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A new class of projectively flat Finsler metrics in terms of hypergeometric functions

By LIBING HUANG (Tianjin) and XIAOHUAN MO (Beijing)

Abstract. In this paper, we study a class of Finsler metrics in the form $F(x,y) = |y|\phi\big(|x|, \frac{\langle x,y\rangle}{|y|}\big)$. We obtain the second-order differential equation for F to be projectively flat and manufacture new projectively flat Finsler metrics on an open subset in $\mathbb{R}^n$ using hypergeometric functions. We also express F in terms of elementary functions in some special cases.

1. Introduction

Hilbert's Fourth Problem (characterize the distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are shortest paths) is particularly attractive [8], [13]. Distance functions induced by Finsler metrics are regarded as smooth ones. Thus Hilbert's Fourth Problem in the smooth case is to characterize and study projectively flat Finsler metrics on an open subset in $\mathbb{R}^n$. In general, projectively flat Finsler metrics are characterized by vanishing Douglas curvature and Weyl curvature. A Randers metric $F = \alpha + \beta$ is projectively flat if and only if α is projectively flat and β is closed. Randers metrics are simplest (α, β)-metrics. Apart from some special case, Z. Shen have obtained the equations which characterizes projectively flat (α, β)-metrics [5], [12]. One of them is ϕ satisfying ODE

$$\left[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2\right] \phi''(s) = (k_1 + k_2 s^2)\left[\phi(s) - s\phi'(s)\right] \tag{1.1}$$

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The second author (Xiaohuan Mo) is the corresponding author.

where k_1, k_2 and k_3 are constants. Recall that an (α,β)-*metric* is a Finsler metric in the form $F=\alpha\phi(\frac{\beta}{\alpha})$ where α is a Riemannian metric and β is a 1-form with $\|\beta_x\|_\alpha<b_o$ and $\phi=\phi(s)$ is a C^∞ positive function on an open interval $(-b_o,b_o)$. By using Shen's equations, many Finslerian geometers manufactured explicitly or classified projectively flat Finsler metrics on an open subset in $\mathbb{R}^n$ [7], [10], [12]. For instance, the second author constructed projectively flat (α,β)-metrics with $k_2\neq 0$ by using hypergeometric functions (see (1.1)) [6]. Hypergeometric functions are solutions of hypergeometric differential equations, which include binomial functions, logarithmic functions and arctangent functions. Lately, a Finsler metric in the form $F=\alpha\phi(\rho,s)$ was studied in [14] (see Section 2 for details). These metrics not only include Randers metrics and (α,β)-metrics, but also include a Bryant metric which is projectively flat on the standard unit sphere $\mathbb{S}^n$ with constant flag curvature $K=1$.

In fact, the following projectively flat Finsler metric is also general (α,β)-metric

$$\begin{aligned}F_\varepsilon(x,y)=&\frac{1}{2}\left\{\frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle}{1-|x|^2}\right\}\\&-\frac{1}{2}\left\{\frac{\varepsilon\sqrt{(1-\varepsilon^2|x|^2)|y|^2+\varepsilon^2\langle x,y\rangle^2}+\varepsilon^2\langle x,y\rangle}{1-\varepsilon^2|x|^2}\right\}\end{aligned}$$

where $\varepsilon<1$. It is easy to see that

$$F_\varepsilon=\alpha\phi_\varepsilon(\rho,s)$$

where

$$\alpha=|y|,\qquad \phi_\varepsilon(\rho,s)=\frac{1}{2}\left\{\frac{\sqrt{1-(\rho-s^2)}+s}{1-\rho}-\frac{\varepsilon\sqrt{1-\varepsilon^2(\rho-s^2)}+\varepsilon^2 s}{1-\varepsilon^2\rho}\right\}$$

where

$$\rho=|x|^2,\qquad s=\frac{\langle x,y\rangle}{|y|}.$$

We know that F is of constant flag curvature $K=-1$. This example was discovered by Z. Shen in 2002 (see [1], [9] and [11, Example 2.5]). In particular, F_{-1} is the famous Klein metric on the unit ball $\mathbb{B}^n$. In [14], a sufficient condition on ϕ, α and β for the general (α,β)-metric to be projectively flat was obtained.

In this paper, we study general (α,β)-metrics in the form

$$F(x,y)=\alpha\left[\epsilon+\rho^\mu f\left(\frac{s}{\rho}\right)\right]$$

where $\epsilon>0$, $\rho=\|\beta\|_\alpha$ and $s=\beta/\alpha$. We obtain the ordinary differential equation on f for F to be projectively flat (see Theorem 3.1 below). Furthermore, we give the solutions of this ODE in terms of hypergeometric functions. Precisely we prove the following:

Theorem 1.1. *Let $f(\lambda)$ be a function defined by*

$$f(\lambda) = \delta\lambda + \text{hypergeom}\left(\left[-\frac{1}{2}, -\frac{\mu}{2}\right], \left[\frac{1}{2}\right], \lambda^2\right)$$

where δ and μ are constants ($\mu \geq 0$). Then the following general (α, β)-metric on an open subset in $\mathbb{R}^n \backslash \{0\}$

$$F = |y| \left\{ \epsilon + |x|^\mu f\left(\frac{\langle x, y\rangle}{|x|\,|y|}\right)\right\}$$

is projectively flat.

Recall that for $a, b, c \in \mathbb{R}$, the *hypergeometric function* is defined by

$$\text{hypergeom}([a, b], [c], t) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} t^n \tag{1.2}$$

where $a, b, c \in \mathbb{R}$ and

$$(d)_n := \begin{cases} 1 & \text{if } n = 0 \\ d(d+1)\ldots(d+n-1) & \text{if } n \geq 1. \end{cases} \tag{1.3}$$

From Theorem 1.1, we construct explicitly a lot of new projectively flat Finsler metrics. In addition, we give the elementary function expression of F for some special μ (see Proposition 4.4 and Proposition 4.5 below).

2. Preliminaries

A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is said to be *projectively flat* if all geodesics are straight in $\mathcal{U}$, equivalently, it satisfies the following system of equations [3],

$$F_{x^j y^i} y^j = F_{x^i}. \tag{2.1}$$

Consider Finsler metrics defined by a Riemannian metric $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and a 1-form with $\|\beta_x\|_\alpha < b_o$. They are expressed in the form $F = \alpha\phi(\rho, s)$, $\rho = \|\beta_x\|_\alpha$, $s = \frac{\beta}{\alpha}$, where $\phi = \phi(\rho, s)$ is a C^∞ positive function on an open interval

$(-b_o, b_o)$. It is known that $F = \alpha\phi\big(\|\beta_x\|_\alpha, \frac{\beta}{\alpha}\big)$ is a Finsler metric for any α and β with $\|\beta_x\|_\alpha < b_o$ if and only if

$$\phi(s) - s\phi_s(s) > 0, \quad \phi(s) - s\phi_s(s) + (\rho^2 - s^2)\phi_{ss}(s) > 0, \quad |s| \leq \rho < b_o$$

where $n \geq 3$ or

$$\phi(s) - s\phi_s(s) + (\rho^2 - s^2)\phi_{ss}(s) > 0, \quad |s| \leq \rho < b_o$$

where $n = 2$ [14]. Such a metric is called a *general* (α, β)-*metric*.

A function ξ defined on $T\mathcal{U}$ can be expressed as $\xi(x^1, \ldots, x^n; y^1, \ldots, y^n)$. We use the following notation

$$\xi_0 = \frac{\partial \xi}{\partial x^i} y^i.$$

By (2.1), we obtain the following

Lemma 2.1. *A Finsler metric $F = F(x, y)$ is projectively flat if and only if it satisfies the following system of equations*

$$(F_0)_{y^i} = 2F_{x^i}. \tag{2.2}$$

3. Reducible differential equation

Consider the general (α, β)-metric $F = \alpha\phi\big(\|\beta_x\|_\alpha, \frac{\alpha}{\beta}\big)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$, where $\phi = \phi(\rho, s)$ is given by

$$\phi = \epsilon + \rho^\mu f\left(\frac{s}{\rho}\right)$$

and

$$\alpha = |y|, \quad \beta = \langle x, y \rangle.$$

In this section, we are going to find a necessary and sufficient condition on f for F to be projectively flat.

Lemma 3.1. *Define*

$$\tilde{F}(x, y) := F(x, y) - \epsilon|y|. \tag{3.1}$$

Then $F = F(x, y)$ is projectively flat if and only if $\tilde{F}$ satisfies the following system of equations

$$(\tilde{F}_0)_{y^i} = 2\tilde{F}_{x^i}.$$

Proof. By (3.1), we obtain

$$\tilde{F}_{x^i} = F_{x^i}.$$

It follows that

$$\tilde{F}_0 = F_0, \quad (\tilde{F}_0)_{y^i} = (F_0)_{y^i}.$$

Together with Lemma 2.1 yields Lemma 3.1. □

We rewrite our general (α, β)-metric as follows:

$$F(x,y) = |y| \left\{\epsilon + |x|^{\mu} f(\lambda)\right\} \tag{3.2}$$

where

$$\lambda = \frac{\langle x, y\rangle}{|x|\,|y|}.$$

Direct calculations yields

$$|x|_{x^i} = \frac{x^i}{|x|}, \qquad |y|_{y^i} = \frac{y^i}{|y|}. \tag{3.3}$$

It follows that

$$\lambda_{x^i} = \frac{1}{|y|}\left(\frac{\langle x, y\rangle}{|x|}\right)_{x^i} = \frac{1}{|y|}\frac{|x|y^i - \langle x, y\rangle \frac{x^i}{|x|}}{|x|^2} = \frac{|x|^2 y^i - \langle x, y\rangle x^i}{|y|\,|x|^3}. \tag{3.4}$$

Similarly, we get

$$\lambda_{y^i} = \frac{|y|^2 x^i - \langle x, y\rangle y^i}{|x|\,|y|^3}. \tag{3.5}$$

By (3.1) and (3.2), we have

$$\tilde{F}(x,y) = |y|\,|x|^{\mu} f(\lambda), \quad \lambda = \frac{\langle x, y\rangle}{|x|\,|y|}.$$

Together with (3.4) yields

$$\begin{aligned}\tilde{F}_{x^i} &= |y|\left(\mu|x|^{\mu-2}x^i f + |x|^{\mu} f' \frac{|x|^2 y^i - \langle x, y\rangle x^i}{|y|\,|x|^3}\right)\\ &= \mu|x|^{\mu-2}|y|x^i f + |x|^{\mu-3}(|x|^2 y^i - \langle x, y\rangle x^i) f' \end{aligned} \tag{3.6}$$

where we have used the following

$$(|x|^{\mu})_{x^i} = \left[\left(|x|^2\right)^{\frac{2}{\mu}}\right]_{x^i} = \mu|x|^{\mu-2}x^i, \quad f' = \frac{\partial f}{\partial \lambda} = f'\left(\frac{\langle x, y\rangle}{|x|\,|y|}\right).$$

It follows that

$$\tilde{F}_0 := \tilde{F}_{x^i} y^i = \mu |x|^{\mu-2}|y|\langle x, y\rangle f + |x|^{\mu-3}(|x|^2|y|^2 - \langle x, y\rangle^2) f'$$
$$= \mu |x|^{\mu-2}|y|\langle x, y\rangle f + |x|^{\mu-1}|y|^2(1-\lambda^2) f'. \tag{3.7}$$

A simple calculation gives the following formula:

$$\left[|y|^2(1-\lambda^2)\right]_{y^i} = 2y^i - \frac{2\langle x, y\rangle x^i}{|x|^2}.$$

Together with (3.5) and (3.7) we obtain

$$\begin{aligned}\left(\tilde{F}_0\right)_{y^i} &= \mu|x|^{\mu-2}\left(|y|\langle x, y\rangle f\right)_{y^i} + |x|^{\mu-1}\left[|y|^2(1-\lambda^2) f'\right]_{y^i} \\ &= \mu|x|^{\mu-2}\left\{\frac{y^i}{|y|}\langle x, y\rangle f + |y| x^i f + \langle x, y\rangle \frac{|y|^2 x^i - \langle x, y\rangle y^i}{|x|\,|y|^2} f'\right\} \\ &\quad + |x|^{\mu-1}\left\{2\left(y^i - \frac{\langle x, y\rangle x^i}{|x|^2}\right) f' + |y|^2(1-\lambda^2) f''\lambda_{y^i}\right\} \\ &= \mu|x|^{\mu-2}\left(|y| x^i + \frac{\langle x, y\rangle}{|y|} y^i\right) f \\ &\quad + |x|^{\mu-3}\left[(\mu-2)\langle x, y\rangle x^i + \left(2|x|^2 - \mu\frac{\langle x, y\rangle^2}{|y|^2}\right) y^i\right] f' \\ &\quad + |x|^{\mu-1}|y|^2(1-\lambda^2)\lambda_{y^i} f''. \end{aligned} \tag{3.8}$$

By (3.6), (3.7) and Lemma 3.1, $F = F(x, y)$ is projectively flat if and only if

$$\begin{aligned}\mu|x|^{\mu-2}&\left(|y| x^i + \frac{\langle x, y\rangle}{|y|} y^i\right) f + |x|^{\mu-1}|y|^2(1-\lambda^2)\lambda_{y^i} f'' \\ &+ |x|^{\mu-3}\left[(\mu-2)\langle x, y\rangle x^i + \left(2|x|^2 - \mu\frac{\langle x, y\rangle^2}{|y|^2}\right) y^i\right] f' \\ &= 2\mu|x|^{\mu-2}|y| x^i f + 2|x|^{\mu-3}(|x|^2 y^i - \langle x, y\rangle x^i) f'. \end{aligned} \tag{3.9}$$

By (3.5), (3.9) holds if and only if

$$\begin{aligned}-|x|^{\mu-1}|y|^2(1-\lambda^2)\lambda_{y^i} f'' - \mu|x|^{\mu-3}\langle x, y\rangle\lambda_{y^i}|x|\,|y| f' \\ + \mu|x|^{\mu-2}\lambda_{y^i}|x|\,|y|^2 f = 0. \end{aligned} \tag{3.10}$$

By the definition of λ, (3.10) holds if and only if

$$|x|^{\mu-1}|y|^2\lambda_{y^i}\left[(\lambda^2-1) f'' - \mu\lambda f' + \mu f\right] = 0. \tag{3.11}$$

Plugging (3.5) into (3.11) yields

$$|x|^{\mu-2}\frac{|y|^2x^i-\langle x,y\rangle y^i}{|y|}\big[(\lambda^2-1)f''-\mu\lambda f'+\mu f\big]=0. \tag{3.12}$$

Contracting (3.12) with x^i yields

$$|x|^{\mu-2}\frac{|y|^2|x|^2-\langle x,y\rangle^2}{|y|}\big[(\lambda^2-1)f''-\mu\lambda f'+\mu f\big]=0.$$

Taking x and y with $x\wedge y\neq 0$. Then

$$(\lambda^2-1)f''-\mu\lambda f'+\mu f=0. \tag{3.13}$$

Thus we have the following

Theorem 3.1. *Let* $F(x,y):=|y|\big\{\epsilon+|x|^\mu f\big(\frac{\langle x,y\rangle}{|x|\,|y|}\big)\big\}$ *be a general* (α,β)*-metric on an open subset* $\mathcal{U}\subset\mathbb{R}^n$*. Then* $F=F(x,y)$ *is projectively flat if and only if* f *satisfies* (3.13).

Note that (3.13) is the Gegenbauer-type ordinary differential equation.

4. Solutions

In order to find projectively flat general (α,β)-metrics we consider the following ordinary differential equation:

$$\begin{cases}(1-\lambda^2)f''(\lambda)=\mu(f-\lambda f')\\ f(0)=1,\quad f'(0)=\delta.\end{cases} \tag{4.1}$$

Lemma 4.1. *The solution of* (4.1) *is*

$$f_\mu(\lambda)=1+\delta\lambda+\mu\int_0^\lambda\int_0^\tau(1-\sigma^2)^{\frac{\mu}{2}-1}d\sigma d\tau. \tag{4.2}$$

Furthermore, if $g=g(\lambda)$ *satisfies*

$$\begin{cases}g''(\lambda)=\mu(1-\lambda^2)^{\frac{\mu}{2}-1}\\ g(0)=1,\quad g'(0)=\delta.\end{cases}$$

Then $g=1+\delta\lambda+\int_0^\lambda\int_0^\tau(1-\sigma^2)^{\frac{\mu}{2}-1}d\sigma d\tau$.

Proof. When $\mu = 0$, our conclusion is obvious. We assume that $\mu \neq 0$. We have the expansion

$$(1+x)^\xi = \sum_{k=0}^{\infty} C_\xi^k x^k, \quad x \in (-1,1)$$

where

$$C_\xi^k := \frac{\xi(\xi-1)\dots(\xi-k+1)}{k!}.$$

By simple calculations, we have

$$\frac{\xi}{k+1} C_{\xi-1}^k = C_\xi^{k+1}, \quad f_\mu(\lambda) = 1 + \delta\lambda + \sum_{k=0}^{\infty} \frac{\mu C_{\xi-1}^k t^k \lambda^{2k+2}}{(2k+1)(2k+2)},$$

$$f'_\mu(\lambda) = \delta + \sum_{k=0}^{\infty} \frac{\mu C_{\xi-1}^k t^k \lambda^{2k+1}}{2k+1}, \quad f''_\mu(\lambda) = \mu(1-\lambda^2)^{\frac{\mu}{2}-1}$$

where $\xi = \frac{\mu}{2}$, $\quad t = -1$. Thus we have

$$f_\mu(0) = 1, \qquad f'_\mu(0) = \delta$$

and

$$\begin{aligned} f_\mu(\lambda) - \lambda f'_\mu(\lambda) &= 1 + \sum_{k=0}^{\infty} \mu \left[\frac{C_{\xi-1}^k t^k}{(2k+1)(2k+2)} - \frac{C_{\xi-1}^k t^k}{2k+1} \right] \lambda^{2k+2} \\ &= 1 + \sum_{k=0}^{\infty} \frac{\xi C_{\xi-1}^k t^{k+1} \lambda^{2k+2}}{k+1} = 1 + \sum_{j=1}^{\infty} C_\xi^j (t\lambda^2)^j \\ &= (1+t\lambda^2)^\xi = \left(1-\lambda^2\right)^{\frac{\mu}{2}} \\ &= \left(\frac{1}{\mu} - \frac{1}{\mu}\lambda^2\right) \mu(1-\lambda^2)^{\frac{\mu}{2}-1} = \left(\frac{1}{\mu} - \frac{1}{\mu}\lambda^2\right) f''_\mu(\lambda). \end{aligned}$$

It follows that f_μ satisfies (4.1). □

Lemma 4.2. *Suppose that f is given in Lemma 4.1. Then*

$$\phi(\rho, s) = \epsilon + \rho^\mu f\left(\frac{s}{\rho}\right), \quad \mu \geq 0$$

satisfies

$$\phi(s) - s\phi_s(s) > 0, \quad \phi(s) - s\phi_s(s) + (\rho^2 - s^2)\phi_{ss}(s) > 0, \quad |s| \leq \rho.$$

PROOF. Direct computations yield

$$\phi_s = \rho^{\mu-1} f'\left(\frac{s}{\rho}\right), \quad \phi_{ss} = \rho^{\mu-2} f''\left(\frac{s}{\rho}\right).$$

It follows that

$$\phi(s) - s\phi_s(s) = \epsilon + \rho^{\mu-1}(\rho f - sf') = \epsilon + \rho^{\mu}(f - \lambda f') \tag{4.3}$$

where we used $\lambda = \frac{s}{\rho}$. Similarly, we get

$$\phi(s) - s\phi_s(s) + (\rho^2 - s^2)\phi_{ss}(s) = \epsilon + \rho^{\mu}(f - \lambda f') + \rho^{\mu}(1 - \lambda^2)f''. \tag{4.4}$$

Assume that $\mu = 0$. In this case

$$f(\lambda) = 1 + \delta\lambda.$$

Then

$$\phi(s) - s\phi_s(s) = \phi(s) - s\phi_s(s) + (\rho^2 - s^2)\phi_{ss}(s) = 1 + \epsilon > 0$$

from (4.3) and (4.4).

Assume that $\mu > 0$. Direct computations yield (cf. proof of Lemma 2.2 in [7])

$$f(\lambda) - \lambda f'(\lambda) = \frac{1}{\mu}(1 - \lambda^2)^{\frac{\mu}{2}}, \quad f''(\lambda) = \mu(1 - \lambda^2)^{\frac{\mu}{2} - 1}.$$

It follows that

$$\phi(s) - s\phi_s(s) = \epsilon + \rho^{\mu}(1 - \lambda^2)^{\frac{\mu}{2}} = \epsilon + (\rho^2 - s^2)^{\frac{\mu}{2}} \geq \epsilon > 0, \quad |s| \leq \rho$$

and

$$\begin{aligned} \phi(s) - s\phi_s(s) + (\rho^2 - s^2)\phi_{ss}(s) &= \epsilon + (1 + \mu)\rho^{\mu}(1 - \lambda^2)^{\frac{\mu}{2}} \\ &= \epsilon + (1 + \mu)(\rho^2 - s^2)^{\frac{\mu}{2}} \geq \epsilon > 0, \qquad |s| \leq \rho. \end{aligned}$$

□

Lemma 4.3. *If $\mu = 2n$, $n \in \mathbb{N}$, then the solution of (4.1) is*

$$f(\lambda) = 1 + \delta\lambda + 2n \sum_{k=0}^{n-1} \frac{(-1)^k C_{n-1}^k \lambda^{2k+2}}{(2k+1)(2k+2)}.$$

PROOF. Similar to the proof of Lemma 4.1 in [7]. □

Proposition 4.4. *Let $f(\lambda)$ be a polynomial function defined by*

$$f(\lambda) = 1 + \delta\lambda + 2n \sum_{k=0}^{n-1} \frac{(-1)^k C_{n-1}^k \lambda^{2k+2}}{(2k+1)(2k+2)}.$$

Then the following general (α, β)-metric on an the open subset at origin in $\mathbb{R}^n \setminus \{0\}$

$$F = |y| \left\{ \epsilon + |x|^{2n} f\left(\frac{\langle x, y \rangle}{|x|\,|y|} \right) \right\}$$

is projectively flat.

Proof. Combine Theorem 3.1, Lemma 4.2 and Lemma 4.3. □

Remark. When $\delta = 0$, then

$$f(\lambda) = (2n+1) H_n(-\lambda^2)$$

where

$$H_n(z) := \Sigma_{i=0}^k \left(\frac{1}{2k+1} C_{k+1}^i - \frac{1}{2i-1} C_k^{i-1} \right) z^i.$$

These general (α, β)-metrics, up to a scaling, were constructed in [4, page 70, Example 4.48].

Proposition 4.5. *Let $f(\lambda)$ be a function defined by*

$$f(\lambda) = \delta\lambda + \frac{(2n-1)!!}{(2n-2)!!} \left[\sqrt{1-\lambda^2} + \lambda \arcsin\lambda - \sum_{k=1}^{n-1} \frac{(2k-2)!!}{(2k+1)!!} (1-\lambda^2)^{\frac{2k+1}{2}} \right].$$

Then the following general (α, β)-metric on an open subset in $\mathbb{R}^n \setminus \{0\}$

$$F = |y| \left\{ \epsilon + |x|^{2n-1} f\left(\frac{\langle x, y \rangle}{|x|\,|y|} \right) \right\}$$

is projectively flat.

Proof. Lemma 4.2 tells us F is a general (α, β)-metric. Similar to proofs of Lemma 5.1 and Theorem 5.4 in [7] where we take $\mu = 2n-1$ in Lemma 4.1. □

5. Hypergeometric functions and proof of Theorem 1.1

In this section, we are going to give the solutions of (4.1) in terms of hypergeometric functions and manufacture new projectively flat Finsler metrics.

Lemma 5.1. *Let μ be a real number. Then*

$$\int_0^\tau (1-x^2)^{\frac{\mu}{2}-1}dx = \tau \text{hypergeom}\left(\left[\frac{1}{2}, 1-\frac{\mu}{2}\right], \left[\frac{3}{2}\right], \tau^2\right).$$

Proof. The hypergeometric function is given in (1.2) and (1.3) (see Section 1). In particular,

$$\text{hypergeom}([-a,b],b,-t) = (1+t)^a. \tag{5.1}$$

where $a, b \in \mathbb{R}$. From (5.1) and (1.2) we have

$$\begin{aligned}\int_0^\tau (1-x^2)^{\frac{\mu}{2}-1}dx &= \int_0^\tau \sum_{n=0}^{\infty} \frac{(1-\frac{\mu}{2})_n}{n!} x^{2n}dx = \sum_{n=0}^{\infty} \frac{(1-\frac{\mu}{2})_n}{n!} \int_0^\tau x^{2n}dx \\ &= \sum_{n=0}^{\infty} \frac{(1-\frac{\mu}{2})_n}{n!} \frac{\tau^{2n+1}}{2n+1} = \tau \sum_{n=0}^{\infty} \frac{(1-\frac{\mu}{2})_n}{n!} \frac{\tau^{2n}}{2n+1}.\end{aligned} \tag{5.2}$$

By using (1.2), we have

$$\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} = \frac{\frac{1}{2}(\frac{1}{2}+1)\dots(\frac{1}{2}+n-1)}{\frac{3}{2}(\frac{3}{2}+1)\dots(\frac{3}{2}+n-1)} = \frac{\frac{1}{2}}{\frac{3}{2}+n-1} = \frac{1}{2n+1}.$$

Plugging this into (5.2) yields

$$\begin{aligned}\int_0^\tau (1-x^2)^{\frac{\mu}{2}-1}dx &= \tau \sum_{n=0}^{\infty} \frac{(1-\frac{\mu}{2})_n}{n!} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} (\tau^2)^n = \tau \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (1-\frac{\mu}{2})_n}{n!\left(\frac{3}{2}\right)_n} (\tau^2)^n \\ &= \tau \,\text{hypergeom}\left(\left[\frac{1}{2}, 1-\frac{\mu}{2}\right], \left[\frac{3}{2}\right], \tau^2\right).\end{aligned} \tag{5.3}$$

□

Lemma 5.2. *Let μ be a non-zero constant. Then*

$$\begin{aligned}&\int_0^\lambda \tau \,\text{hypergeom}\left(\left[\frac{1}{2}, 1-\frac{\mu}{2}\right], \left[\frac{3}{2}\right], \tau^2\right) d\tau \\ &\qquad = \frac{1}{\mu} + \frac{1}{\mu} \text{hypergeom}\left(\left[-\frac{1}{2}, -\frac{\mu}{2}\right], \left[\frac{1}{2}\right], \lambda^2\right).\end{aligned}$$

Proof. By (5.3) we obtain

$$\int_0^\lambda \tau \,\text{hypergeom}\left(\left[\frac{1}{2}, 1-\frac{\mu}{2}\right], \left[\frac{3}{2}\right], \tau^2\right) d\tau = \int_0^\lambda \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(1-\frac{\mu}{2}\right)_n}{n!\left(\frac{3}{2}\right)_n} \tau^{2n+1} d\tau$$
$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(1-\frac{\mu}{2}\right)_n}{n!\left(\frac{3}{2}\right)_n} \int_0^\lambda \tau^{2n+1} d\tau = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(1-\frac{\mu}{2}\right)_n}{n!\left(\frac{3}{2}\right)_n} \frac{\lambda^{2n+2}}{2n+2}.$$

Taking $m = n+1$ we obtain

$$\int_0^\lambda \tau \,\text{hypergeom}\left(\left[\frac{1}{2}, 1-\frac{\mu}{2}\right], \left[\frac{3}{2}\right], \tau^2\right) d\tau$$
$$= \frac{1}{2} \sum_{m=1}^{\infty} \frac{\left(\frac{1}{2}\right)_{m-1} \left(1-\frac{\mu}{2}\right)_{m-1}}{m!\left(\frac{3}{2}\right)_{m-1}} \left(\lambda^2\right)^m. \quad (5.4)$$

By straightforward computations one obtains

$$\left(\frac{1}{2}\right)_{m-1} = -2 \times \left(-\frac{1}{2}\right)_m, \quad \left(1-\frac{\mu}{2}\right)_{m-1} = -\frac{2}{\mu}\left(-\frac{\mu}{2}\right)_m,$$
$$\left(\frac{3}{2}\right)_{m-1} = 2 \times \left(\frac{1}{2}\right)_m. \quad (5.5)$$

Substituting (5.5) into (5.4) yields

$$\int_0^\lambda \tau \,\text{hypergeom}\left(\left[\frac{1}{2}, 1-\frac{\mu}{2}\right], \left[\frac{3}{2}\right], \tau^2\right) d\tau$$
$$= \frac{1}{2} \sum_{m=1}^{\infty} \frac{-2\left(-\frac{1}{2}\right)_m \left(-\frac{2}{\mu}\right)\left(-\frac{\mu}{2}\right)_m}{m!2\left(\frac{1}{2}\right)_m} \left(\lambda^2\right)^m = \frac{1}{\mu} \sum_{m=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_m \left(-\frac{\mu}{2}\right)_m}{m!\left(\frac{1}{2}\right)_m} \left(\lambda^2\right)^m$$
$$= -\frac{1}{\mu} + \frac{1}{\mu} \text{hypergeom}\left(\left[-\frac{1}{2}, -\frac{\mu}{2}\right], \left[\frac{1}{2}\right], \lambda^2\right). \qquad \square$$

Proof of Theorem 1.1. From Lemma 4.1, the solution of (4.1) is

$$f(\lambda) = 1 + \delta\lambda + \mu \int_0^\lambda \int_0^\tau (1-\sigma^2)^{\frac{\mu}{2}-1} d\sigma d\tau.$$

Combining this with Lemma 5.1 and Lemma 5.2 we obtain

$$f(\lambda) = 1 + \delta\lambda + \mu \int_0^\lambda \tau \,\text{hypergeom}\left(\left[\frac{1}{2}, 1-\frac{\mu}{2}\right], \left[\frac{3}{2}\right], \tau^2\right) d\tau$$

$$= 1 + \delta\lambda + \mu\left[-\frac{1}{\mu} + \frac{1}{\mu}\,\text{hypergeom}\left(\left[-\frac{1}{2}, -\frac{\mu}{2}\right], \left[\frac{1}{2}\right], \lambda^2\right)\right]$$

$$= \delta\lambda + \text{hypergeom}\left(\left[-\frac{1}{2}, -\frac{\mu}{2}\right], \left[\frac{1}{2}\right], \lambda^2\right) \tag{5.6}$$

for arbitrary nonzero μ. When $\mu = 0$, (5.6) is automatically true. Lemma 4.2 tells us that F is a general (α, β)-metric when $\mu \geq 0$. By Theorem 3.1, the general (α, β)-metric

$$F = |y|\left\{\epsilon + |x|^{\mu} f\left(\frac{\langle x, y\rangle}{|x|\,|y|}\right)\right\}$$

is projectively flat. □

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LIBING HUANG
SCHOOL OF MATHEMATICAL SCIENCES
NANKAI UNIVERSITY
TIANJIN 300071
P.R. CHINA

E-mail: huanglb@nankai.edu.cn

XIAOHUAN MO
KEY LABORATORY OF PURE
AND APPLIED MATHEMATICS
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING 100871
P.R. CHINA

E-mail: moxh@pku.edu.cn

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