz),$$

where $a > 0$ and $b > 0$ are real numbers.

With respect to a suitable basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\mathfrak{m} = \mathfrak{g}$, the invariant Riemannian metric and Lie brackets are determined by

$\langle\, ,\, \rangle$	X_1	X_2	Y_1	Y_2	W
X_1	$\frac{2}{3}a^2$	0	$\frac{1}{3}a^2$	0	0
X_2	0	$2b^2+1$	0	b^2-1	b^2-1
Y_1	$\frac{1}{3}a^2$	0	$\frac{2}{3}a^2$	0	0
Y_2	0	b^2-1	0	$2b^2+1$	$-(b^2-1)$
W	0	b^2-1	0	$-(b^2-1)$	$2b^2+1$

(2.13)

and

$[\, ,\,]$	X_1	X_2	Y_1	Y_2	W
X_1	0	$-X_2$	0	0	W
X_2	X_2	0	X_2	0	0
Y_1	0	$-X_2$	0	Y_2	0
Y_2	0	0	$-Y_2$	0	0
W	$-W$	0	0	0	0

(2.14)

3. Homogeneous contact metric structures: types which are not naturally reductive

As we already mentioned in the Introduction, generalized symmetric spaces which are not naturally reductive are the ones of type $2, 3, 4, 7, 8a, 8b$ and 9. We shall first specify which of these examples do not carry any invariant contact metric structure, obtaining the following result.

Proposition 3.1. *Generalized symmetric spaces of type* 2, 4 *and* 7 *do not admit any invariant contact structure. In particular, they are not homogeneous contact metric manifolds.*

Proof. The complete proof follows from a case by case study. For types 2, 4 and 7, it is easily seen that there no exist any 1-form $\eta : \mathfrak{m} \to \mathbb{R}$, such that $d\eta \wedge d\eta \neq 0$. In particular, no invariant contact forms may occur.

As an example, we report the calculations for type 2. We start from the basis $\{e_1, e_2, e_3, e_4, e_5\} := \{X_1, Y_1, X_2, Y_2, W\}$ of $\mathfrak{m}$ and consider the 1-forms $\{e^i\}$ dual to $\{e_i\}$. From (2.7), we get

$$de^1 = \lambda_1 e^1 \wedge e^5, \quad de^2 = \lambda_2 e^2 \wedge e^5, \quad de^3 = -\lambda_1 e^3 \wedge e^5, \quad de^4 = -\lambda_2 e^4 \wedge e^5, \quad de^5 = 0$$

and so, $de^i \wedge de^j = 0$ for all indices $i, j = 1, \dots, 5$. Consequently, as any 1-form $\eta : \mathfrak{m} \to \mathbb{R}$ is given by $\eta = \sum_i a_i e^i$, for some real constants $a_1, \dots, a_5$, we have $d\eta \wedge d\eta = 0$. Therefore, no invariant contact structures occur. □

Proposition 3.1 leaves us to consider types $3, 8a, 8b, 9$. All of them do admit invariant contact metric structures, under some restrictions on the parameters describing these spaces. The results are the following.

Theorem 3.2. *A generalized symmetric space of type* 3 *admits a homogeneous contact metric structure if and only if* $a^4 - \gamma^2 = 1/4$. *Thus, there exists a two-parameter family of locally non-isometric invariant contact metric structures on five-dimensional generalized symmetric spaces of type* 3.

Proof. We start from the basis $\{e_1, \dots, e_5\} := \{X_1, X_2, Y_1, Y_2, W\}$ of $\mathfrak{m}$ used in (2.8) and (2.9) and consider the dual basis $\{e^1, \dots, e^5\}$ of 1-forms over $\mathfrak{m}$. From (2.9), we get

$$\begin{aligned} &de^1 = e^1 \wedge e^5, \qquad && de^2 = e^2 \wedge e^5, \qquad && de^3 = -e^3 \wedge e^5, \\ &de^4 = -e^4 \wedge e^5, \qquad && de^5 = e^1 \wedge e^4 - e^2 \wedge e^3. && \end{aligned} \tag{3.1}$$

Consider now a 1-form $\eta : \mathfrak{m} \to \mathbb{R}$. Then, $\eta = \sum_i a_i e^i$, for some real constants $a_1, \dots, a_5$. As we are only interested in invariant contact forms, we now consider the isotropy representation

$$\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m}), \quad \rho(x)(y) = [x, y]_{\mathfrak{m}} \quad \text{for all } x \in \mathfrak{h},\ y \in \mathfrak{m}.$$

By (2.9), $\mathfrak{h}$ is spanned by A, to which corresponds the isotropy representation

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{e_i\}$. Denoting by $E = (a_i)$ the column matrix of cofficients of η with respect to the dual basis $\{e^i\}$, we then have that $\eta : \mathfrak{m} \to \mathbb{R}$ is invariant if and only if $X \cdot E = 0$, which yields $a_1 = a_2 = a_3 = a_4 = 0$. Hence, $\eta = a_5 e^5$, for some real constant a_5. By (3.1) it easily follows that $\eta \wedge (d\eta)^2 \neq 0$ (and so, η is a contact form) whenever $a_5 \neq 0$. Consider now the characteristic vector field $\xi = \sum_i \xi_i e_i$. Then, from $\eta(\xi) = 1$ and $g(\xi, X) = \eta(X)$, we easily deduce $\xi = \frac{1}{a_5} e_5$ and $a_5^2 = b^2$, that is, up to sign,

$$\eta = b\, e^5, \qquad \xi = \frac{1}{b}\, e_5.$$

Notice that $\{e_1, \ldots, e_4\}$ is then a basis of $\ker \eta$. We can now determine the tensor φ using the condition $2\Phi = \mathrm{d}\eta$. We find

$$\begin{aligned} \varphi e_1 &= \frac{1}{2(\gamma^2 - a^4)}(\gamma\, e_1 + a^2\, e_4), \qquad & \varphi e_2 &= \frac{1}{2(\gamma^2 - a^4)}(\gamma\, e_2 - a^2\, e_3), \\ \varphi e_3 &= \frac{1}{2(\gamma^2 - a^4)}(a^2\, e_2 - \gamma\, e_3), \qquad & \varphi e_4 &= -\frac{1}{2(\gamma^2 - a^4)}(a^2\, e_1 + \gamma\, e_4). \end{aligned} \tag{3.2}$$

From the second equation in (2.1), we have $\varphi^2 e_1 = -e_1 + \eta(e_1)\xi$, which yields at once $a^4 - \gamma^2 = 1/4$. It is easy to check that under this restriction, equations (2.1) and (2.2) are satisfied by η, φ and ξ as described above. Taking $a^4 - \gamma^2 = 1/4$ into account, (3.2) now becomes

$$\begin{aligned} \varphi e_1 &= -2(\gamma\, e_1 + a^2\, e_4), \qquad & \varphi e_2 &= -2(\gamma\, e_2 - a^2\, e_3), \\ \varphi e_3 &= -2(a^2\, e_2 - \gamma\, e_3), \qquad & \varphi e_4 &= 2(a^2\, e_1 + \gamma\, e_4), \end{aligned} \tag{3.3}$$

which completes the description of these invariant contact metric structures. Because of condition $a^4 - \gamma^2 = 1/4$, they form a two-parameter family of non-isometric structures, which depend on the values of $a, b > 0$. □

Theorem 3.3. *A generalized symmetric space of type* $8a, 8b$ *admits a homogeneous contact metric structure if and only if* $c = 2b$. *Thus, locally non-isometric invariant contact metric structures on five-dimensional generalized symmetric spaces of type* $8a$ *and* $8b$ *form a one-parameter family.*

PROOF. We describe in detail the case corresponding to type $8a$ (type $8b$ can be treated in the same way, up to some needed changes of sign). Let $\{e_1,\dots,e_5\} := \{X_1,X_2,Y_1,Y_2,W\}$ denote the basis of $\mathfrak{m}$ we used in (2.12). Consider the dual basis $\{e^1,\dots,e^5\}$ of 1-forms over $\mathfrak{m}$. From (2.12), we find

$$de^1=0,\ de^2=e^1\wedge e^5,\ de^3=0,\ de^4=-e^3\wedge e^5,\ de^5=-e^1\wedge e^2+e^3\wedge e^4. \quad (3.4)$$

Consider now a 1-form $\eta:\mathfrak{m}\to\mathbb{R}$. Then, $\eta=\sum_i a_ie^i$, for some real constants $a_1,\dots,a_5$. By (2.12), $\mathfrak{h}$ is spanned by A, to which corresponds the isotropy representation

$$X=\begin{pmatrix}0&0&1&0&0\\0&0&0&-1&0\\-1&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\end{pmatrix}$$

with respect to $\{e_i\}$. Denoting by $E=(a_i)$ the column of cofficients of η with respect to the dual basis $\{e^i\}$, we then have that $\eta:\mathfrak{m}\to\mathbb{R}$ is invariant if and only if $X\cdot E=0$, that is, $a_1=a_2=a_3=a_4=0$. Thus, $\eta=a_5e^5$, for some real constant a_5 and (3.4) yields that η is a contact form if and only if $a_5\neq 0$. Let $\xi=\sum_i\xi_ie_i$ denote the characteristic vector field. Then, equations $\eta(\xi)=1$ and $g(\xi,X)=\eta(X)$ give $\xi=\frac{1}{a_5}e_5$ and $a_5^2=c^2$. Therefore, up to sign, $\eta=c\,e^5$, $\xi=\frac{1}{c}\,e_5$.

Next, we describe the tensor φ using the condition $2\Phi=\mathrm{d}\eta$. With respect to the basis $\{e_1,\dots,e_4\}$ of $\ker\eta$, we get

$$\varphi e_1=\frac{c}{2}\,e_2,\quad \varphi e_2=-\frac{c}{2b^2}\,e_1,\quad \varphi e_3=-\frac{c}{2}\,e_4,\quad \varphi e_4=\frac{c}{2b^2}\,e_3. \quad (3.5)$$

From $\varphi^2e_1=-e_1+\eta(e_1)\xi$ it now follows $4b^2=c^2$, that is, $c=2b$. Under this restriction, equations (2.1) and (2.2) are satisfied by η, φ and ξ. Summarizing, we now have

$$\eta=2b\,e^5,\qquad \xi=\frac{1}{2b}\,e_5$$

and from (3.5) we get

$$\varphi e_1=b\,e_2,\quad \varphi e_2=-\frac{1}{b}\,e_1,\quad \varphi e_3=-b\,e_4\quad \varphi e_4=\frac{1}{b}\,e_3. \quad (3.6)$$

So, we completed the description of invariant contact metric structures on generalized symmetric spaces of type $8a$. These structures form a one-parameter family, depending on the value of $b>0$. □

Theorem 3.4. *A generalized symmetric space of type* 9 *admits a homogeneous contact metric structure if and only if* $b = 1$. *Thus, there exists a one-parameter family of locally non-isometric (left-)invariant contact metric structures on five-dimensional generalized symmetric spaces of type* 9.

Proof. We start from the basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\mathfrak{m} = \mathfrak{g}$ used in (2.13) and (2.14). Then, from (2.13) we deduce that

$$\begin{aligned}
e_1 &:= \frac{1}{a\sqrt{2}}(X_1 + Y_1), \qquad e_2 := \frac{1}{\sqrt{2b^2+1}}X_2,\\
e_3 &:= \frac{\sqrt{3}}{a\sqrt{2}}(X_1 - Y_1), \qquad e_4 := \frac{1}{b\sqrt{6}}(Y_2 - W),\\
e_5 &:= \frac{1}{3b\sqrt{2(2b^2+1)}}\{-2(b^2-1)X_2 + (2b^2+1)(Y_2 + W)\} \qquad (3.7)
\end{aligned}$$

is an orthonormal basis of $\mathfrak{m}$. Using (2.14), we can now calculate the Lie brackets $[e_i, e_j]$ and then de^k, for all indices i, j, k. We get

$$\begin{aligned}
de^1 &= 0, \qquad de^2 = \frac{\sqrt{2}}{a}e^1 \wedge e^2 - \frac{\sqrt{b^2-1}}{ab}e^1 \wedge e^5 + \frac{\sqrt{b^2-1}}{ab\sqrt{2b^2+1}}e^3 \wedge e^4,\\
de^3 &= 0, \qquad de^4 = -\frac{1}{a\sqrt{2}}e^1 \wedge e^4 + \frac{\sqrt{2b^2+1}}{a\sqrt{2}}e^3 \wedge e^5,\\
de^5 &= -\frac{1}{a\sqrt{2}}e^1 \wedge e^5 + \frac{3\sqrt{2}}{a2\sqrt{2b^2+1}}e^3 \wedge e^4. \qquad (3.8)
\end{aligned}$$

Consider now a 1-form $\eta : \mathfrak{m} \to \mathbb{R}$. Then, $\eta = \sum_i a_i e^i$, for some real constants $a_1, \dots, a_5$. Using (3.8), a direct calculation yields

$$\eta \wedge (d\eta)^2 = \frac{5a_2}{2a^2b\sqrt{2b^2+1}}\left(2\sqrt{2}(b^2-1)a_2a_5 + 3ba_5^2 - b(2b^2+1)a_4^2\right)e^1 \wedge \dots \wedge e^5.$$

Therefore, η is a left-invariant contact form on M if and only if

$$a_2 \neq 0 \quad \text{and} \quad 2\sqrt{2}(b^2-1)a_2a_5 + 3ba_5^2 - b(2b^2+1)a_4^2 \neq 0. \qquad (3.9)$$

We now suppose that there exists a left-invariant contact metric structure on M, having such 1-form η as contact form, and we calculate tensor φ by condition $2\Phi = \mathrm{d}\eta$. We find

$$\varphi e_1 = -\frac{1}{a\sqrt{2}}a_2\, e_2 + \frac{1}{2a\sqrt{2}}a_4\, e_4 + \frac{1}{4ab}\left(2(b^2-1)a_2 + \sqrt{2}ba_5\right)e_5,$$

$$\varphi e_2 = -\frac{1}{a\sqrt{2}} a_2\, e_1,$$

$$\varphi e_3 = -\frac{1}{4ab\sqrt{2b^2+1}} \left(2(b^2-1)a_2 + 3\sqrt{2}ba_5\right)\, e_4 - \frac{\sqrt{2b^2+1}}{2a\sqrt{2}} a_5\, e_5,$$

$$\varphi e_4 = -\frac{1}{2a\sqrt{2}} a_4\, e_1 + \frac{1}{4ab\sqrt{2b^2+1}} \left(2(b^2-1)a_2 + 3\sqrt{2}ba_5\right)\, e_3,$$

$$\varphi e_5 = -\frac{1}{4ab} \left(2(b^2-1)a_2 + \sqrt{2}ba_5\right)\, e_1 + \frac{\sqrt{2b^2+1}}{2a\sqrt{2}} a_4\, e_3. \tag{3.10}$$

Let $\xi = \sum_i \xi_i e_i$ denote the characteristic vector field of this contact metric structure. Applying equations (2.1) and (2.2) to the above description of η, φ and ξ and taking restrictions (3.9) into account, a long but straightforward calculation yields that (η, φ, ξ) is a (left-invariant) contact structure if and only if $b = 1$,

$$a_1 = a_3 = a_5 = 0, \qquad a_2^2 = \frac{4}{3}a^2, \qquad a_4^2 = \frac{8}{3}a^2,$$

$$\xi_1 c = \xi_3 = \xi_5 = 0, \qquad \xi_2 = \frac{1}{a_2}\left(1 - \frac{a_2^2}{a^2}\right), \qquad \xi_4 = \frac{a_4}{4a^2},$$

that is, up to sign,

$$\eta = \frac{2a}{\sqrt{3}}(e^2 + \sqrt{2}e^4), \qquad \xi = \frac{1}{2\sqrt{3}a}(e_2 + \sqrt{2}e_4).$$

Thus, $\ker \eta = \mathrm{Span}(E_1, E_2, E_3, E_4)$, where we put

$$E_1 := e_1, \quad E_2 := \frac{1}{2\sqrt{3}a}(\sqrt{2}e_2 - e_4), \quad E_3 := e_3, \quad E_4 := e_5.$$

As $b = 1$, from (3.10) we now easily get

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = -E_4, \quad \varphi E_4 = E_3, \tag{3.11}$$

which completes the description of left-invariant contact metric structures on generalized symmetric spaces of type 9. As $b = 1$, these structures form a one-parameter family, depending on $a > 0$. Notice that, by (3.11), $\{\xi, E_1, \dots, E_4\}$ is what is called a φ-basis [2] of the contact metric manifold. □

4. Contact metric geometry of five-dimensional generalized symmetric spaces

Let $(M = G/H, g)$ denote any homogeneous (pseudo-)Riemannian manifold. Consider the corresponding decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of Lie algebra of G, and denote by $\{e_i\}$, $\{h_r\}$ a basis of $\mathfrak{m}$ and $\mathfrak{h}$ respectively.

The invariant metric g on $\mathfrak{m}$ uniquely defines its invariant linear Levi–Civita connection, described in terms of the corresponding homomorphism of $\mathfrak{h}$-modules $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$, where $\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$ (see for example [7]). Explicitly, one has

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \quad \text{for all } x, y \in \mathfrak{g}, \tag{4.1}$$

where $v : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is the $\mathfrak{h}$-invariant symmetric mapping uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_m, [z, y]_m) + g(y_m, [z, x]_m), \quad \text{for all } x, y, z \in \mathfrak{g}. \tag{4.2}$$

The curvature tensor is then determined by the mapping $R : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$, such that $R(x, y) = [\Lambda(x), \Lambda(y)] - \Lambda([x, y])$, for all $x, y \in \mathfrak{m}$.

Finally, the Ricci tensor ϱ of g, described in terms of its components with respect to $\{u_i\}$, is given by

$$\operatorname{Ric}(u_i, u_j) = \sum_{r=1}^{4} R_{ri}(u_r, u_j), \quad i, j = 1, \dots, 4 \tag{4.3}$$

and the Ricci operator Q is then easily determined by equation $g(QX, Y) = \operatorname{Ric}(X, Y)$. We can apply this procedure to describe the Levi–Civita connection and curvature of any five-dimensional generalized symmetric space. With regard to the examples carrying some invariant contact metric structures, we prove the following results.

Theorem 4.1. *Let $(M = G/H, \eta, g)$ be a homogeneous contact metric manifold corresponding to a generalized symmetric space of type 3. Then, (M, η, g) is H-contact, and never K-contact (in particular, never Sasakian).*

Moreover, (M, η, g) is η-Einstein if and only if $4a^4 = b^4 + 1$. Thus, there exists a one-parameter family of locally non-isometric invariant η-Einstein contact metric structures on five-dimensional generalized symmetric spaces of type 3.

Proof. We first calculate tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ with respect to the basis $\{e_1, \dots, e_4\}$ of $\ker \eta$. Starting from the description of tensor φ given in (3.3) and we easily get

$$he_1 = \frac{2a^2}{b}\, e_4, \quad he_2 = -\frac{2a^2}{b}\, e_3, \quad he_3 = -\frac{2a^2}{b}\, e_2, \quad he_4 = \frac{2a^2}{b}\, e_1.$$

As $a > 0$, we conclude that $h \neq 0$ and so, (M, η, g) is not K-contact.

Next, we use equations (2.8) (taking into account $a^4 - \gamma^2 = 1/4$) and (2.9) and apply formulae (4.1), (4.2) and (4.3) to determine the Ricci tensor of (M, η, g) with respect to $\{e_i\}$. A long but straightforward calculation gives

$$Q = \begin{pmatrix} 4\gamma - 2b^2 & 0 & 0 & \frac{4a^2(b^2-2\gamma)}{b^2} & 0 \\ 0 & 4\gamma - 2b^2 & \frac{4a^2(2\gamma - b^2)}{b^2} & 0 & 0 \\ 0 & \frac{4a^2(2\gamma - b^2)}{b^2} & 4\gamma - 2b^2 & 0 & 0 \\ \frac{4a^2(b^2-2\gamma)}{b^2} & 0 & 0 & 4\gamma - 2b^2 & 0 \\ 0 & 0 & 0 & 0 & \frac{4(b^4-4a^4)}{b^2} \end{pmatrix} \tag{4.4}$$

where γ is a solution of $a^4 - \gamma^2 = 1/4$. From (4.4) it follows at once that e_5 (and hence, ξ) is a Ricci eigenvector. Therefore, (M, η, g) is a (homogeneous) H-contact manifold.

Calculating the Ricci eigenvalues from (4.4), we conclude that they never coincide. Hence, (M, η, g) is not Einstein. Finally, as $\eta = be^5$, it easily follows from (4.4) that equation (2.4) holds if and only if $b^2 = 2\gamma$, which, compared with $a^4 - \gamma^2 = 1/4$, yields $4a^4 = b^4 + 1$. Hence, (M, η, g) is η-Einstein if and only if $4a^4 = b^4 + 1$. In this case, (2.4) holds for $\lambda = 0$ and $\nu = \frac{4(b^4-4a^4)}{b^2}$. □

Theorem 4.2. *Let $(M = G/H, \eta, g)$ be a homogeneous contact metric manifold corresponding to a generalized symmetric space of type 8a. Then, (M, η, g) is H-contact, and never K-contact (in particular, Sasakian). Moreover, the following conditions are equivalent:*

(i) *(M, η, g) is Ricci-flat;*

(ii) *(M, η, g) is η-Einstein;*

(iii) $c(= 2b) = 1$.

A similar result holds for homogeneous contact metric manifolds corresponding to a generalized symmetric space of type 8b.

Proof. Calculating h with respect to the basis $\{e_1, \dots, e_4\}$ of $\ker\eta$, from (3.6) we find

$$he_1 = \frac{1}{4b^2}\, e_1, \quad he_2 = -\frac{1}{4b^2}\, e_2, \quad he_3 = \frac{1}{4b^2}\, e_3, \quad he_4 = -\frac{1}{4b^2}\, e_4.$$

So, $h \neq 0$, that is, (M, η, g) is not K-contact.

With regard to the Ricci curvature, starting from the description of generalized symmetric spaces of type $8a$ given in Section 2, taking into account $c = 2b$ we find

$$Q = \begin{pmatrix} \frac{(4b^2-1)^2}{2b^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1-16b^4}{2b^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{4b^2-1}{8b^4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-4b^2}{2b^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{16b^4-1}{4b^4} \end{pmatrix}, \tag{4.5}$$

with respect to the orthogonal basis $\{e_1, \dots, e_5\}$. Therefore, $\xi = \frac{1}{2b} e_5$ is a Ricci eigenvector, that is, (M, η, g) is a (homogeneous) H-contact manifold. The last statement of Theorem 4.2 easily follows from (4.5). □

Theorem 4.3. *Let $(M = G/H, \eta, g)$ be a homogeneous contact metric manifold corresponding to a generalized symmetric space of type 9. Then, (M, η, g) is H-contact, and never K-contact (in particular, Sasakian).*

Moreover, (M, η, g) is never η-Einstein (in particular, Einstein).

Proof. Consider the φ-basis $\{E_1, \dots, E_4\}$ of $\ker\eta$. The Lie brackets $[\xi, E_i]$ can be easily deduced from the definition of $\{E_i\}$ and (3.8). Starting from (3.11), we then find

$$he_1 = -\frac{1}{4a^2}\, e_1, \quad he_2 = \frac{1}{4a^2}\, e_2, \quad he_3 = -\frac{1}{4a^2}\, e_3, \quad he_4 = \frac{1}{4a^2}\, e_4.$$

So, $h \neq 0$, that is, (M, η, g) is not K-contact.

We then start from (2.13) (with $b = 1$) and (2.14) and apply formulae (4.1), (4.2) and (4.3). After a long but direct calculation, we find that with respect to $\{X_1, Y_1, X_2, Y_2, W\}$, the Ricci operator is completely described by

$$QX_1 = -\frac{3}{a^2} X_1, \quad QY_1 = -\frac{3}{a^2} Y_1, \quad QX_2 = QY_2 = QW = 0. \tag{4.6}$$

By (3.7), for the orthonormal basis $\{e_i\}$ we correspondingly have

$$Qe_1 = -\frac{3}{a^2}e_1, \quad Qe_3 = -\frac{3}{a^2}e_3, \quad Qe_2 = Qe_4 = Qe_5 = 0.$$

As $\xi = \frac{1}{2\sqrt{3}a}(e_2 + \sqrt{2}e_4)$, we conclude at once that $Q\xi = 0$. Thus, ξ is a Ricci eigenvector, that is, (M, η, g) is a (homogeneous) H-contact manifold.

Finally, $\eta = \frac{2a}{\sqrt{3}}(e^2 + \sqrt{2}e^4)$ and (4.6) easily imply that (M, η, g) is never η-Einstein. □

5. A rigidity result for the naturally reductive examples

The aim of this Section is to prove the following rigidity result.

Theorem 5.1. *Let (η, g) be any invariant contact metric structure over a (naturally reductive) five-dimensional generalized symmetric space $M = G/H$ of type either 1, 5 or 6. Then, (M, η, g) is a globally φ-symmetric space.*

Proof. We start from the case of a generalized symmetric space M of type 1. Let $\{X_1, Y_1, X_2, Y_2, W\}$ denote the basis used in (2.6). We then consider the orthonormal basis $\{e_1, \ldots, e_5\} := \{X_1, Y_1, X_2, Y_2, (1/\rho)W\}$ and the dual basis $\{e^i\}$ of 1-forms. Consider a 1-form $\eta = \sum_i a_i e^i$, for some real constants $a_1, \ldots, a_5$. From (2.6) we get $de^i = 0$ for $i = 1, \ldots, 4$ and so it is easily seen that η is a contact form if and only if $a_5 \neq 0$.

Consider the corresponding invariant contact structure (φ, η, ξ), having g as an associated metric. Since $\{e_i\}$ is orthonormal, $\eta(X) = g(X, \xi)$ easily yields $\xi = \sum_i a_i e_i$. With regard to tensor φ, as usual it is determined by condition $2\Phi = \mathrm{d}\eta$, which in this case gives

$$\varphi e_1 = -\frac{a_5\rho}{2}e_3, \quad \varphi e_2 = \frac{a_5\rho}{2}e_4, \quad \varphi e_3 = \frac{a_5\rho}{2}e_1, \quad \varphi e_4 = -\frac{a_5\rho}{2}e_2, \quad \varphi e_5 = 0.$$

Applying the second equation of (2.1) to vector fields $e_1, \ldots, e_5$, we then get $a_1 = a_2 = a_3 = a_4 = 0$, $\rho^2 a_5^2 = 4$ and $a_5^2 = 1$. Thus, $\rho = 2$ and up to sign an invariant contact metric structure on a generalized symmetric space of type 1 is necessarily of the form

$$\eta = e^5, \quad \xi = e_5, \quad \varphi e_1 = -e_3, \quad \varphi e_2 = e_4.$$

In order to show that this contact metric structure is φ-symmetric, we calculate $\bar{\nabla}$ and $\bar{R}$ with respect to $\{e_i\}$. Applying (2.5), we find that $\bar{\nabla}$ is completely determined by the following non-vanishing derivatives:

$$\bar{\nabla}_{e_5} e_1 = 2e_3, \quad \bar{\nabla}_{e_5} e_2 = -2e_4, \quad \bar{\nabla}_{e_5} e_3 = -2e_1, \quad \bar{\nabla}_{e_5} e_4 = 2e_2 \tag{5.1}$$

and the non-vanishing components of $\bar{R}$ with respect to $\{e_i\}$ are given by

$$\bar{R}_{1313} = \bar{R}_{1414} = -\bar{R}_{1324} = 4. \tag{5.2}$$

Starting from (5.1) and (5.2), it is now easy to check that $\bar{\nabla}\bar{R} = 0$. Hence, (M, η, g) is φ-symmetric. Precisely, it corresponds to case III in the classification obtained in [10].

Next, we consider the case corresponding to type $5a$ (type $5b$ can be treated in a very similar way). Let $\{e_1, \ldots, e_5\} := \{X_1, X_2, Y_1, Y_2, W\}$ be the basis used in (2.10). The isotropy representation following from (2.10) implies at once that an invariant contact metric structure must be of the form $\eta = a_5 e^5$, for some real constant $a_5 \neq 0$. Following the same argument we already used for type 1, we then consider the corresponding invariant contact structure (φ, η, ξ), having g as an associated metric. Applying (2.1) and (2.2) to vector fields $e_1, \ldots, e_5$, we conclude that an invariant contact metric structure on a generalized symmetric space of type $5a$ exists if and only if $a_5 = c = \pm 1/2$ and (up to sign) is necessarily given by

$$\begin{aligned}
&\eta = \frac{1}{2}\, e^5, && \xi = 2e^5, \\
&\varphi e_1 = -\frac{a^2 - b^2}{2a^2b^2}\, e_2 + \frac{a^2 + b^2}{2a^2b^2}\, e_3, && \varphi e_2 = -\frac{a^2 - b^2}{2a^2b^2}\, e_1 + \frac{a^2 + b^2}{2a^2b^2}\, e_4, \\
&\varphi e_3 = -\frac{a^2 + b^2}{2a^2b^2}\, e_1 + \frac{a^2 - b^2}{2a^2b^2}\, e_4, && \varphi e_4 = -\frac{a^2 + b^2}{2a^2b^2}\, e_2 + \frac{a^2 - b^2}{2a^2b^2}\, e_3.
\end{aligned}$$

Thus, in this case there is a two-parameter family of invariant contact metric structures. A long but standard calculation gives $\bar{\nabla}\bar{R} = 0$. Therefore, these invariant contact metric structures are φ-symmetric. Indeed, they correspond to case I in the classification obtained in [10].

We end the proof considering type $6a$, leaving the similar case $6b$ to the reader. Starting from the basis $\{e_1, \ldots, e_5\} := \{X_1, Y_1, X_2, Y_2, W\}$ used in (2.11), the isotropy representation easily yelds that an invariant contact metric structure has to be of the form $\eta = a_5 e^5$, for some real constant $a_5 \neq 0$. We then describe the corresponding invariant contact structure (φ, η, ξ), having g as an associated metric. We apply (2.1) and (2.2) to vector fields $e_1, \ldots, e_5$ and we find that an invariant contact metric structure on a generalized symmetric space of type $6a$ exists if and only if $c = \pm 2a^2$ and (up to sign) is necessarily given by

$$\eta = 2a^2\, e^5, \quad \xi = \frac{1}{2a^2} e^5, \quad \varphi e_1 = e_3, \quad \varphi e_2 = -e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = e_2.$$

Thus, there exists a one-parameter family of invariant contact metric structures. It is easy to check that $\bar{\nabla}\bar{R} = 0$ and so, these invariant contact metric structures are φ-symmetric. In fact, they correspond to case IV in the classification given in [10]. □

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