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Slant submanifolds of quaternionic space forms

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Abstract. In this paper we establish some inequalities concerning the k-Ricci curvature of a slant submanifold in a quaternionic space form. We also obtain obstructions to the existence of quaternionic slant immersions in quaternionic space forms with unfull first normal bundle.

1. Introduction

According to B.-Y. Chen [4], one of the most important problems in submanifold theory is "*to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold*". In [5], B.-Y. Chen established a sharp inequality between the k-Ricci curvature, one of the main intrinsic invariants, and the squared mean curvature, the main extrinsic invariant, for a submanifold in a real space form with arbitrary codimension. Also, in the same spirit, B.-Y. Chen obtained an optimal inequality between the k-Ricci curvature and the shape operator for submanifolds in real space forms. These inequalities were further extended to many classes of submanifolds in different ambient spaces: complex space forms [25], [26], cosymplectic space forms [23], [24], [38], Sasakian space forms [9], [15], [29], [32], locally conformal Kähler space forms [3], [12], generalized complex space forms [11], [17], [27], locally conformal almost cosymplectic manifolds [16], [37], (κ, μ)-contact space forms [33], Kenmotsu space forms [1], [22], S-space forms [10], [18].

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In quaternionic setting, such inequalities were obtained for quaternionic and totally-real submanifolds [20], [21], [36]. But there are two classes of submanifolds which generalize both quaternionic and totally real submanifolds of quaternionic Kähler manifolds, with no inclusion between them: quaternionic CR-submanifolds (see [2]) and slant submanifolds (see [30]). Some recent results concerning quaternionic CR-submanifolds can be found in [13], [34] and an inequality involving Ricci curvature and squared mean curvature for quaternionic CR-submanifolds in quaternionic space forms was proved in [28]. On the other hand, some optimal inequalities involving scalar curvature, Ricci curvature and squared mean curvature for slant submanifolds in quaternionic space forms were obtained recently in [31], [35]. The main purpose of this paper is to obtain two kinds of inequalities for slant submanifolds in quaternionic space forms: between the k-Ricci curvature and the squared mean curvature and between the k-Ricci curvature and the shape operator. Moreover, we investigate the existence of quaternionic slant immersions in quaternionic space forms with unfull first normal bundle.

2. Preliminaries

Let $\overline{M}$ be a differentiable manifold and assume that there is a rank 3-subbundle σ of $\mathrm{End}(T\overline{M})$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying for all $\alpha \in \{1, 2, 3\}$:

$$J_\alpha^2 = -Id, J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{1}$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3. Then the bundle σ is called an almost quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called a canonical local basis of σ. Moreover, $(\overline{M}, \sigma)$ is said to be an almost quaternionic manifold. It is easy to see that any almost quaternionic manifold is of dimension $4m$.

A Riemannian metric $\overline{g}$ on $\overline{M}$ is said to be adapted to the almost quaternionic structure σ if it satisfies:

$$\overline{g}(J_\alpha X, J_\alpha Y) = \overline{g}(X, Y), \quad \forall \alpha \in \{1, 2, 3\} \tag{2}$$

for all vector fields X,Y on $\overline{M}$ and any canonical local basis $\{J_1, J_2, J_3\}$ of σ. Moreover, $(\overline{M}, \sigma, \overline{g})$ is said to be an almost quaternionic Hermitian manifold.

If the bundle σ is parallel with respect to the Levi–Civita connection $\overline{\nabla}$ of $\overline{g}$, then $(\overline{M}, \sigma, \overline{g})$ is said to be a quaternionic Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that we have for all $\alpha \in \{1, 2, 3\}$:

$$\overline{\nabla}_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}, \tag{3}$$

for any vector field X on $\overline{M}$, where the indices are taken from $\{1, 2, 3\}$ modulo 3 (see [14]).

Let $(\overline{M}, \sigma, \overline{g})$ be a quaternionic Kähler manifold and let X be a non-null vector on $\overline{M}$. Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $Q(X)$, is called a quaternionic 4-plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say c. It is well-known that a quaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is a quaternionic space form, denoted $\overline{M}(c)$, if and only if its curvature tensor is given by (see [14]):

$$\overline{R}(X, Y)Z = \frac{c}{4}\Big\{\overline{g}(Z, Y)X - \overline{g}(X, Z)Y + \sum_{\alpha=1}^{3}[\overline{g}(Z, J_\alpha Y)J_\alpha X - \\ - \overline{g}(Z, J_\alpha X)J_\alpha Y + 2\overline{g}(X, J_\alpha Y)J_\alpha Z]\Big\} \tag{4}$$

for all vector fields X, Y, Z on $\overline{M}$ and any local basis $\{J_1, J_2, J_3\}$ of σ.

For a submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$, we denote by g the metric tensor induced on M. If ∇ is the covariant differentiation induced on M, the Gauss and Weingarten formulas are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM) \tag{5}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad \forall X \in \Gamma(TM), \ \forall N \in \Gamma(TM^{\perp}) \tag{6}$$

where h is the second fundamental form of M, $\nabla^{\perp}$ is the connection on the normal bundle and A_N is the shape operator of M with respect to N. The shape operator A_N is related to h by:

$$g(A_N X, Y) = \overline{g}(h(X, Y), N), \tag{7}$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

If we denote by $\overline{R}$ and R the curvature tensor fields of $\overline{\nabla}$ and ∇ we have the Gauss equation:

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + \overline{g}(h(X, W), h(Y, Z)) - \overline{g}(h(X, Z), h(Y, W)) \tag{8}$$

for all $X, Y, Z, W \in \Gamma(TM)$.

For the second fundamental form h, we define the covariant derivative $\overline{\nabla} h$ of h with respect to the connection on $TM \oplus T^{\perp}M$ by

$$(\overline{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{9}$$

where D is the linear connection induced on the normal bundle of M in $\overline{M}$. Then the equation of Codazzi is given by

$$(\overline{R}(X,Y)Z)^{\perp} = (\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z). \tag{10}$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM and $\{e_{n+1}, \ldots, e_{4m}\}$ is an orthonormal basis of $T_p^{\perp}M$, where $p \in M$, we denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i,j \in \{1, \ldots, n\}, \ r \in \{n+1, \ldots, 4m\}$$

and

$$\|h\|^2(p) = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

A submanifold M of a quaternionic Kähler manifold $\overline{M}$ is called a quaternionic submanifold (resp. totally real submanifold) if each tangent space of M is carried into itself (resp. into the normal space) by each section in σ. Recently, Şahin [30] introduced the slant submanifolds of quaternionic Kähler manifolds, as a natural generalization of both quaternionic and totally real submanifolds. A submanifold M of a quaternionic Kähler manifold $\overline{M}$ is said to be a slant submanifold if for each non-null vector X tangent to M at p, the angle $\theta(X)$ between $J_\alpha(X)$ and T_pM, $\alpha \in \{1,2,3\}$, is constant, i.e. it does not depend on choice of $p \in M$ and $X \in T_pM$. We can easily see that quaternionic submanifolds are slant submanifolds with $\theta = 0$ and totally-real submanifolds are slant submanifolds with $\theta = \frac{\pi}{2}$. A slant submanifold of a quaternionic Käler manifold is said to be proper (or θ-slant proper) if it is neither quaternionic nor totally real.

If M is a slant submanifold of a quaternionic Kähler manifold $\overline{M}$, then for any $X \in \Gamma(TM)$ we have the decomposition

$$J_\alpha X = P_\alpha X + F_\alpha X, \tag{11}$$

where $P_\alpha X$ denotes the tangential component of $J_\alpha X$ and $F_\alpha X$ denotes the normal component of $J_\alpha X$.

Similarly for any $U \in \Gamma(TM^{\perp})$ we have

$$J_\alpha U = B_\alpha U + C_\alpha U, \tag{12}$$

where $B_\alpha U$ is the tangential component of $J_\alpha U$ and $C_\alpha U$ is the normal component of $J_\alpha U$.

We recall now the following results which we shall need in the sequel.

Theorem 2.1 ([30]). *Let M be a submanifold of a quaternionic Kähler manifold $\overline{M}$. Then M is slant if and only if there exists a constant $\lambda \in [-1, 0]$ such that:*

$$P_\beta P_\alpha X = \lambda X, \quad \forall X \in \Gamma(TM),\ \alpha, \beta \in \{1, 2, 3\}. \tag{13}$$

Furthermore, in such case, if θ is the slant angle of M, then it satisfies $\lambda = -\cos^2\theta$.

Corollary 2.2 ([30]). *Let M be a slant submanifold of a quaternionic Kähler manifold $\overline{M}$, with slant angle θ. Then we have*

$$P_\alpha^2 X = -\cos^2\theta X, \tag{14}$$

$$B_\alpha F_\alpha X = -\sin^2\theta X, \tag{15}$$

for any $X \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$.

From the above Theorem we deduce that if M is a θ-slant submanifold of a quaternionic Kähler manifold $\overline{M}$, then we have for any $X, Y \in \Gamma(TM)$:

$$g(P_\alpha X, P_\beta Y) = \cos^2\theta g(X, Y),\ \alpha, \beta \in \{1, 2, 3\} \tag{16}$$

and

$$\overline{g}(F_\alpha X, F_\beta Y) = \sin^2\theta g(X, Y),\ \alpha, \beta \in \{1, 2, 3\}. \tag{17}$$

Moreover, we can remark that every proper slant submanifold of a quaternionic Kähler manifold is of even dimension $n = 2s$, because we can choose a canonical orthonormal local frame $\{e_1, \sec\theta P_\alpha e_1, \ldots, e_s, \sec\theta P_\alpha e_s\}$ of T_pM, $p \in M$, called an adapted slant frame, where α is settled in $\{1, 2, 3\}$.

For an n-dimensional Riemanian manifold (M, g) we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{1 \le i < j \le n} K_{ij}, \tag{18}$$

where K_{ij} denotes the sectional curvatures of the 2-plane section spanned by e_i and e_j.

For a k-plane section L of T_pM, $p \in M$, and X a unit vector in L, we choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$. The Ricci curvature of L at X, denoted $\mathrm{Ric}_L(X)$, is defined by

$$\mathrm{Ric}_L(X) = \sum_{j=2}^{k} K_{1j}. \tag{19}$$

We note that such a curvature is called a k-Ricci curvature. The scalar curvature of a k-plane section L is given by

$$\tau(L) = \sum_{1\le i<j\le k} K_{ij}, \tag{20}$$

For an integer k, $2 \le k \le n$, B.-Y. Chen introduced a Riemannian invariant Θ_k defined by

$$\Theta_k(p) = \frac{1}{k-1}\inf\{\mathrm{Ric}_L(X)|L, X\}, \quad p \in M, \tag{21}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L (see e.g. [8]).

3. k-Ricci curvature and the squared mean curvature

Theorem 3.1. *Let M^n be a θ-slant proper submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then, for any $p \in M$ and any integer k, $2 \le k \le n$, one has:*

$$\|H\|^2(p) \ge \Theta_k(p) - \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right). \tag{22}$$

Proof. We choose an adapted slant basis of T_pM at $p \in M$:

$$\{e_1, e_2 = \sec\theta P_\alpha e_1, \ldots, e_{2s-1}, e_{2s} = \sec\theta P_\alpha e_{2s-1}\},$$

where $2s = n$, and $\{e_{n+1}, \ldots, e_{4m}\}$ an orthonormal basis of $T_p^\perp M$, such that the normal vector e_{n+1} is in the direction of the mean curvature vector $H(p)$ and $\{e_1, \ldots, e_n\}$ diagonalize the shape operator A_{n+1}.

Taking now $X = Z = e_i$, $Y = W = e_j$ in the equation of Gauss (8), by summing and using (4), we obtain:

$$n^2\|H\|^2(p) = 2\tau(p) + \|h\|^2(p) - \frac{n(n-1)c}{4} - \frac{3c}{4}\sum_{\beta=1}^{3}\sum_{i,j=1}^{n} g^2(P_\beta e_i, e_j). \tag{23}$$

On the other hand, because $\{e_1, \ldots, e_{2s}\}$ is an adapted slant basis of T_pM, using (13) and (16) we can see that we have:

$$g^2(P_\beta e_i, e_{i+1}) = g^2(P_\beta e_{i+1}, e_i) = \cos^2\theta, \quad \text{for } i = 1, 3, \ldots, 2s-1 \tag{24}$$

and

$$g(P_\beta e_i, e_j) = 0, \quad \text{for } (i,j) \notin \{(2l-1, 2l), (2l, 2l-1) | l \in \{1, 2, \dots, s\}\}. \tag{25}$$

From (23), (24) and (25) we derive:

$$n^2 \|H\|^2(p) = 2\tau(p) + \|h\|^2(p) - \frac{c}{4}\left[n(n-1) + 9n\cos^2\theta\right]. \tag{26}$$

On the other hand, due to the choosing of the basis of T_pM and $T_p^\perp M$, the shape operators have the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}, \tag{27}$$

$$A_r = (h_{ij}^r)_{i,j=\overline{1,n}}, \ \text{trace} A_r = \sum_{i=1}^{n} h_{ii}^r = 0, \ \forall r \in \{n+2, \dots, 4m\}. \tag{28}$$

Now, using (27) and (28) in (26) we obtain:

$$n^2 \|H\|^2(p) = 2\tau(p) + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \frac{c}{4}\left[n(n-1) + 9n\cos^2\theta\right]. \tag{29}$$

On the other hand, because we have the inequality

$$(n-1)\sum_{i=1}^{n} a_i^2 \geq 2\sum_{i<j} a_i a_j,$$

from

$$n^2 \|H\|^2(p) = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{1 \leq i < j \leq n} a_i a_j$$

we derive

$$\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2(p). \tag{30}$$

Using now (30) in (29) we obtain:

$$n(n-1)\|H\|^2(p) \geq 2\tau(p) - \frac{c}{4}\left[n(n-1) + 9n\cos^2\theta\right]. \tag{31}$$

But, from (18) and (20), it follows that for any k-plane section $L_{i_1\dots i_k}$ spanned by $\{e_{i_1},\dots,e_{i_k}\}$, one has:

$$\tau(L_{i_1\dots i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\dots,i_k\}} \mathrm{Ric}_{L_{i_1\dots i_k}}(e_i) \tag{32}$$

and

$$\tau(p) = \frac{(k-2)!(n-k)!}{(n-2)!}\sum_{1\le i_1<\dots<i_k\le n} \tau(L_{i_1\dots i_k}). \tag{33}$$

From (32) and (33) we obtain:

$$\tau(p) \ge \frac{n(n-1)}{2}\cdot\Theta_k(p) \tag{34}$$

and finally, from (31) and (34) one derives (22). □

Applying Theorem 3.1 we may obtain, as a particular case, the corresponding inequality for totally-real submanifolds in quaternionic space forms, established in [21].

Corollary 3.2. *Let M^n be a totally-real submanifold of a quaternionic space form $\overline{M}^{4m}(c)$. Then, for any $p\in M$ and any integer k, $2\le k\le n$, one has:*

$$\|H\|^2(p) \ge \Theta_k(p) - \frac{c}{4}. \tag{35}$$

4. *k*-Ricci curvature and shape operator

Theorem 4.1. *Let $x : M \to \overline{M}^{4m}(c)$ be an isometric immersion of an n-dimensional θ-slant proper submanifold M into a $4m$-dimensional quaternionic space form $\overline{M}(c)$. Then, for any $p\in M$ and any integer k, $2\le k\le n$, one has:*

i. *If $\Theta_k(p) \ne \frac{c}{4}\left(1+\frac{9}{n-1}\cos^2\theta\right)$, then the shape operator at the mean curvature satisfies*

$$A_H > \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4}\left(1+\frac{9}{n-1}\cos^2\theta\right)\right] I_n, \tag{36}$$

at p, where I_n denotes the identity map of T_pM.

ii. *If $\Theta_k(p) = \frac{c}{4}\left(1+\frac{9}{n-1}\cos^2\theta\right)$, then $A_H \ge 0$ at p.*

iii. *A unit vector $X \in T_pM$ satisfies*

$$A_H X = \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right)\right] X \tag{37}$$

if and only if $\Theta_k(p) = \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right)$ *and X belongs to the relative null space of M at p:*

$$\mathcal{N}_p = \{Z \in T_pM | h(Z,Y) = 0,\ \forall Y \in T_pM\}.$$

iv. *The identity*

$$A_H = \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right)\right] I_n, \tag{38}$$

holds at p if and only if p is a totally geodesic point.

Proof. i. We choose an adapted slant basis of T_pM at $p \in M$:

$$\{e_1, e_2 = \sec\theta P_\alpha e_1, \dots, e_{2s-1}, e_{2s} = \sec\theta P_\alpha e_{2s-1}\},$$

where $2s = n$, and $\{e_{n+1}, \dots, e_{4m}\}$ an orthonormal basis of $T_p^\perp M$, such that the normal vector e_{n+1} is in the direction of the mean curvature vector $H(p)$ and $\{e_1, \dots, e_n\}$ diagonalize the shape operator A_{n+1}. Consequently, the shape operators have the forms (27) and (28).

One can distinguishes two cases:

Case I: $H(p) = 0$. In this situation it follows from (22) that $\Theta_k(p) \neq \frac{c}{4}\left(1 + \frac{9}{n-1}\cos^2\theta\right)$ and the conclusion follows.

Case II: $H(p) \neq 0$. Taking $X = Z = e_i$ and $Y = W = e_j$ in the Gauss equation and using (4), we obtain:

$$a_i a_j = K_{ij} - \frac{c}{4}\left[1 + 3\sum_{\beta=1}^{3} g^2(P_\beta e_i, e_j)\right] - \sum_{r=n+2}^{4m}\left[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\right]. \tag{39}$$

From (39) we derive:

$$\begin{aligned} a_1(a_{i_2} + \dots + a_{i_k}) = \mathrm{Ric}_{L_{1i_2\dots i_k}}(e_1) - \frac{(k-1)c}{4} - \frac{3c}{4}\sum_{\beta=1}^{3}\sum_{j=2}^{k} g^2(P_\beta e_1, e_{i_j}) \\ - \sum_{r=n+2}^{4m}\sum_{j=2}^{k}\left[h_{11}^r h_{i_j i_j}^r - (h_{1i_j}^r)^2\right] \end{aligned} \tag{40}$$

which implies

$$a_1(a_2+\cdots+a_n) = \frac{(k-2)!(n-k)!}{(n-2)!}\sum_{2\le i_2<\cdots<i_k\le n}\mathrm{Ric}_{L_{1i_2\dots i_k}}(e_1) - \frac{(n-1)c}{4} - \frac{3c}{4}\sum_{\beta=1}^{3}\sum_{j=2}^{n} g^2(P_\beta e_1, e_j) + \sum_{r=n+2}^{4m}\sum_{j=1}^{n}(h_{1j}^r)^2. \tag{41}$$

and taking into account (21), we obtain:

$$a_1(a_2+\cdots+a_n) \ge (n-1)\theta_k(p) - \frac{(n-1)c}{4} - \frac{3c}{4}\sum_{\beta=1}^{3}\sum_{j=2}^{n} g^2(P_\beta e_1, e_j). \tag{42}$$

Using (24) and (25) in (42) we obtain:

$$a_1(a_2+\cdots+a_n) \ge (n-1)\Theta_k(p) - \frac{(n-1)c}{4} - \frac{9c}{4}\cos^2\theta \tag{43}$$

and we find:

$$a_1(a_1+a_2+\cdots+a_n) = a_1^2 + a_1(a_2+\cdots+a_n) \ge (n-1)\left[\Theta_k(p) - \frac{c}{4}\left(1+9\cos^2\theta\right)\right]. \tag{44}$$

Similar inequalities hold when the index 1 is replaced by $j \in \{2,\dots,n\}$. Hence, we have

$$a_j(a_1+a_2+\cdots+a_n) \ge (n-1)\left[\Theta_k(p) - \frac{c}{4}\left(1+9\cos^2\theta\right)\right], \tag{45}$$

for all $j \in \{1,\dots,n\}$, and because $n\|H\| = a_1+\cdots+a_n$ we find

$$A_H \ge \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4}\left(1+\frac{9}{n-1}\cos^2\theta\right)\right] I_n. \tag{46}$$

We remark that the equality does not hold because we are in the case $H(p) \neq 0$.

ii. The statement is clear from i.

iii. If $X \in T_pM$ is a unit vector such that (37) holds, then we have equalities both in (42) and (44). Consequently, we obtain $a_1 = 0$ and $h_{1j}^r = 0$, for all $j \in \{1,\dots,n\}$ and $r \in \{n+2,\dots,4m\}$, which implies $\Theta_k(p) = \frac{c}{4}\left(1+\frac{9}{n-1}\cos^2\theta\right)$ and $X \in \mathcal{N}_p$. The converse part is clear.

iv. The equality (41) holds for any $X \in T_pM$ if and only if $\mathcal{N}_p = TpM$, i.e. p is a totally geodesic point. This completes the proof of the theorem. □

Corollary 4.2. *Let $x : M \to \overline{M}^{4m}(c)$ be an isometric immersion of an n-dimensional totally-real submanifold M into a $4m$-dimensional quaternionic space form $\overline{M}(c)$. Then, for any $p \in M$ and any integer k, $2 \leq k \leq n$, one has:*

i. *If $\Theta_k(p) \neq \frac{c}{4}$, then the shape operator at the mean curvature satisfies*

$$A_H > \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4}\right] I_n, \tag{47}$$

at p, where I_n denotes the identity map of T_pM.

ii. *If $\Theta_k(p) = \frac{c}{4}$, then $A_H \geq 0$ at p.*

iii. *A unit vector $X \in T_pM$ satisfies*

$$A_H X = \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4}\right] X \tag{48}$$

if and only if $\Theta_k(p) = \frac{c}{4}$ and $X \in \mathcal{N}_p$.

iv. *The identity*

$$A_H = \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4}\right] I_n, \tag{49}$$

holds at p if and only if p is a totally geodesic point.

5. Quaternionic slant submanifolds with unfull first normal bundle

Let M be a submanifold isometrically immersed in a Riemannian manifold $(\overline{M}, \overline{g})$. If p is a point of M, then the first normal space at p is defined to be $\operatorname{Im} h_p$, the image space of the second fundamental form h at p. Moreover, $\operatorname{Im} h$ is called the first normal bundle of M in $\overline{M}$. The submanifold is said to have full first normal bundle if $\operatorname{Im} h_p = T_pM^{\perp}$, for any $p \in M$ (see [6], [7]).

The existence of Kählerian slant submanifolds of smallest possible codimension in complex space forms, having unfull first normal bundle, has been investigated in [19]. Next we'll study this problem in the context of slant submanifolds in quaternionic space forms. The quaternionic version of Kählerian slant submanifolds has been introduced in [30], under the name of quaternionic slant submanifolds. Therefore a proper slant submanifold M of a quaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is said to be quaternionic slant submanifold if it satisfies the condition

$$\overline{\nabla}_X P_\alpha = \omega_{\alpha+2}(X) P_{\alpha+1} - \omega_{\alpha+1}(X) P_{\alpha+2}, \tag{50}$$

for any vector field X on $\overline{M}$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

We have the following characterization of quaternionic slant submanifolds.

Theorem 5.1 ([30]). *Let M be a proper slant submanifold of a quaternionic Kähler manifold $\overline{M}$. Then M is quaternionic slant submanifold if and only if*

$$A_{F\alpha Y}Z = A_{F\alpha Z}Y \tag{51}$$

for all $Y, Z \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}$.

Lemma 5.2. *Let M be a slant submanifold of a quaternionic Kähler manifold $\overline{M}$. Then we have*

$$P_\alpha^2 = -Id - B_\alpha F_\alpha, \tag{52}$$

$$C_\alpha F_\alpha + F_\alpha P_\alpha = 0, \tag{53}$$

$$C_\alpha^2 = -Id - F_\alpha B_\alpha, \tag{54}$$

$$P_\alpha B_\alpha + B_\alpha C_\alpha = 0 \tag{55}$$

for $\alpha \in \{1,2,3\}$.

Proof. For any $X \in \Gamma(TM)$, taking into account (1) and (11), we derive

$$-X = J_\alpha^2 X = P_\alpha^2 X + F_\alpha P_\alpha X + B_\alpha F_\alpha X + C_\alpha F_\alpha X.$$

Equating the tangent and normal parts of both the sides we obtain (52) and (53).

On the other hand, taking into account (1) and (12) we conclude that for any $U \in \Gamma(TM^\perp)$ we have

$$-U = J_\alpha^2 U = P_\alpha B_\alpha U + F_\alpha B_\alpha U + B_\alpha C_\alpha U + C_\alpha^2 U.$$

Equating now the tangent and normal parts of both the sides we obtain (54) and (55). □

Lemma 5.3. *Let M be a θ-slant proper submanifold of a quaternionic Kähler manifold $\overline{M}$. Then for any vectors $U, V \in T_pM^\perp$, $p \in M$, we have*

$$\overline{g}(C_\alpha U, C_\alpha V) = \cos^2\theta\,\overline{g}(U,V), \quad \alpha = 1,2,3. \tag{56}$$

Proof. Because M is a θ-slant proper submanifold of $\overline{M}$, it follows that there exist $X_\alpha, Y_\alpha \in T_pM$ such that $U = F_\alpha X_\alpha$, $V = F_\alpha Y_\alpha$. Then, by using (16), (17) and (53), we derive

$$\begin{aligned}\overline{g}(C_\alpha U, C_\alpha V) &= \overline{g}(C_\alpha F_\alpha X_\alpha, C_\alpha F_\alpha Y_\alpha) = \overline{g}(F_\alpha P_\alpha X_\alpha, F_\alpha P_\alpha Y_\alpha)\\ &= \sin^2\theta\,\overline{g}(P_\alpha X_\alpha, P_\alpha Y_\alpha) = \sin^2\theta\cos^2\theta\,\overline{g}(X_\alpha, Y_\alpha)\\ &= \cos^2\theta\,\overline{g}(F_\alpha X_\alpha, F_\alpha Y_\alpha) = \cos^2\theta\,\overline{g}(U,V).\end{aligned}$$

□

From Theorem 5.1 and Lemma 5.3, using the same techniques as in [19], we can state now the following result.

Lemma 5.4. *Let M be a quaternionic slant submanifold of a quaternionic Kähler manifold $\overline{M}$. Then*

$$B_\alpha(\operatorname{Im} h_p)^\perp = \mathcal{N}_p, \quad \alpha = 1,2,3,$$

where $(\operatorname{Im} h_p)^\perp$ denotes the orthogonal complementary subspace of $\operatorname{Im} h_p$ in $T_pM^\perp$ and $\mathcal{N}_p$ is the relative null space of M at p.

Proof. For $Z \in B_\alpha(\operatorname{Im} h_p)^\perp$ it follows that there exists $U \in (\operatorname{Im} h_p)^\perp$ such that $Z = B_\alpha U$. Then, by using (7), (51), (54) and (56), we obtain for all vector $X, Y \in T_pM$ and $\alpha = 1,2,3$:

$$\begin{aligned}\overline{g}(h(X,Z), F_\alpha Y) &= g(A_{F_\alpha Y}Z, X) = g(A_{F_\alpha Z}Y, X) = \overline{g}(h(X,Y), F_\alpha Z)\\ &= \overline{g}(h(X,Y), F_\alpha B_\alpha U) = \sin^2\theta \overline{g}(h(X,Y),U) = 0.\end{aligned}$$

Therefore it follows that $h(X,Z) = 0$, for any $X \in T_pM$ and thus we obtain $Z \in \mathcal{N}_p$.

If we take now $Z \in \mathcal{N}_p$, it is clear that for any $X, Y \in T_pM$ and $\alpha = 1,2,3$ we have

$$\overline{g}(h(X,Y), F_\alpha Z) = \overline{g}(h(Z,X), F_\alpha Y) = 0.$$

Thus it follows $F_\alpha Z \in (\operatorname{Im} h_p)^\perp$ and therefore we derive

$$B_\alpha F_\alpha Z \in B_\alpha(\operatorname{Im} h_p)^\perp. \tag{57}$$

From (15) and (57) we conclude that $Z \in B_\alpha(\operatorname{Im} h_p)^\perp$ and the proof is now complete. □

Theorem 5.5. *Let $x : M \to \overline{M}(c)$ be an isometric immersion of a quaternionic slant submanifold M of minimal codimension into a quaternionic space form $\overline{M}(c)$. If the first normal bundle is not full, then $c = 0$.*

Proof. First of all we remark that if the dimension of $\overline{M}(c)$ is $4m$, then the minimal codimension of a proper slant submanifold M of $\overline{M}(c)$ is $2m$; in this case we can choose an adapted slant basis of T_pM at $p \in M$:

$$\{e_1, e_2 = \sec\theta P_\alpha e_1, \ldots, e_{2m-1}, e_{2m} = \sec\theta P_\alpha e_{2m-1}\},$$

and an orthonormal basis of $T_p^\perp M$:

$$\{e_{2m+1} = \operatorname{cosec}\theta F_\alpha e_1, e_{2m+2} = \operatorname{cosec}\theta F_\alpha e_2, \ldots, e_{4m} = \operatorname{cosec}\theta F_\alpha e_{2m}\},$$

where α is settled in $\{1,2,3\}$.

Moreover, if the first normal bundle is not full, then it follows that there exists a unit normal vector $U \in T_pM^{\perp}$ at a point $p \in M$ such that $\overline{g}(h(X,Y),U) = 0$, for any vector $X, Y \in T_pM$ and without loss of generality we can suppose $e_{4m} = U$. Applying Lemma 5.4 it follows $B_\alpha e_{4m} \in \mathcal{N}_p$, $\alpha = 1,2,3$, and from (15) we conclude $e_{2m} \in \mathcal{N}_p$. Thus we have

$$h(e_i, e_{2m}) = 0, \quad i = 1, \ldots, 2m-1. \tag{58}$$

By using now (9) and (58) in (10) we obtain for $i = 1, \ldots, 2m-1$:

$$(\overline{R}(e_i, e_{2m})e_{2m})^{\perp} = h(e_i, \nabla_{e_{2m}} e_{2m})$$

and taking into account (58) and the definition of the Christoffel symbols Γ_{ij}^k:

$$\nabla_{e_i} e_j = \sum_{k=1}^{2m} \Gamma_{ij}^k e_k$$

we obtain

$$(\overline{R}(e_i, e_{2m})e_{2m})^{\perp} = \sum_{1 \leq k,l < 2m} \Gamma_{2m2m}^k h_{ik}^{2m+l} e_{2m+l}. \tag{59}$$

On the other hand, from (4) we obtain

$$\overline{R}(e_i, e_{2m})e_{2m} = \frac{c}{4}\left[e_i + 3\sum_{\beta=1}^{3} \overline{g}(e_i, J_\beta e_{2m}) J_\beta e_{2m}\right]$$

and therefore

$$(\overline{R}(e_i, e_{2m})e_{2m})^{\perp} = \frac{3c}{4}\sum_{\beta=1}^{3} g(e_i, P_\beta e_{2m}) F_\beta e_{2m}. \tag{60}$$

But, since M is a slant submanifold, we can easily remark that

$$P_1X = P_2X = P_3X, \ X \in T_pM. \tag{61}$$

On the other hand, using (17) we obtain for all $\beta \in \{1,2,3\}$ and $k \in \{1, \ldots, 2m\}$:

$$\overline{g}(F_\beta e_{2m}, e_{2m+k}) = \operatorname{cosec}\theta \overline{g}(F_\beta e_{2m}, F_\alpha e_k) = \operatorname{cosec}\theta sin^2\theta g(e_{2m}, e_k) = \sin\theta \delta_{2mk},$$

where δ_{ij} denotes the Kronecker delta. Thus we derive

$$F_1 e_{2m} = F_2 e_{2m} = F_3 e_{2m} = \sin\theta e_{4m}. \tag{62}$$

From (60), (61) and (62) we derive

$$(\overline{R}(e_i, e_{2m})e_{2m})^{\perp} = \frac{9c}{4} g(e_i, P_\alpha e_{2m}) \sin\theta e_{4m}$$

and considering the decomposition of $P_\alpha e_{2m}$ with respect to the adapted slant basis of T_pM:

$$P_\alpha e_{2m} = \sum_{j=1}^{2m-1} \lambda_j e_j$$

we obtain

$$(\overline{R}(e_i, e_{2m})e_{2m})^{\perp} = \frac{9c}{4} \lambda_i \sin\theta e_{4m}. \tag{63}$$

Comparing now (59) and (63) we derive

$$9c\lambda_i \sin\theta = 0, \quad i = 1, \dots, 2m-1,$$

and since M is a proper slant submanifold of $\overline{M}$ and $\sum_{i=1}^{2m-1} \lambda_i^2 \neq 0$, we conclude that $c = 0$. □

Corollary 5.6. *There do not exist quaternionic slant immersions of minimal codimension in $P^m(\mathbb{H})$ with unfull first normal bundle.*

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