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Local distribution of the parts of unequal partitions in arithmetic progressions II

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1. Introduction

This paper contains the main parts of the proofs of the results announced in [6]. We recall below some notations and the main result of this paper but we recommend to the reader to study first [6]. Let $d \in \mathbb{N}^*$, $\mathcal{D}$ a non-empty subset of $\{1, \dots, d\}$ and $\mathcal{D}^c = \{1, \dots, d\} \setminus \mathcal{D}$ its complement. Let $\mathcal{R}_\mathcal{D} = \{N_r : r \in \mathcal{D}\}$ be a multiset of $|\mathcal{D}|$ non-negative integers. The main goal of our work is to obtain an asymptotic formula for $\Pi_d^*(n, \mathcal{R}_\mathcal{D})$, the number of unequal partitions of n with exactly N_r parts congruent to r modulo d for all $r \in \mathcal{D}$. We adopt the convention $\Pi_d^*(0, \mathcal{R}_\mathcal{D}) = 1$ if $\mathcal{R}_\mathcal{D} = \{0, \dots, 0\}$ and 0 otherwise.

Recall that if $n \geq 1$ and $\Pi_d^*(n, \mathcal{R}_\mathcal{D}) \geq 1$ then n satisfies

$$n \equiv R_\mathcal{D} \pmod{\delta}, \tag{1.1}$$

where $R_\mathcal{D} = \sum_{r \in \mathcal{D}} r N_r$ and δ is the *g. c. d.* of the elements of $\mathcal{D}^c \cup \{d\}$. In the introduction of [6], we observed that the N_r, $r \in \mathcal{D}$ may be expected to be close to k_0 with

$$k_0 := \frac{2\sqrt{3}\log 2}{\pi} \frac{\sqrt{n}}{d}. \tag{1.2}$$

More precisely we suppose that for all $r \in \mathcal{D}$ we have

$$|N_r - k_0| \leq \frac{n^{\frac{1}{4}}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3} w(n)}, \tag{1.3}$$

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where $w(n)$ is a non-decreasing function such that $w(n) \to \infty$ if $n \to \infty$. Let us recall the main result of [6].

Theorem 1.1. *Let $\varepsilon > 0$. The following two propositions hold.*

(i) *Let $d \le n^{1/4-\varepsilon}$, $\mathcal{D} = \{1, \dots, d\}$ and $n \equiv R_{\mathcal{D}} \pmod d$. Let $\mathcal{R} = \mathcal{R}_{\mathcal{D}} = \{N_1, \dots, N_d\}$ be a multiset of integers satisfying* (1.3). *Then we have*

$$\Pi_d^*(n, \mathcal{R}_D) = (1 + o(1))q(n)\frac{d}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}}\left(\frac{d}{2\sqrt{3n}}\right)^{d/2}$$
$$\times \exp\left\{ - \frac{2\sqrt{3}\log^2 2}{\pi\left(1 - \frac{12(\log 2)^2}{\pi^2}\right)\sqrt{n}}\left(\sum_{r=1}^{d}(N_r - k_0)\right)^2 - \frac{\pi d}{2\sqrt{3n}}\sum_{r=1}^{d}(N_r - k_0)^2\right\}.$$

(ii) *We suppose now that $d \le n^{1/6-\varepsilon}$ and $\mathcal{D} \subset \{1, \dots, d\}$. Then under* (1.1) *and* (1.3) *we have*

$$\Pi_d^*(n, \mathcal{R}_D) = q(n)\frac{\delta(1 + o(1))}{\sqrt{1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\left(\frac{d}{2\sqrt{3n}}\right)^{|\mathcal{D}|/2}$$
$$\times \exp\left(- \frac{2\sqrt{3}(\log 2)^2}{\pi\left(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right)\sqrt{n}}\left(\sum_{r\in\mathcal{D}}(N_r - k_0)\right)^2 - \frac{\pi d}{2\sqrt{3n}}\sum_{r\in\mathcal{D}}(N_r - k_0)^2\right).$$

First we complete the proof of Theorem 1.1 in the case $\mathcal{D} = \{1, \dots, d\}$, after we will handle the complementary case when $\mathcal{D}^c \ne \emptyset$. The last sections are devoted to the proofs of the different corollaries of [6].

2. The term S_2

We begin to assume that

$$d \le n^{\frac{1}{2}-\varepsilon} \tag{2.1}$$

with some fixed positive ε and

$$|k - k_0| = o\left(\frac{\sqrt{n}}{d}\right). \tag{2.2}$$

Let

$$x_0 := \frac{\pi}{2\sqrt{3n}}, \quad t := dx_0. \tag{2.3}$$

Then

$$k_0 t = k_0 d x_0 = \log 2. \tag{2.4}$$

We also suppose that

$$|N_r - k_0| = o\left(\frac{\sqrt{n}}{d}\right) \quad (r = 1, \dots, d). \tag{2.5}$$

In Section 4 of [6] we proved that as $n \to \infty$ then we have

$$\prod_{r=1}^{d} g_{N_r}(dx_0) = \exp\left(\frac{\pi^2}{12x_0} + \frac{(\log 2)^2}{2x_0} + o(\sqrt{n}\,)\right). \tag{2.6}$$

According to the notations of [6] Sections 3 and 4, we have

$$|S_2| \le \frac{d}{2\pi}\int_{3\pi x_0 \le |y| \le \pi/d} \left\{\prod_{r=1}^{d} |g_{N_r}(d(x_0+iy))|\right\} \exp((n-R-Q)x_0)\mathrm{d}y.$$

The main part of this section is the following lemma.

Lemma 2.1. *Under the notations and hypotheses* (1.2), (2.1), (2.2), (2.3), (2.4), *and* $3\pi x_0 \le |y| \le \pi/d$, *we have*

$$|g_k(d(x_0+iy))| \le g_k(dx_0)\exp\left(\frac{-1+o(1)}{4dx_0}\right).$$

Proof. This time we start out from the first expression of g_k and develop the logarithms:

$$\begin{aligned} g_k(w) &= \exp\left(\sum_{\nu=1}^{k} \log\frac{1}{1-\exp(-\nu w)}\right) = \exp\left(\sum_{\nu=1}^{k}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-\nu m w)\right) \\ &= \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\nu=1}^{k}\exp(-\nu m w)\right) \\ &= \exp\left(\sum_{\nu=1}^{k}\exp(-\nu w) + \sum_{m=2}^{\infty}\frac{1}{m}\sum_{\nu=1}^{k}\exp(-\nu m w)\right). \end{aligned} \tag{2.7}$$

We take the moduli

$$\begin{aligned} |g_k(w)| &\le \exp\left(\Big|\sum_{\nu=1}^{k}\exp(-\nu w)\Big| + \sum_{m=2}^{\infty}\frac{1}{m}\sum_{\nu=1}^{k}\exp(-\nu m t)\right) \\ &= g_k(t)\exp\left(\Big|\sum_{\nu=1}^{k}\exp(-\nu w)\Big| - \sum_{\nu=1}^{k}\exp(-\nu t)\right) \end{aligned}$$

$$= g_k(t)\exp\left(\frac{|1-\exp(-kw)|}{|\exp(w)-1|} - \frac{1-\exp(-kt)}{\exp(t)-1}\right)$$
$$\le g_k(t)\exp\left(\frac{1+\exp(-kt)}{|\exp(w)-1|} - \frac{1-\exp(-kt)}{\exp(t)-1}\right).$$

When $|\Im m\, w| \le \pi$,

$$|\exp(w)-1|^2 = (\exp(t)-1)^2 + 4\mathrm{e}^t(\sin(b/2))^2 \ge 4(\sin(b/2))^2 \ge \frac{4|b|^2}{\pi^2},$$

thus

$$|g_k(w)| \le g_k(t)\exp\left(\frac{1+\exp(-kt)}{\frac{2}{\pi}|\Im m\, w|} - \frac{1-\exp(-kt)}{\exp(t)-1}\right)$$

if $|\Im m\, w| \le \pi$. Therefore, $3\pi x_0 \le |y| \le \pi/d$ implies that

$$|g_k(d(x_0+iy))| \le g_k(dx_0)\exp\left(\frac{1+\exp(-kdx_0)}{\frac{2}{\pi}d|y|} - \frac{1-\exp(-kdx_0)}{\exp(dx_0)-1}\right)$$
$$\le g_k(dx_0)\exp\left(\frac{1+\exp(-kdx_0)}{6dx_0} - \frac{1-\exp(-kdx_0)}{\exp(dx_0)-1}\right).$$

By (1.2), (2.1), (2.2), (2.3), (2.4),

$$|g_k(d(x_0+iy))| \le g_k(dx_0)\exp\left(\frac{\frac{3}{2}+o(1)}{6dx_0} - \frac{\frac{1}{2}+o(1)}{\exp(dx_0)-1}\right)$$
$$= g_k(dx_0)\exp\left(\frac{\frac{3}{2}+o(1)}{6dx_0} - \frac{\frac{1}{2}+o(1)}{dx_0} + O(1)\right) = g_k(dx_0)\exp\left(-\frac{1+o(1)}{4dx_0}\right).$$

This ends the proof of Lemma 2.1. □

By (2.5) we obtain for S_2,

$$|S_2| \le \left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp\left(-\frac{1+o(1)}{4x_0}\right)\exp((n-R-Q)x_0)$$
$$= \exp\left(\frac{\pi\sqrt{n}}{\sqrt{3}} - \frac{\sqrt{3n}}{2\pi} + o(\sqrt{n})\right), \tag{2.8}$$

by (2.6) and according to the estimates of Q and R obtained in Sections 2 and 4 of [6].

3. The term S_1

Next, we will try to give a similar and simple estimation for $n^{-\frac{5}{8}+\varepsilon} \le |y| \le 3\pi x_0$.

Lemma 3.1. (i) *Under the notations and hypotheses* (1.2), (2.1), (2.2), (2.3), (2.4), *for* $n^{-\frac{5}{8}+\varepsilon} \le |y| \le 3\pi x_0$ *we have*

$$|g_k(d(x_0+iy))| \le g_k(dx_0)\exp\left(-\frac{\sqrt{3}n^{1/4+2\varepsilon}}{27\pi^5 d}\right).$$

(ii) *Under the notations and hypotheses* (1.2), (2.1), (2.2), (2.3), (2.4), *for* $n^{-\frac{3}{4}+\frac{\varepsilon}{3}} \le |y| \le n^{-\frac{5}{8}+\varepsilon}$ *we have*

$$|g_k(d(x_0+iy))| \le g_k(dx_0)\exp\left(-\frac{\sqrt{3}n^{2\varepsilon/3}}{27\pi^5 d}\right).$$

Proof. First we prove (i). We suppose that $n^{-\frac{5}{8}+\varepsilon} \le |y| \le 3\pi x_0$. By (2.7) we have

$$g_k(w) = \exp\left(\sum_{\nu=1}^{k}\exp(-\nu w) + \sum_{m=2}^{\infty}\frac{1}{m}\sum_{\nu=1}^{k}\exp(-\nu m w)\right).$$

We study again $|g_k(w)|$:

$$\begin{aligned}
|g_k(w)| &\le \exp\left(\Re e\left(\sum_{\nu=1}^{k}\exp(-\nu w)\right) + \sum_{m=2}^{\infty}\frac{1}{m}\exp(-\nu m t)\right)\\
&= g_k(t)\exp\left(\sum_{\nu=1}^{k}\Re e\left(\exp(-\nu w)\right) - \sum_{\nu=1}^{k}\exp(-\nu t)\right)\\
&= g_k(t)\exp\left(\sum_{\nu=1}^{k}\exp(-\nu t)(\Re e\,\exp(-\nu i b) - 1)\right)\\
&= g_k(t)\exp\left(\sum_{\nu=1}^{k}\exp(-\nu t)(\cos(\nu b) - 1)\right)\\
&= g_k(t)\exp\left(-2\sum_{\nu=1}^{k}\exp(-\nu t)\sin^2\left(\frac{\nu|b|}{2}\right)\right).
\end{aligned}$$

Let $K_0 := \left\lfloor \frac{k_0}{3\log 2}\right\rfloor$. If $k = k_0 + o\left(\frac{\sqrt{n}}{d}\right)$ then $k > K_0$ for n large enough:

$$|g_k(w)| \le g_k(t)\exp\left(-2\sum_{\nu=1}^{K_0}\exp(-\nu t)\sin^2\left(\frac{\nu|b|}{2}\right)\right)$$

$$\le g_k(t)\exp\left(-2\sum_{\nu=1}^{K_0}\exp(-k_0 t)\sin^2\left(\frac{\nu|b|}{2}\right)\right) = g_k(t)\exp\left(-\sum_{\nu=1}^{K_0}\sin^2\left(\frac{\nu|b|}{2}\right)\right).$$

Writing $w = d(x_0 + iy)$ and using the inequality $|\sin t| \geq \frac{2|t|}{\pi}$ for $|t| \leq \pi/2$, we obtain

$$|g_k(d(x_0 + iy))| \leq g_k(dx_0) \exp\Bigg(- \sum_{\nu=1}^{K_0} \Big(\frac{\nu d|y|}{\pi}\Big)^2\Bigg)$$

since

$$\frac{\nu d|y|}{2} \leq \frac{K_0 d 3\pi x_0}{2} \leq \frac{k_0 d\pi x_0}{2\log 2} = \frac{\pi}{2}.$$

Since $\sum_{\nu=1}^{K_0} \nu^2 \geq \frac{K_0^3}{3}$, we have

$$|g_k(d(x_0+iy))| \leq g_k(dx_0) \exp\Bigg(- \Big(\frac{d|y|}{\pi}\Big)^2 \frac{K_0^3}{3}\Bigg) \leq g_k(dx_0) \exp\Bigg(- \frac{\sqrt{3} n^{1/4+2\varepsilon}}{\pi^5 3^3 d}\Bigg).$$

(ii) can be obtained similarly. $\square$

By (2.5),

$$\begin{aligned}
&\Bigg| \frac{d}{2\pi} \int_{n^{-\frac{5}{8}+\varepsilon} \leq |y| \leq 3\pi x_0} \Bigg\{ \prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\Bigg\} \exp((n-R-Q)(x_0+iy)) \mathrm{d}y \Bigg| \\
&\quad \leq \frac{d}{2\pi} \int_{n^{-\frac{5}{8}+\varepsilon} \leq |y| \leq 3\pi x_0} \Bigg| \prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\Bigg| \exp((n-R-Q)x_0) \mathrm{d}y \\
&\quad \leq \Bigg\{ \prod_{r=1}^{d} g_{N_r}(dx_0)\Bigg\} \exp\Bigg(- \frac{\sqrt{3} n^{\frac{1}{4}+2\varepsilon}}{\pi^5 3^3}\Bigg) \exp((n-R-Q)x_0).
\end{aligned}$$

We have to stop here since the previously error term $o(\sqrt{n}\,)$ is rough. Otherwise the above proof can be applied, *e.g.*, for $n^{-\frac{3}{4}+\frac{\varepsilon}{3}} \leq |y| \leq n^{-\frac{5}{8}+\varepsilon}$ and results that

$$\begin{aligned}
&\Bigg| \frac{d}{2\pi} \int_{n^{-\frac{3}{4}+\frac{\varepsilon}{3}} \leq |y| \leq n^{-\frac{5}{8}+\varepsilon}} \Bigg\{ \prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\Bigg\} \exp((n-R-Q)(x_0+iy)) \mathrm{d}y \Bigg| \\
&\qquad \leq \Bigg\{ \prod_{r=1}^{d} g_{N_r}(dx_0)\Bigg\} \exp\Bigg(- \frac{\sqrt{3} n^{2\varepsilon/3}}{\pi^5 3^3}\Bigg) \exp((n-R-Q)x_0).
\end{aligned}$$

Finally we obtain for S_1:

$$|S_1| \ll \Bigg\{ \prod_{r=1}^{d} g_{N_r}(dx_0)\Bigg\} \exp\Bigg(- \frac{\sqrt{3} n^{2\varepsilon/3}}{\pi^5 3^3}\Bigg) \exp((n-R-Q)x_0). \tag{3.1}$$

4. The function g_k in the range $|y| < y_1$

Let $|y| \le y_1 = n^{-\frac{3}{4}+\frac{\varepsilon}{3}}$, $w = t + ib = dx_0 + idy$. Now $\frac{|b|}{t} = O\big(n^{-\frac{1}{4}+\frac{\varepsilon}{3}}\big)$.

In this section we work with a general subset $\mathcal{D} \subset \{1, \dots, d\}$. Instead of (2.2) and (2.5), we suppose that

$$|k - k_0| \le \frac{n^{\frac{1}{4}}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)} \quad \text{and} \quad |N_r - k_0| \le \frac{n^{\frac{1}{4}}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)} \quad (r \in \mathcal{D}) \tag{4.1}$$

where $w(n)$ is a non-decreasing function such that $w(n) \to \infty$ if $n \to \infty$. The aim of this paragraph is to obtain an asymptotic formula for $g_k(x_0 + iy)$ for $|y| \le y_1$. Instead of (2.1), we suppose that

$$d \le n^{\frac{1}{4}-2\varepsilon}. \tag{4.2}$$

Thus (4.1) implies (2.2) and (2.5). We will prove the following Lemma:

Lemma 4.1. *Under* (4.1) *we have*

$$g_k(w) = f(w)\exp\Bigg\{-\frac{C_2}{t} + (k-k_0)\log 2 - \frac{(k-k_0)^2 t}{2} + \frac{\log 2}{2}$$
$$+ib\left(\frac{C_2}{t^2} + \frac{k_0\log 2}{t} - k_0(k-k_0)\right) + b^2\left(\frac{C_2}{t^3} + \frac{k_0\log 2}{t^2} + \frac{k_0^2}{2t}\right) + o\left(\frac{n^{-\varepsilon}}{|\mathcal{D}|}\right)\Bigg\}.$$

We have again by Lemma 4.1 of [6]

$$g_k(w) = f(w)\exp\left\{-\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmw) + \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw) + O(k^{-1})\right\}.$$

By (1.2) and (2.2),

$$\frac{1}{k} = O\left(\frac{d}{\sqrt{n}}\right) = O\left(\frac{1}{d}\frac{d^2}{\sqrt{n}}\right).$$

Then $\frac{1}{k} = O(n^{-2\varepsilon}/d)$ and $\frac{|b|}{t} = o(n^{-\varepsilon}d^{-1})$. Since now $|y| \le y_1$, it is possible to replace w by t in the different $\exp(-kmw)$, in cost of an admissible error term:

Lemma 4.2. (i) *We have*

$$\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw) = \sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + O\left(\frac{|b|}{t}\right).$$

(ii) *We have*

$$\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmw) = \frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) - \frac{kib}{w}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)$$
$$-\frac{k^2b^2}{2w}\sum_{m=1}^{\infty}\exp(-kmt) + O\left(\frac{k^3|b|^3}{|w|}\right)\sum_{m=1}^{\infty}m\exp(-kmt).$$

Proof. By standard approximations we have

$$\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw)=\sum_{m=1}^{\infty}\frac{1}{m}(1-(1-\exp(-kmib)))\exp(-kmt)$$
$$=\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)+\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)O(km|b|)$$
$$=\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)+O\left(\frac{k|b|}{e^{kt}-1}\right)=\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)+O\left(\frac{|b|}{t}\right).$$

This proves (i). Next we prove (ii). We have

$$\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmw)=\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt)\exp(-ikmb)$$
$$=\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt)\left\{1-ikmb-\frac{(kmb)^2}{2}+O((km|b|)^3)\right\}.$$

It remains to develop to end the proof of Lemma 4.2. □

We also have

$$\sum_{m=1}^{\infty}m\exp(-kmt)=\frac{\exp(-kt)}{(1-\exp(-kt))^2}=\frac{1}{(\exp(kt)-1)(1-\exp(-kt))}\le\frac{1}{(kt)^2},$$

since for $u>0$,

$$\mathrm{e}^u-1=u\sum_{n=0}^{\infty}\frac{u^n}{(n+1)!}>u\sum_{n=0}^{\infty}\frac{u^n}{n!2^n}=u\mathrm{e}^{\frac{u}{2}},$$

thus $(1-\mathrm{e}^{-u})>u\mathrm{e}^{-u/2}$ and $(\mathrm{e}^u-1)(1-\mathrm{e}^{-u})>u^2$.

This gives for $g_k(w)$:

$$g_k(w)=f(w)\exp\Bigg\{-\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt)+\frac{ikb}{w}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)$$
$$+\frac{k^2b^2}{2w}\sum_{m=1}^{\infty}\exp(-kmt)+\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)+o\left(\frac{n^{-\varepsilon}}{d}\right)+O\bigg(k\left(\frac{|b|}{t}\right)^3\bigg)\Bigg\}.$$

The next step of the proof of Lemma 4.1 consists of "replacing" $\frac{1}{w}$ by $\frac{1}{t}$ and computing the terms arisen by this manipulation. We use the formula

$$\frac{1}{w}=\frac{1}{t(1-(-i\frac{b}{t}))}=\frac{1}{t}\left(1-i\frac{b}{t}-\frac{b^2}{t^2}+O\left(\frac{|b|^3}{t^3}\right)\right). \tag{4.3}$$

This gives for $g_k(w)$

$$\begin{aligned} g_k(w) = f(w)\exp\Bigg\{ &-\frac{1}{t}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + \frac{ib}{t^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) \\ &+ \frac{b^2}{t^3}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + O\left(\frac{|b|^3}{t^4}\right) + \frac{ikb}{t}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) \\ &+ \frac{kb^2}{t^2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + O\left(\frac{k|b|^3}{t^3}\log\left(1+\frac{1}{\mathrm{e}^{kt}-1}\right)\right) \\ &+ \frac{k^2b^2}{2t}\sum_{m=1}^{\infty}\exp(-kmt) + O\left(\frac{k^2|b|^3}{t^2(\mathrm{e}^{kt}-1)}\right) \\ &+ \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + o\left(\frac{n^{-\varepsilon}}{d}\right) + O\left(\frac{k|b|^3}{t^3}\right)\Bigg\}. \end{aligned}$$

We collect the terms with ib, the terms with b^2:

$$\begin{aligned} g_k(w) = f(w)\exp\Bigg\{ &-\frac{1}{t}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) \\ &+ ib\Bigg(\frac{1}{t^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + \frac{k}{t}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)\Bigg) \\ &+ b^2\Bigg(\frac{1}{t^3}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + \frac{k}{t^2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + \frac{k^2}{2t}\sum_{m=1}^{\infty}\exp(-kmt)\Bigg) \\ &+ o\left(\frac{n^{-\varepsilon}}{d}\right) + O\left(\left(k+\frac{1}{t}\right)\frac{|b|^3}{t^3}\right)\Bigg\}. \end{aligned}$$

Now we compute the different summations over m. By (4.1), $\exp(-kmt)$ is close to $\exp(-k_0mt)$ if m is not too large, but we have again some computations to do to control this approximation. For $s = 0, 1, 2$:

$$\sum_{m=1}^{\infty}\frac{1}{m^s}\exp(-kmt) = \sum_{m=1}^{\infty}\frac{1}{m^s}\exp(-k_0mt)\exp(-(k-k_0)mt).$$

Since $\mathrm{e}^x = \sum_{n=0}^{M}\frac{x^n}{n!} + \sum_{n=M+1}^{\infty}\frac{x^n}{n!}$, we have

$$\left|\mathrm{e}^x - \sum_{n=0}^{M}\frac{x^n}{n!}\right| \le |x|^{M+1}\sum_{n=M+1}^{\infty}\frac{|x|^{n-M-1}}{n!} \le |x|^{M+1}\mathrm{e}^{|x|}.$$

Thus we obtain

$$\sum_{m=1}^{\infty}\frac{1}{m^s}\exp(-kmt)=\sum_{m=1}^{\infty}\frac{1}{m^s}\exp(-k_0mt)\Big\{1-(k-k_0)mt+\frac{1}{2}(k-k_0)^2m^2t^2 - \frac{1}{6}(k-k_0)^3m^3t^3+O(|k-k_0|^4m^4t^4\exp(|k-k_0|mt)).$$

By (2.2), $\exp(|k-k_0|mt)\le\exp(mk_0t/2)$.Next we use the fact that $k_0t=\log 2$:

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{1}{m^s}\exp(-kmt)&=\sum_{m=1}^{\infty}\frac{1}{m^s2^m}\bigg(1-(k-k_0)mt+\frac{1}{2}(k-k_0)^2m^2t^2\\&\quad-\frac{1}{6}(k-k_0)^3m^3t^3+O(|k-k_0|^4m^4t^42^{m/2})\bigg)\\&=\sum_{m=1}^{\infty}\frac{1}{m^s2^m}-(k-k_0)t\sum_{m=1}^{\infty}\frac{2^{-m}}{m^{s-1}}+\frac{1}{2}(k-k_0)^2t^2\sum_{m=1}^{\infty}\frac{2^{-m}}{m^{s-2}}\\&\quad-\frac{(k-k_0)^3}{6}t^3\sum_{m=1}^{\infty}\frac{2^{-m}}{m^{s-3}}+O\bigg(|k-k_0|^4t^4\sum_{m=1}^{\infty}\frac{2^{-m/2}}{m^{s-4}}\bigg).\end{aligned}$$

We obtain for the function g_k

$$\begin{aligned}g_k(w)=f(w)\exp\bigg\{&-\frac{1}{t}\sum_{m=1}^{\infty}\frac{1}{m^22^m}+(k-k_0)\sum_{m=1}^{\infty}\frac{1}{m2^m}-\frac{(k-k_0)^2t}{2}\sum_{m=1}^{\infty}2^{-m}\\&+O(|k-k_0|^3t^2)+\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m2^m}+O(|k-k_0|t)\\&+ib\bigg(\frac{1}{t^2}\sum_{m=1}^{\infty}\frac{1}{m^22^m}-\frac{(k-k_0)}{t}\sum_{m=1}^{\infty}\frac{1}{m2^m}+O(|k-k_0|^2)\\&+\frac{k}{t}\sum_{m=1}^{\infty}\frac{1}{m2^m}-k(k-k_0)\sum_{m=1}^{\infty}2^{-m}+O(kt|k-k_0|^2)\bigg)\\&+b^2\bigg(\frac{1}{t^3}\sum_{m=1}^{\infty}\frac{1}{m^22^m}+O\left(\frac{|k-k_0|}{t^2}\right)+\frac{k}{t^2}\sum_{m=1}^{\infty}\frac{1}{m2^m}+O\left(\frac{k|k-k_0|}{t}\right)\\&+\frac{k^2}{2t}\sum_{m=1}^{\infty}2^{-m}+O(k^2|k-k_0|)\bigg)+o(n^{-\varepsilon}d^{-1})+O\left(\left(k+\frac{1}{t}\right)\frac{|b|^3}{t^3}\right)\bigg\}.\end{aligned}$$

Next we compute the different sums on m:

$$g_k(w)=f(w)\exp\bigg\{-\frac{C_2}{t}+(k-k_0)\log 2-\frac{(k-k_0)^2t}{2}+\frac{1}{2}\log 2$$

$$+\, ib\left(\frac{C_2}{t^2}-\frac{(k-k_0)}{t}\log 2+\frac{k\log 2}{t}-k(k-k_0)\right)+b^2\left(\frac{C_2}{t^3}+\frac{k}{t^2}\log 2+\frac{k^2}{2t}\right)$$
$$+\, O\left(|k-k_0|^3t^2+|k-k_0|t+|k-k_0|^2|b|+\frac{|k-k_0|b^2}{t^2}\right)$$
$$+\, o\left(\frac{n^{-\varepsilon}}{d}\right)+O\left(\left(k+\frac{1}{t}\right)\frac{|b|^3}{t^3}\right)\Bigg\}.$$

Then by (4.1) the above error terms give $o(n^{-\varepsilon}|\mathcal{D}|^{-1})$ and we can replace $-k(k-k_0)$ with $-k_0(k-k_0)$ in the coefficient of ib and analogously in that of b^2. Finally,

$$g_k(w)=f(w)\exp\Bigg\{-\frac{C_2}{t}+(k-k_0)\log 2-\frac{(k-k_0)^2t}{2}+\frac{\log 2}{2}$$
$$+ib\left(\frac{C_2}{t^2}+\frac{k_0\log 2}{t}-k_0(k-k_0)\right)+b^2\left(\frac{C_2}{t^3}+\frac{k_0\log 2}{t^2}+\frac{k_0^2}{2t}\right)+o\left(\frac{n^{-\varepsilon}}{|\mathcal{D}|}\right)\Bigg\},$$

as claimed in Lemma 4.1. □

5. The term S_0, end of the proof of Theorem 1.1 in the case $\mathcal{D}=\{1,\ldots,d\}$

As a special case of Lemma 4.1 applied with $\mathcal{D}=\{1,\ldots,d\}$, we remark that

$$g_k(dx_0)=f(dx_0)\exp\left\{-\frac{C_2}{dx_0}+(k-k_0)\log 2-\frac{(k-k_0)^2dx_0}{2}+\frac{\log 2}{2}+o\left(\frac{n^{-\varepsilon}}{d}\right)\right\},$$

and

$$\prod_{r=1}^{d}g_{N_r}(dx_0)=f^d(dx_0)\exp\Bigg\{-\frac{C_2}{x_0}+\sum_{r=1}^{d}(N_r-k_0)\log 2$$
$$-\frac{dx_0}{2}\sum_{r=1}^{d}(N_r-k_0)^2+\frac{d\log 2}{2}+o(n^{-\varepsilon})\Bigg\}. \tag{5.1}$$

Since $f(w)=\exp\left(\frac{\pi^2}{6w}+\frac{1}{2}\log\frac{w}{2\pi}+O(|w|)\right)$ for $w\to 0$ in $|\arg w|\le\kappa<\pi/2$ and $\Re e\, w>0$, we have for $|y|\le y_1\le n^{-\frac{3}{4}+\frac{\varepsilon}{3}}$:

$$f(d(x_0+iy))=\exp\left(\frac{\pi^2}{6d(x_0+iy)}+\frac{1}{2}\log\left(\frac{d(x_0+iy)}{2\pi}\right)+O(dx_0)\right),$$
$$f^d(d(x_0+iy))=\exp\left(\frac{\pi^2}{6(x_0+iy)}+\frac{d}{2}\log\left(\frac{d(x_0+iy)}{2\pi}\right)+O(d^2x_0)\right)$$

$$= \exp\left(\frac{\pi^2}{6x_0}\left(1-\frac{iy}{x_0}-\frac{y^2}{x_0^2}+O\left(\frac{|y|^3}{x_0^3}\right)\right)+\frac{d}{2}\log\left(\frac{dx_0}{2\pi}\right)\right.$$
$$\left.+\frac{d}{2}\log\left(1+\frac{iy}{x_0}\right)+O(d^2x_0)\right)$$
$$= \exp\left(\frac{\pi^2}{6x_0}-iy\frac{\pi^2}{6x_0^2}-\frac{\pi^2y^2}{6x_0^3}+\frac{d}{2}\log\left(\frac{dx_0}{2\pi}\right)+o(n^{-\varepsilon})\right).$$

We obtain for the integrand

$$P := \left\{\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\right\}\exp((n-R-Q)(x_0+iy))$$
$$= \exp\left\{\frac{\pi^2}{6x_0}-iy\frac{\pi^2}{6x_0^2}-\frac{\pi^2y^2}{6x_0^3}+\frac{d}{2}\log\left(\frac{dx_0}{2\pi}\right)+o(n^{-\varepsilon})\right.$$
$$+(n-R-Q)(x_0+iy)-\frac{C_2}{x_0}+\sum_{r=1}^{d}(N_r-k_0)\log 2-\frac{1}{2}\sum_{r=1}^{d}(N_r-k_0)^2dx_0$$
$$+\frac{d}{2}\log 2+idy\Bigg(\frac{C_2}{dx_0^2}+\frac{k_0\log 2}{x_0}-k_0\sum_{r=1}^{d}(N_r-k_0)\Bigg)$$
$$\left.+d^2y^2\left(\frac{C_2}{d^2x_0^3}+\frac{k_0\log 2}{dx_0^2}+\frac{k_0^2}{2x_0}\right)+o(n^{-\varepsilon})\right\}.$$

We collect terms in iy, y^2:

$$P = \exp\left\{\frac{\pi^2}{6x_0}+\frac{d}{2}\log\left(\frac{dx_0}{2\pi}\right)+(n-R-Q)x_0\right.$$
$$-\frac{C_2}{x_0}+\sum_{r=1}^{d}(N_r-k_0)\log 2-\frac{1}{2}\sum_{r=1}^{d}(N_r-k_0)^2dx_0+\frac{d\log 2}{2}$$
$$+iy\Bigg(n-R-Q-\frac{\pi^2}{6x_0^2}+\frac{C_2}{x_0^2}+\frac{dk_0\log 2}{x_0}-dk_0\sum_{r=1}^{d}(N_r-k_0)\Bigg)$$
$$\left.+y^2\Bigg(-\frac{\pi^2}{6x_0^3}+\frac{C_2}{x_0^3}+\frac{dk_0\log 2}{x_0^2}+\frac{d^2k_0^2}{2x_0}\Bigg)+o(n^{-\varepsilon})\right\}.$$

As a special case, we obtained that

$$\left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp((n-R-Q)x_0) = \exp\left\{\frac{\pi^2}{6x_0}+\frac{d}{2}\log\left(\frac{dx_0}{2\pi}\right)+(n-R-Q)x_0\right.$$
$$\left.-\frac{C_2}{x_0}+\sum_{r=1}^{d}(N_r-k_0)\log 2-\frac{1}{2}\sum_{r=1}^{d}(N_r-k_0)^2dx_0+\frac{d\log 2}{2}+o(n^{-\varepsilon})\right\}. \qquad (5.2)$$

Consequently,

$$\left\{\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\right\}\exp((n-R-Q)(x_0+iy))$$
$$=\left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp((n-R-Q)x_0)$$
$$\times\exp\left\{iy\left(n-R-Q-\frac{\pi^2}{6x_0^2}+\frac{1}{x_0^2}\sum_{m=1}^{\infty}\frac{1}{m^2 2^m}+\frac{dk_0\log 2}{x_0}-dk_0\sum_{r=1}^{d}(N_r-k_0)\right)\right.$$
$$\left.+y^2\left(-\frac{\pi^2}{6x_0^3}+\frac{1}{x_0^3}\sum_{m=1}^{\infty}\frac{1}{m^2 2^m}+\frac{dk_0\log 2}{x_0^2}+\frac{d^2k_0^2}{2x_0}\right)+o(n^{-\varepsilon})\right\}. \tag{5.3}$$

The coefficient of y^2 in (5.3) is

$$\frac{1}{x_0^3}\left(-\frac{\pi^2}{6}+\frac{\pi^2}{12}-\frac{\log^2 2}{2}+dk_0x_0\log 2+\frac{d^2k_0^2x_0^2}{2}\right)=-\frac{2\sqrt{3}}{\pi}n^{\frac{3}{2}}\left(1-\frac{12\log^2 2}{\pi^2}\right),$$

where $\frac{12\log^2 2}{\pi^2}<\frac{12\cdot 0.49}{\pi^2}<\frac{6}{\pi^2}<1$.

The coefficient of iy in (5.3) is

$$n-R-Q-\frac{1}{x_0^2}\left(\frac{\pi^2}{6}-\sum_{m=1}^{\infty}\frac{1}{m^2 2^m}-dk_0x_0\log 2\right)-dk_0\sum_{r=1}^{d}(N_r-k_0)$$
$$=-R-Q+\frac{\log^2 2}{2x_0^2}-dk_0\sum_{r=1}^{d}(N_r-k_0)=-2dk_0\sum_{r=1}^{d}(N_r-k_0)+O(n^{\frac{3}{4}-2\varepsilon}).$$

Since $|y|O(n^{\frac{3}{4}-2\varepsilon})=o(n^{-\varepsilon})$ we infer from (5.3) that

$$\left\{\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\right\}\exp((n-R-Q)(x_0+iy))$$
$$=\left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp((n-R-Q)x_0)$$
$$\times\exp\left\{-iy2dk_0\sum_{r=1}^{d}(N_r-k_0)-y^2\frac{2\sqrt{3}n^{\frac{3}{2}}}{\pi}\left(1-\frac{12\log^2 2}{\pi^2}\right)+o(n^{-\varepsilon})\right\}. \tag{5.4}$$

Let $A=\frac{2\sqrt{3}n^{\frac{3}{2}}}{\pi}\left(1-\frac{12\log^2 2}{\pi^2}\right)$, $(A>0)$ and $B=2dk_0\sum_{r=1}^{d}(N_r-k_0)$. Then, from (5.4)

$$S_0=\frac{d}{2\pi}\int_{-y_1}^{y_1}\left\{\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\right\}\exp((n-R-Q)(x_0+iy))\mathrm{d}y$$

$$= \frac{d}{2\pi}\left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp((n-R-Q)x_0)\int_{-y_1}^{y_1}\exp(-iyB-Ay^2+o(n^{-\varepsilon}))\mathrm{d}y.$$

Lemma 5.1. *We have:*

$$\int_{-y_1}^{y_1}\exp(-iyB-Ay^2+o(n^{-\varepsilon}))\mathrm{d}y$$
$$= \sqrt{\frac{\pi}{A}}\exp\left(-\frac{B^2}{4A}\right)\left\{1+o(n^{-\varepsilon})\exp\left(\frac{B^2}{4A}\right)\right\}. \quad (5.5)$$

Proof. These are standard manipulations on Gaussian integrals thus we won't write all the details. Let $I_{AB}(y_1)$ be the integral of the left hand side of (5.5). Since for $|y| \le y_1$, $\exp(-iyB-Ay^2+o(n^{-\varepsilon})) = (1+o(n^{-\varepsilon}))\exp(-iyB-Ay^2))$, we have:

$$I_{AB}(y_1) = \int_{-\infty}^{+\infty}\exp(-iyB-Ay^2)\mathrm{d}y$$
$$+\,O\Big(\int_{y_1}^{+\infty}\exp(-Ay^2)\mathrm{d}y\Big) + o(n^{-\varepsilon})\int_{-\infty}^{+\infty}\exp(-Ay^2)\mathrm{d}y.$$

The main term is a Gaussian integral:

$$\int_{-\infty}^{+\infty}\exp(-iyB-Ay^2)\mathrm{d}y = \sqrt{\frac{\pi}{A}}\exp\left(-\frac{B^2}{4A}\right).$$

For the error terms we have

$$\int_{y_1}^{+\infty}\exp(-Ay^2)\mathrm{d}y \ll \frac{1}{Ay_1}\exp(-Ay_1^2) \quad\text{and}\quad \int_{-\infty}^{+\infty}\exp(-Ay^2) \ll \frac{1}{\sqrt{A}},$$

the Lemma follows.

Furthermore

$$\frac{B^2}{4A} = O\left(n^{-\frac{3}{2}}\left(\sqrt{n}d\frac{n^{\frac{1}{4}}\sqrt{\log n}}{dw(n)}\right)^2\right) = o(\log n).$$

Thus the error term in Lemma 5.1 is:

$$o(n^{-\varepsilon})\exp\left(\frac{B^2}{4A}\right) = o(1).$$

Therefore

$$S_0 = (1+o(1))\sqrt{\frac{\pi}{A}}\exp\left(-\frac{B^2}{4A}\right)\frac{d}{2\pi}\left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp((n-R-Q)x_0).$$

Adding the estimates for the trivial parts (see (2.8), (3.1)) we obtain that for $n \equiv R \pmod d$,

$$\begin{aligned}\Pi_d^*(n,\mathcal{R}) &= \left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp((n-R-Q)x_0)\\ &\times\left\{(1+o(1))\sqrt{\frac{\pi}{A}}\exp\left(-\frac{B^2}{4A}\right)\frac{d}{2\pi}+O\left(\exp\left(-\frac{1+o(1)}{4x_0}\right)\right)\right.\\ &\left.+O\left(\exp\left(-\frac{\sqrt{3}}{\pi^5 3^3}n^{\frac14+2\varepsilon}\right)\right)+O\left(\exp\left(-\frac{\sqrt{3}}{\pi^5 3^3}n^{2\varepsilon/3}\right)\right)\right\}\\ &= (1+o(1))\sqrt{\frac{\pi}{A}}\exp\left(-\frac{B^2}{4A}\right)\frac{d}{2\pi}\left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp((n-R-Q)x_0). \quad (5.6)\end{aligned}$$

To end the proof, it remains to insert the classical formula

$$q(n) = (1+o(1))\frac{1}{4\cdot 3^{1/4}n^{3/4}}\exp\left(\frac{\pi\sqrt{n}}{\sqrt{3}}\right), \quad (5.7)$$

and our previous results on the $g_{N_r}(dx_0)$ (see (5.2)), and to do the convenient computations. We obtain

$$\begin{aligned}\Pi_d^*(n,\mathcal{R}) &= (1+o(1))4\cdot 3^{1/4}q(n)\frac{1}{\sqrt{1-\frac{12(\log 2)^2}{\pi^2}}}\frac{d}{2\sqrt{2}3^{1/4}}\left(\frac{d}{2\sqrt{3n}}\right)^{d/2}\exp\left(-\frac{B^2}{4A}\right.\\ &\left.+\sum_{r=1}^{d}(N_r-k_0)\log 2 - x_0\left(R+Q-\frac{\log^2 2}{2x_0^2}\right)-\frac12\sum_{r=1}^{d}(N_r-k_0)^2 dx_0\right). \quad (5.8)\end{aligned}$$

By formulae (2.11) and (2.12) of [6] and by (2.4), we have:

$$R+Q-\frac{\log^2 2}{2x_0^2} = \frac{dk_0}{2}+dk_0\sum_{r=1}^{d}(N_r-k_0)+\frac{d}{2}\sum_{r=1}^{d}(N_r-k_0)^2+o(\sqrt{n}).$$

Thus the argument of the exponential in (5.8) is

$$\exp\left(-\frac{B^2}{4A}+\dots\right) = \exp\left(-\frac{B^2}{4A}-\frac{\log 2}{2}-dx_0\sum_{r=1}^{d}(N_r-k_0)^2+o(1)\right).$$

Inserting this in (5.8) ends the proof of Theorem 1.1 for $\mathcal{D}=\{1,\dots,d\}$. $\square$

6. First steps of the proof of Theorem 1.1 for $\mathcal{D} \neq \{1, \ldots, d\}$

Like in Section 3 of [6], we apply Lemma 2.1 of [6], the Cauchy formula and write $z = x_0 + iy$:

$$\Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big\{ \prod_{r \in \mathcal{D}^c} h_r(x_0 + iy) \Big\} \\ \times \Big\{ \prod_{r \in \mathcal{D}} g_{N_r}(d(x_0 + iy)) \Big\} \exp\big((n - \mathcal{R}_{\mathcal{D}} - Q_{\mathcal{D}})(x_0 + iy)\big) \mathrm{d}y, \quad (6.1)$$

with

$$h_r(z) = \prod_{j=0}^{\infty} (1 + \exp(-(r + jd)z)).$$

When $\mathcal{D}^c \neq \emptyset$ and $\mathcal{D}^c \neq \{d\}$, the functions h_r are not $2\pi/d$-periodic but we still split the integral in intervales of length $2\pi/d$ in order to use our previous work on the functions g_k.

$$\Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) = \frac{1}{2\pi} \sum_{|\lambda| \leq \lfloor \frac{d-1}{2} \rfloor} \int_{-\frac{\pi}{d} + \frac{2\lambda\pi}{d}}^{\frac{\pi}{d} + \frac{2\lambda\pi}{d}} \cdots + B,$$

with

$$B = \begin{cases} 0 & \text{if } d \text{ is odd} \\ \displaystyle\int_{-\pi}^{-\pi + \frac{\pi}{d}} \cdots + \int_{\pi - \frac{\pi}{d}}^{\pi} \cdots = \int_{\pi - \frac{\pi}{d}}^{\pi + \frac{\pi}{d}} & \text{if } d \text{ is even.} \end{cases}$$

Next we do some convenient change of variables :

$$\Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) = \frac{1}{2\pi} \sum_{-\frac{d}{2} < \lambda \leq \frac{d}{2}} \int_{-\frac{\pi}{d}}^{\frac{\pi}{d}} \Big\{ \prod_{r \in \mathcal{D}^c} h_r\left(x_0 + iy + i\frac{2\lambda\pi}{d}\right) \Big\} \Big\{ \prod_{r \in \mathcal{D}} g_{N_r}(d(x_0 + iy)) \Big\} \\ \times \exp\left((n - \mathcal{R}_{\mathcal{D}} - Q_{\mathcal{D}})(x_0 + iy) + (n - \mathcal{R}_{\mathcal{D}}) \frac{2i\lambda\pi}{d} \right) \mathrm{d}y = \sum_{-\frac{d}{2} < \lambda \leq \frac{d}{2}} S(\lambda),$$

say. We will write the splitting

$$S(\lambda) = S_0(\lambda) + S_1(\lambda) + S_2(\lambda),$$

where in $S_0(\lambda)$ the range of integration for y is $|y| \leq y_1$, in $S_1(\lambda)$ it is for $y_1 \leq |y| \leq y_2$ (with $y_2 = 3\pi x_0$) and in $S_2(\lambda)$ we take $y_2 \leq |y| \leq \frac{\pi}{d}$ (cf. (4.11) of [6]).

7. Upper bounds of $S_1(\lambda)$ and $S_2(\lambda)$

To obtain a convenient upper bound of these terms we first remark that

$$\left| h_r\left(x_0 + iy + \frac{2i\pi\lambda}{d}\right)\right| \leq h_r(x_0). \tag{7.1}$$

Let $j \in \mathbb{N}$. To prove (7.1) it is enough to prove that each $T_j \leq 1$ with

$$T_j := \frac{\left|1 + \exp\left(- (r+jd)(x_0 + iy + \frac{2i\pi\lambda}{d})\right)\right|}{|1+\exp(-(r+jd)x_0)|}.$$

By a simple computation we have

$$T_j^2 = 1 - \frac{4\exp(-x_0(r+jd))\sin^2\left(\frac{y}{2}(r+jd) + \frac{\pi\lambda r}{d}\right)}{(1+\exp(-x_0(r+jd)))^2} \leq 1.$$

By Lemma 2.1 we have

$$|S_2(\lambda)| \leq \left\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\right\}\left\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\right\} \times \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0)\exp\left(\frac{|\mathcal{D}|(-1+o(1))}{4dx_0}\right).$$

By Lemma 3.1 we also have

$$|S_1(\lambda)| \leq \left\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\right\}\left\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\right\} \times \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0)\exp\left(-\frac{|\mathcal{D}|\sqrt{3}n^{2\varepsilon/3}}{27\pi^5 d}\right).$$

If $|\mathcal{D}|$ is small, *i.e.*, if $|\mathcal{D}| \leq dn^{-\varepsilon/3}$ this last estimate for $S_1(\lambda)$ is not sufficient. However, by Lemma 3.1 (i), the contribution of the range $n^{-5/8+\varepsilon} \leq |y| \leq 3\pi x_0$ to $S_1(\lambda)$ is

$$\leq \left\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\right\}\left\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\right\}\exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0)\exp\left(-\frac{|\mathcal{D}|\sqrt{3}n^{1/4+2\varepsilon}}{27\pi^5 d}\right).$$

Thus it remains to handle the case $|\mathcal{D}| \leq dn^{-\varepsilon/3}$ in the range $y_1 \leq |y| < n^{-5/8+\varepsilon}$.

• First we study the case $\lambda = 0$. We use a similar argument as in the proof of Lemma 3.1.

For any $1 \le r \le d$, let J_r denote the set of the integers j such that

$$\frac{1}{2x_0} \le r + jd \le \frac{1}{x_0}. \tag{7.2}$$

Then for $j \in J_r$ we have

$$\frac{\exp(-x_0(r+jd))}{(1+\exp(-x_0(r+jd)))^2} \ge \frac{1}{\mathrm{e}(1+\mathrm{e}^{-1/2})^2}. \tag{7.3}$$

This gives for the correspondent T_j :

$$T_j^2 \le 1 - \frac{4}{\mathrm{e}(1+\mathrm{e}^{-1/2})^2} \sin^2\Big(\frac{y}{2}(r+jd)\Big).$$

Next, quite like in the proof of Lemma 3.1, we have for n large enough

$$\begin{aligned} |h_r(x_0+iy)| &\le h_r(x_0)\bigg(1 - \frac{4}{\mathrm{e}(1+\mathrm{e}^{-1/2})^2}\frac{y_1^2}{4\pi^2 x_0^2}\bigg)^{|J_r|/2} \\ &\le h_r(x_0)\exp\bigg(-\frac{y_1^2}{48\pi^2 d x_0^3}\bigg) \le h_r(x_0)\exp\bigg(-\frac{\sqrt{3}n^{2\varepsilon/3}}{2\pi^5 d}\bigg). \end{aligned}$$

This upper bound combined with Lemma 3.1 is sufficient to obtain

$$|S_1(0)| \le \bigg\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\bigg\}\bigg\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\bigg\}\exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0)\exp\bigg(-\frac{\sqrt{3}n^{2\varepsilon/3}}{27\pi^5}\bigg).$$

• Now we suppose that $\lambda \ne 0$. We write $\frac{\lambda}{d} = \frac{\lambda'}{d'}$ with $(\lambda', d') = 1$ and $d' > 0$.

First we suppose that $d' \ge n^{\varepsilon/4}$. Since $(\lambda', d') = 1$, there are $\frac{d'}{4} + O(1)$ integers $r_0 \in \{1, \dots, d'\}$ such that $\frac{\lambda' r_0}{d'} \mod 1 \in \big[\frac14, \frac12\big]$. Thus there are $\frac{d}{4} + O(\frac{d}{d'})$ integers $r \in \{1, \dots, d\}$ such that $\frac{\lambda' r}{d'} \mod 1 \in \big[\frac14, \frac12\big]$ (again for n large enough).

Since $|\mathcal{D}| \le dn^{-\varepsilon/3} < \frac{d}{4} + O\big(\frac{d}{d'}\big)$ for n large enough, there exists $r_1 \in \mathcal{D}^c$ such that $\frac{\lambda' r_1}{d'} \mod 1 \in \big[\frac14, \frac12\big]$. For $j \in J_{r_1}$, $|(r_1 + jd)y| \ll n^{-1/8+\varepsilon}$. Thus $\sin^2\big((r_1 + jd)\frac{y}{2} + \frac{\lambda r_1 \pi}{d}\big) \ge \sin^2\frac{\pi}{6} = \frac14$.

This gives

$$\prod_{j\in J_{r_1}} |T_j(r_1)|^2 \le \bigg(\frac{11}{12}\bigg)^{|J_{r_1}|} \le \exp\bigg(-\frac{1}{30dx_0}\bigg),$$

which is a sufficient upper bound.

Now we suppose that $2 \le d' < n^{\varepsilon/4}$. There are $d - \frac{d}{d'}$ integers r such that $d' \nmid r$. For these integers r and $j \in J_r$, we have

$$\sin^2\left((r+jd)\frac{y}{2} + \frac{\pi\lambda r}{d}\right) \ge \sin^2\left(\frac{\pi}{2d'}\right) \ge \frac{1}{d^2}.$$

Since $|\mathcal{D}| \le dn^{-\varepsilon/3}$, there are at least $d/3$ such integers $r \in \mathcal{D}^c$ such that $d' \nmid r$.

Thus we have:

$$\prod_{r\in\mathcal{D}^c} \frac{\left|h_r\left(x_0+iy+\frac{2i\pi\lambda}{d}\right)\right|}{h_r(x_0)} \le \prod_{\substack{r\in\mathcal{D}^c\\ r\not\equiv 0 \pmod{d'}}} \prod_{j\in J_r}\left(1-\frac{1}{6d^2}\right) \le \exp\left(-\frac{d}{48d^3x_0}\right),$$

which is a sufficient upper bound when $d \le n^{1/4-2\varepsilon}$.

8. The terms $S_0(\lambda)$ for $\lambda \neq 0$

We have to consider the integrals

$$S_0(\lambda) = \frac{1}{2\pi}\int_{|y|\le y_1}\left\{\prod_{r\in\mathcal{D}^c} h_r\left(x_0+iy+\frac{2i\pi\lambda}{d}\right)\right\}\left\{\prod_{r\in\mathcal{D}} g_{N_r}(d(x_0+iy))\right\}$$
$$\times \exp\left((n-R_{\mathcal{D}}-Q_{\mathcal{D}})\left(x_0+iy+\frac{2i\pi\lambda}{d}\right)\right)\mathrm{d}y.$$

• First we suppose that there exists $r \in \mathcal{D}^c$ such that $\lambda r \not\equiv 0 \pmod{d}$. In the previous section we remarked that

$$\left|h_r\left(x_0+iy+\frac{2i\pi\lambda}{d}\right)\right|$$
$$= h_r(x_0)\prod_{j=0}^{\infty}\left(1-\frac{4\exp(-x_0(r+jd))\sin^2\left(\frac{y}{2}(r+jd)+\frac{\pi\lambda r}{d}\right)}{(1+\exp(-x_0(r+jd)))^2}\right)^{1/2}.$$

We work again with the sets J_r defined by (7.2) in the previous section. For $j \in J_r$, $\lambda r \equiv a \pmod{d}, 1 \le |a| \le d/2$ and $|y| \le y_1$ we have:

$$\sin^2\left(\frac{y}{2}(r+jd)+\frac{\pi\lambda r}{d}\right)$$
$$= \sin^2\left(\frac{\pi|a|}{d} \pm \frac{|y|}{2}(r+jd)\right) \ge \sin^2\left(\frac{\pi|a|}{d} - \frac{y_1}{x_0}\right) \ge \sin^2\left(\frac{\pi}{d} - \frac{y_1}{x_0}\right) \ge \frac{3}{d^2},$$

for n large enough (we recall that $d \le n^{1/4-2\varepsilon}$).

Since $|J_r| = \frac{1}{2dx_0} + O(1) \ge \frac{1}{3dx_0}$, we have:

$$\left|h_r\left(x_0 + iy + \frac{2i\pi\lambda}{d}\right)\right| \le h_r(x_0) \prod_{j\in J_r} \left(1 - \frac{12}{\mathrm{e}d^2(1+\mathrm{e}^{-1/2})^2}\right)^{1/2}$$

$$\le h_r(x_0)\exp\left(\frac{1}{6dx_0}\log\left(1 - \frac{12}{\mathrm{e}d^2(1+\mathrm{e}^{-1/2})^2}\right)\right) \le h_r(x_0)\exp\left(-\frac{2}{\mathrm{e}d^3x_0(1+\mathrm{e}^{-1/2})^2}\right).$$

Thus if there exists $r_0 \in \mathcal{D}^c$ such that $\lambda r_0 \not\equiv 0 \pmod d$ then

$$|S_0(\lambda)| \le \left\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\right\}\left\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\right\}\exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0) \times \exp\left(-\frac{2}{\mathrm{e}d^3x_0(1+\mathrm{e}^{-1/2})^2}\right). \tag{8.1}$$

This upper bound is sufficient only for $d \le n^{1/6-\varepsilon}$.

• We suppose now that

$$\lambda r \equiv 0 \pmod d \quad \text{for all } r \in D^c. \tag{8.2}$$

If $\mathcal{D}^c = \{d\}$ then by (1.1), $n - R_D - Q_{\mathcal{D}} \equiv 0 \pmod d$ and $S_0(\lambda) = S_0(0)$ for all λ.

In the general case, if (8.2) holds then we must have $\frac{d}{(\lambda,d)}|r$. Thus $\frac{d}{(\lambda,d)}|\delta$ and again by (1.1), we have $n - R_{\mathcal{D}} - Q_{\mathcal{D}} \equiv 0 \pmod{\frac{d}{(\lambda,d)}}$. Thus $\exp\big((n - R_{\mathcal{D}} - Q_{\mathcal{D}})\frac{2i\pi\lambda}{d}\big) = 1$ and $S_0(\lambda) = S_0(0)$.

For $\mathcal{D}^c$ given, there exists δ integers λ modulo d such that $r\lambda \equiv 0 \pmod d$ for all $r \in \mathcal{D}^c$.

We summarize these observations in the following lemma.

Lemma 8.1. *For $d \le n^{1/6-\varepsilon}$ we have:*

$$\sum_{-\frac{d}{2}<\lambda\le\frac{d}{2}} S_0(\lambda) = \delta S_0(0)(1 + o(1)).$$

9. The function h_r^{-1} in the range $|y| \le y_1$

The generating function associated to unequal partitions is $h(z) = \prod_{j=1}^{\infty}(1+z^j)$. For $S_0(0)$ it remains to handle the integral

$$\frac{1}{2\pi}\int_{|y|\le y_1} h(\exp(-(x_0+iy)))\left\{\prod_{r\in\mathcal{D}} \frac{g_{N_r}(d(x_0+iy))}{h_r(x_0+iy)}\right\}\exp((n-R_{\mathcal{D}}-Q_{\mathcal{D}})(x_0+iy))\mathrm{d}y.$$

In this section we will state an asymptotic estimate related to h_r^{-1} in the range $|y| \le y_1$.

The following method looks like the general method of Meinardus [7] for studying generating functions associated to partitions (this method is presented in details in the book of Andrews [1]). In fact we were also inspired by the chapter on the application of saddle point method to the partitions function in the Master course of Tenenbaum [8].

In Lemma 4.1, we obtained an estimation of $g_k(dw)$ in function of $f(dw)$. This leads us to try in fact to obtain an estimation of $U_r(z) := f(dz)h_r^{-1}(z)$ for $|y| \le y_1$ instead of $h_r^{-1}(z)$. This change will make some computations easier. Thus we consider

$$U_r(z) = \left\{ \prod_{j=1}^{\infty} (1-\exp(-jdz))^{-1} \right\} \left\{ \prod_{j=0}^{\infty} (1+\exp(-(r+jd)z))^{-1} \right\}.$$

The main result of this section is the following lemma.

Lemma 9.1. *Let $\eta > 0$. For $|y| \le y_1$ and $1 \le r \le d$, we have*

$$U_r(z) = \exp\left(\frac{\pi^2}{12dz} + \left(\frac{r}{d} - 1\right)\log 2 + \frac{1}{2}\log\left(\frac{dz}{\pi}\right) + O(d^{1+\eta}r^{-\eta}|z|) + O(d^{-1}n^{-2\varepsilon}) \right).$$

In the next section we will apply this lemma with $\eta > 0$ small enough such that $d^{2+\eta}|z| \le n^{-\varepsilon}$.

Proof. If $r = d$, there are quite no work to do since

$$U_d(z) = \prod_{j=1}^{\infty} (1+\exp(-jdz))^{-1}(1-\exp(-jdz))^{-1} = \prod_{j=1}^{\infty} (1-\exp(-2jdz))^{-1} = f(2dz).$$

Thus

$$U_d(z) = \exp\left(\frac{\pi^2}{12dz} + \frac{1}{2}\log\left(\frac{2dz}{2\pi}\right) + O(d|z|) \right).$$

Now we suppose that $r \ne d$. We prefer to work with

$$\begin{aligned}\tilde{U}_r(z) &:= \prod_{j=1}^{\infty} (1-\exp(-jdz))^{-1}(1+\exp(-(r+jd)z))^{-1} \\ &= U_r(z)(1+\exp(-rz)) = U_r(z)u_r(z).\end{aligned}$$

We easily see that

$$u_r(z) = (2 + O(r|z|)). \tag{9.1}$$

Let $F(v,s) = \sum_{k=1}^{\infty} \frac{\exp(-kv)}{k^s}$. If $\Re e\, v > 0$, then $s \mapsto F(v,s)$ is analytic on $\mathbb{C}$.

The first step of the proof is the following result.

Lemma 9.2. *For $r \neq d$, $\eta > 0$, $z = x_0 + iy$ with $|y| \leq y_1$, we have*

$$\begin{aligned}\log(\tilde{U}_r(z)) = &\frac{1}{dz}\left(\frac{\pi^2}{6} + F(rz + i\pi, 2)\right) + \frac{1}{2}\log\left(\frac{dz}{2\pi}\right)\\ &- \frac{1}{2}F(rz + i\pi, 1) + \frac{dz}{12}\left(-\frac{1}{2} + F(z + i\pi, 0)\right) + O(|z| r^{-\eta} d^{1+\eta}).\end{aligned}$$

The last term $\frac{dz}{12}\big(-\frac{1}{2} + F(z+i\pi, 0)\big)$ in the above formula could be removed because it is $O(|z|r^{-\eta}d^{1+\eta})$. The following proof uses Mellin formula and looks like the general method of Meinardus for studying generating functions associated to partitions.

We begin by some standard manipulations

$$\begin{aligned}\tilde{U}_r(z) &= \exp\left\{-\sum_{j=1}^{\infty}(\log(1 + \exp(-(r + jd)z)) + \log(1 - \exp(-jdz))\right\}\\ &= \exp\left\{\sum_{j,k\geq 1}\frac{\exp(-jkdz)}{k}((-1)^k\exp(-krz) + 1)\right\}.\end{aligned}$$

Let us write

$$\log(\tilde{U}_r(z)) = \sum_{m=1}^{\infty}\frac{\beta(m)}{m}\exp(-mdz),$$

with

$$\beta(m) = \sum_{jk=m} j((-1)^k\exp(-krz) + 1).$$

By the Mellin transform formula we have:

$$\log(\tilde{U}_r(z)) = \sum_{m\geq 1}\frac{\beta(m)}{m}\cdot\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{\Gamma(s)}{(mdz)^s}\mathrm{d}s.$$

But the Dirichlet series is

$$\begin{aligned}\sum_{m\geq 1}\frac{\beta(m)}{m^{s+1}} &= \sum_{j\geq 1}\frac{1}{j^s}\sum_{k\geq 1}\frac{(1 + (-1)^k\exp(-krz))}{k^{s+1}}\\ &= \zeta(s)(\zeta(s+1) + F(rz + i\pi, s+1)).\end{aligned}$$

Thus we have

$$\log(\tilde{U}_r(z)) = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz + i\pi, s+1))}{(dz)^s}\mathrm{d}s.$$

Let $\eta \in]0,1[$. We move the integral until the line $\Re e\, s = -1-\eta$. This gives

$$\log(\tilde{U}_r(z)) = \frac{1}{2i\pi}\int_{-1-\eta-i\infty}^{-1-\eta+i\infty} \frac{\Gamma(s)\zeta(s)(\zeta(s+1)+F(rz+i\pi,s+1))}{(dz)^s}\mathrm{d}s$$
$$+\operatorname{Res}(1)+\operatorname{Res}(0)+\operatorname{Res}(-1)+E,$$

where E is the error term arising from the horizontal branches.

For $\Re e\, s \in [-1-\eta, 2]$ we have:

$$F(zr+i\pi,s+1) = \sum_{\ell\geq 1} \mathrm{e}^{-2\ell zr}\left(\frac{1}{(2\ell)^{s+1}} - \frac{\mathrm{e}^{rz}}{(2\ell-1)^{s+1}}\right)$$
$$= \sum_{\ell\geq 1} \mathrm{e}^{-2\ell zr}\left(\frac{1}{(2\ell)^{s+1}} - \frac{1}{(2\ell-1)^{s+1}}\right) + O\left(r|z|\sum_{\ell\geq 1}\frac{\mathrm{e}^{-2\ell r x_0}}{\ell^{1+\Re e s}}\right).$$

When ℓ is small, the term $\mathrm{e}^{-\ell r x_0}$ is $O(1)$. Thus for $-1-\eta \leq \Re e\, s \leq 2$, we have :

$$\sum_{\ell\geq 1}\frac{\mathrm{e}^{-2\ell r x_0}}{\ell^{1+\Re e\, s}} \ll \sum_{\ell\leq 100/rx_0} \ell^\eta + \sum_{\ell>100/rx_0} \ell^\eta \mathrm{e}^{-2\ell r x_0} \ll (rx_0)^{-1-\eta},$$

since for the second sum we have

$$\sum_{\ell>100/rx_0} \ell^\eta \mathrm{e}^{-2\ell r x_0} \ll \sum_{\ell>100/rx_0} \ell \mathrm{e}^{-2\ell r x_0}(rx_0)^{1-\eta} \ll \frac{(rx_0)^{1-\eta}}{(1-\exp(-2rx_0))^2}.$$

For the other term we have

$$\left|\sum_{\ell\geq 1}\mathrm{e}^{-2\ell r z}\left(\frac{1}{(2\ell)^{1+s}} - \frac{1}{(2\ell-1)^{1+s}}\right)\right| \ll (1+|s|)\sum_{\ell\geq 1}\frac{\mathrm{e}^{-2\ell r x_0}}{\ell^{1-\eta}} \ll \frac{1+|s|}{(rx_0)^\eta}.$$

Thus for $-1-\eta \leq \Re e\, s \leq 2$, we have

$$|F(zr+i\pi,s+1)| \ll |s|(rx_0)^{-\eta} + r|z|(rx_0)^{-1-\eta}. \tag{9.2}$$

We will also use the following classical results for the functions ζ and Γ in vertical strips:

– there exists $H>0$ such that $|\zeta(s)| \ll |\Im m\, s|^H$ for $-3 \leq \Re e\, s \leq 3$ (in fact more generally for $\Re e\, s \in [\sigma_1, \sigma_2]$) and $|\Im m\, s| \geq 1$ (cf. [2] Theorem 12.23 p. 270 for a more precise formulation);

– for $-3 \le \Re e\, s \le 3$ (or $\Re e\, s \in [\sigma_1, \sigma_2]$) and $|\Im m\, s| \to +\infty$, ([9] Corollaire II.0.13 p. 182) we have

$$\Gamma(s) = (1 + O(|\Im m\, s|^{-1}))\sqrt{2\pi}|\Im m\, s|^{\Re e\, s - \frac{1}{2}} \mathrm{e}^{-\pi|\Im m\, s|/2} \mathrm{e}^{i\alpha(s)},$$

with $\alpha(s) = (\Im m\, s)\log|\Im m\, s| - \Im m\, s + \frac{1}{2}\pi(\Re e\, s - \frac{1}{2})\,\mathrm{sgn}(\Im m\, s)$.

With this two formulae and by (9.2) we easily see that

$$\lim_{T\to+\infty} \int_{-1-\eta\pm iT}^{2\pm iT} \frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz+i\pi, s+1))}{(dz)^s} \mathrm{d}s = 0, \tag{9.3}$$

and

$$\left| \int_{\Re e\, s=-1-\eta} \frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz+i\pi, s+1))}{(dz)^s} \mathrm{d}s \right| \ll r^{-\eta} d^{1+\eta}|z|. \tag{9.4}$$

Now we compute the different residues:

We have

$$\mathrm{Res}(1) = \frac{\Gamma(1)\big(\zeta(2) + F(rz+i\pi, 2)\big)}{dz} = \frac{1}{dz}\left(\frac{\pi^2}{6} + F(rz+i\pi, 2)\right). \tag{9.5}$$

In $s = 0$, we have two poles, one from Γ, the other from the function $s \mapsto \zeta(s+1)$. We use the well known results $\Gamma'(1) = -\gamma$, $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$).

Thus $\mathrm{Res}(0)$ is the coefficient in s^{-1} in the following formula:

$$\left(\frac{1 - \gamma s + O(|s|^2)}{s}\right)(\zeta(0) + s\zeta'(0) + O(|s|^2))$$
$$\times \left[\frac{1}{s} + \gamma + O(|s|) + F(rz+i\pi, 1)\right](1 - s\log(dz) + O(|s|^2)). \tag{9.6}$$

Since $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, we find

$$\mathrm{Res}(0) = \frac{1}{2}\left(\log\left(\frac{dz}{2\pi}\right) - F(rz+i\pi, 1)\right). \tag{9.7}$$

In $s = -1$, Γ has a simple pole with residue -1 thus we have:

$$\mathrm{Res}(-1) = -\zeta(-1)(\zeta(0) + F(rz+i\pi, 0))dz.$$

Since $\zeta(-1) = -\frac{1}{12}$, we obtain

$$\mathrm{Res}(-1) = \frac{dz}{12}\left(-\frac{1}{2} + F(rz+i\pi, 0)\right). \tag{9.8}$$

Formulae (9.5), (9.7), (9.8), (9.3), (9.4) end the proof of Lemma 9.2. □

Now we have to study the terms $F(*, s)$ with $s = 0, 1, 2$. First we have

$$F(rz + i\pi, 0) = \sum_{m=1}^{\infty} (-1)^m \exp(-mrz) = -\frac{1}{1 + \exp(rz)} = -\frac{1}{2} + O(rn^{-1/2}),$$

for $|y| \le y_1$. For the contribution of $F(rz + i\pi, 1)$ we have

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m} \exp(-mrz) = \frac{1}{2} \log\Big(1 + \exp(-rz)\Big) = \frac{1}{2} \log 2 + O(r|z|). \quad (9.9)$$

The contribution of $F(rz + i\pi, 2)$ is more difficult to handle because of the factor $(dz)^{-1}$. We will prove the following lemma

Lemma 9.3. *For $|y| \le y_1$, we have for $d \le n^{1/4-2\varepsilon}$*

$$\frac{1}{dz} F(rz + i\pi, 2) = -\frac{\pi^2}{12dz} + \frac{r}{d} \log 2 + O(d^{-1} n^{-2\varepsilon}).$$

Proof. Let M be an odd integer. We have

$$\frac{1}{dz} F(rz+i\pi, 2) = \frac{1}{dz} \sum_{m=1}^{M} \frac{(-1)^m \exp(-mrz)}{m^2} + \sum_{m=M+1}^{+\infty} \frac{(-1)^m \exp(-mrz)}{dzm^2} = T_1 + T_2,$$

say. We begin with T_2. Since the sum is absolutely convergent, we can regroup the terms $m = 2\ell$ with the terms $m = 2\ell + 1$

$$\begin{aligned} |T_2| &\le \frac{1}{d|z|} \sum_{\ell=\frac{M+1}{2}}^{+\infty} \exp(-2\ell x_0) \left| \frac{1}{4\ell^2} - \frac{\exp(-rz)}{(2\ell+1)^2} \right| \le \frac{1}{d|z|} \sum_{\ell=\frac{M+1}{2}}^{+\infty} \left| \frac{1}{4\ell^2} - \frac{1+O(r|z|)}{(2\ell+1)^2} \right| \\ &\le \frac{1}{d|z|} \sum_{\ell=\frac{M+1}{2}}^{+\infty} \left(\frac{4\ell+1}{4\ell^2(2\ell+1)^2} + O\Big(\frac{r|z|}{(2\ell+1)^2}\Big) \right) \ll \frac{1}{d|z|M^2} + \frac{r}{dM}. \end{aligned}$$

This leads us to choose M the smallest odd integer $\ge n^{1/4+\varepsilon}$. For T_1 we use the fact that $\exp(-mrz)$ is near 1 if M is not too large. Let $J \in \mathbb{N}$ to be specified later:

$$T_1 = \frac{1}{dz} \sum_{m=1}^{M} \frac{(-1)^m}{m^2} (1 - mrz + \sum_{j=2}^{J-1} \frac{(-1)^j m^j r^j z^j}{j!} + O(m^J r^J |z|^J)).$$

By the same type of argument as for T_2 (which are standard manipulations on alternating series) we show that

$$\frac{1}{dz}\sum_{m=1}^{M}\frac{(-1)^m}{m^2}=\frac{1}{dz}\sum_{m=1}^{+\infty}\frac{(-1)^m}{m^2}+O\left(\frac{1}{d|z|M^2}\right)=-\frac{\pi^2}{12dz}+O\left(\frac{1}{d|z|M^2}\right),$$

and

$$\frac{r}{d}\sum_{m=1}^{M}\frac{(-1)^m}{m}=\frac{r}{d}\sum_{m=1}^{+\infty}\frac{(-1)^m}{m}+O\left(\frac{r}{dM}\right)=-\frac{r}{d}\log 2+O\left(\frac{r}{dM}\right).$$

Next, for each $2\le j\le J-1$ we have (recall that $r\le d\le n^{1/4-2\varepsilon}$)

$$\frac{r^j|z|^{j-1}}{d}\left|\sum_{m=1}^{M}(-1)^m m^{j-2}\right|\ll\frac{r^j|z|^{j-1}M^{j-2}}{d}\ll\frac{n^{-(2+j)\varepsilon}}{d}.$$

Finally for the last error term we have:

$$r^J|z|^{J-1}d^{-1}\sum_{m=1}^{M}m^{J-2}\ll(M|z|)^{J-1}r^Jd^{-1}\ll n^{\frac{1}{4}-J\varepsilon}d^{-1},$$

which is small enough for $J=\lceil\frac{5}{\varepsilon}\rceil$.

This ends the proof of Lemma 9.3.

If we insert (9.9) and Lemma 9.3 in Lemma 9.2 and don't forget (9.1) we obtain Lemma 9.1 in the case $r\ne d$. □

10. The term $S_0(0)$, end of the proof of Theorem 1.1

Recall that

$$h(\exp(-z))=\exp\left(\frac{\pi^2}{12z}-\frac{1}{2}\log 2+O(|z|)\right),$$

if $z\to 0$ with $|\arg z|\le\kappa<\pi/2$. By Lemma 9.1 we have for $|y|\le y_1$:

$$h(\exp(-z))\prod_{r\in\mathcal{D}}U_r(z)=\exp\left(\frac{\pi^2}{12z}\left(1+\frac{|\mathcal{D}|}{d}\right)+\frac{|\mathcal{D}|}{2}\log z-\frac{\log 2}{2}\right.$$
$$\left.+\sum_{r\in\mathcal{D}}\left(\frac{r}{d}-1\right)\log 2+\frac{|\mathcal{D}|}{2}\log\left(\frac{d}{\pi}\right)+O(n^{-\varepsilon})\right).$$

As in the case $\mathcal{D}=\{1,\dots,d\}$ we insert above the two formulae:

$$\frac{1}{z}=\frac{1}{x_0}-\frac{iy}{x_0^2}-\frac{y^2}{x_0^3}+O\left(\frac{|y|^3}{x_0^4}\right)\quad\text{and}\quad\log z=\log x_0+O\left(\frac{|y|}{x_0}\right),$$

and we apply Lemma 4.1:

$$S_0(0)=\frac{1}{2\pi}\int_{-y_1}^{y_1}\exp(C_{\mathcal{D}}+iy\tilde{B}_{\mathcal{D}}-y^2A_{\mathcal{D}})\mathrm{d}y,$$

with

$$\begin{aligned}C_{\mathcal{D}}=&\frac{\pi^2}{12x_0}\left(1+\frac{|\mathcal{D}|}{d}\right)+\frac{|\mathcal{D}|}{2}\log x_0-\frac{\log 2}{2}+\sum_{r\in\mathcal{D}}\left(\frac{r}{d}-1\right)\log 2+\frac{|\mathcal{D}|}{2}\log\left(\frac{d}{\pi}\right)\\&-\frac{C_2|\mathcal{D}|}{dx_0}+\frac{|\mathcal{D}|\log 2}{2}+\sum_{r\in\mathcal{D}}(N_r-k_0)\log 2-\frac{dx_0}{2}\sum_{r\in\mathcal{D}}(N_r-k_0)^2\\&+(n-R_{\mathcal{D}}-Q_{\mathcal{D}})x_0+O(n^{-\varepsilon}),\end{aligned}\tag{10.1}$$

$$\begin{aligned}\tilde{B}_{\mathcal{D}}=&-\frac{\pi^2}{12x_0^2}\left(1+\frac{|\mathcal{D}|}{d}\right)+\frac{C_2|\mathcal{D}|}{dx_0^2}+\frac{|\mathcal{D}|k_0\log 2}{x_0}-dk_0\sum_{r\in\mathcal{D}}(N_r-k_0)\\&+n-R_{\mathcal{D}}-Q_{\mathcal{D}},\end{aligned}\tag{10.2}$$

$$A_{\mathcal{D}}=\frac{\pi^2}{12x_0^3}\left(1+\frac{|\mathcal{D}|}{d}\right)-\frac{C_2|\mathcal{D}|}{dx_0^3}-\frac{|\mathcal{D}|k_0\log 2}{x_0^2}-\frac{d|\mathcal{D}|k_0^2}{2x_0}.\tag{10.3}$$

Since $k_0dx_0=\log 2$, and $C_2=\frac{\pi^2}{12}-\frac{(\log 2)^2}{2}$, $A_{\mathcal{D}}$, $\tilde{B}_{\mathcal{D}}$ and $C_{\mathcal{D}}$ may be simplified:

$$A_{\mathcal{D}}=\frac{\pi^2}{12x_0^3}-\frac{|\mathcal{D}|(\log 2)^2}{dx_0^3}=\frac{2\sqrt{3}n^{\frac{3}{2}}}{\pi}\left(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\right).$$

Recall that $k_0=\frac{2\sqrt{3n}\log 2}{\pi d}$ and $|N_r-k_0|\le\frac{n^{\frac{1}{4}}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)}$.

By Lemma 2.2 of [6], for $|y|\le y_1$, we have

$$y\tilde{B}_{\mathcal{D}}=-2ydk_0\sum_{r\in\mathcal{D}}(N_r-k_0)+O\big(n^{-\varepsilon/3}\big)=yB_{\mathcal{D}}+O\big(n^{-\varepsilon/3}\big),$$

say.

We end the computations as in the case $\mathcal{D}=\{1,\dots,d\}$. Finally we obtain

$$\Pi_d^*(n,\mathcal{R}_{\mathcal{D}})=(1+o(1))\frac{\delta}{2\pi}\sqrt{\frac{\pi}{A_{\mathcal{D}}}}\exp\left(C_{\mathcal{D}}-\frac{B_{\mathcal{D}}^2}{4A_{\mathcal{D}}}\right).\tag{10.4}$$

To simplify $C_{\mathcal{D}}$ we need some more precise estimations of $R_{\mathcal{D}}$ and $Q_{\mathcal{D}}$:

$$
\begin{aligned}
x_0 R_{\mathcal{D}} &= x_0 k_0 \sum_{r\in\mathcal{D}} r + O(d^{2/3}|\mathcal{D}|^{1/3} n^{-1/4+\varepsilon}),\\
Q_{\mathcal{D}} x_0 &= \frac{dx_0}{2}\sum_{r\in\mathcal{D}}(N_r-k_0)^2 + k_0 dx_0 \sum_{r\in\mathcal{D}}(N_r-k_0)\\
&\quad + \frac{x_0 dk_0^2|\mathcal{D}|}{2} - \frac{d|\mathcal{D}|k_0x_0}{2} + O(dn^{-1/4+\varepsilon}).
\end{aligned}
$$

It remains to insert these different formulae in (10.4) to finish the proof of Theorem 1.1.

11. Local stability of $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}})$

In this section we settle a result analogous to Corollary 9.1 of [5]. If $\mathcal{R}_{\mathcal{D}} = \{N_r : r \in \mathcal{D}\}$ and $\mathcal{R}^*_{\mathcal{D}} = \{N^*_r : r \in \mathcal{D}\}$ are two sets and such that N^*_r is near N_r on average then in the estimation of $\Pi^*_d(n, \mathcal{R}_D)$ given by Theorem 1.1, we may replace the N_r by N^*_r. Like in [5] this corollary will be useful for the proofs of the different corollaries announced in the introduction of [6].

Corollary 11.1. *Let $0 < \varepsilon < 10^{-2}$, $n \ge n_0$, $d^3|\mathcal{D}| \le n^{1/2-3\varepsilon}$ and two sets $\mathcal{R}_{\mathcal{D}} = \{N_r : r \in \mathcal{D}\} \in \mathbb{Z}^{|\mathcal{D}|}$, $\mathcal{R}^*_{\mathcal{D}} = \{N^*_r : r \in \mathcal{D}\} \in \mathbb{R}^{|\mathcal{D}|}$ such that*

(i) (1.1) *is satisfied for $\mathcal{R}_{\mathcal{D}}$;*

(ii) $|N_r - k_0| \le \frac{n^{1/4}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)}$ *for all $r \in \mathcal{D}$ where $w(n)$ is a non-decreasing function such that $\lim_{u\to\infty} w(u) = \infty$;*

(iii) $\sum_{r\in\mathcal{D}} |N_r - N^*_r| \le \delta + |\mathcal{D}| - 1$, $\sum_{r\in\mathcal{D}} |N_r - N^*_r|^2 \le \delta^2 + |\mathcal{D}| - 1$.

Then we have

$$
\Pi^*_d(n, \mathcal{R}_D) = q(n)\frac{\delta(1+o(1))}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\left(\frac{d}{2\sqrt{3n}}\right)^{|\mathcal{D}|/2}
$$
$$
\times \exp\left(-\frac{2\sqrt{3}(\log 2)^2}{\pi\big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\big)\sqrt{n}}\Big(\sum_{r\in\mathcal{D}}(N^*_r-k_0)\Big)^2 - \frac{\pi d}{2\sqrt{3n}}\sum_{r\in\mathcal{D}}(N^*_r-k_0)^2\right).
$$

Proof. By (iii), $\sum_{r\in\mathcal{D}} |N_r - N^*_r| \le 2d$ and we have

$$
\left|\Big(\sum_{r\in\mathcal{D}}(N_r-k_0)\Big)^2 - \Big(\sum_{r\in\mathcal{D}}(N^*_r-k_0)\Big)^2\right|
$$

$$\leq \Big(\sum_{r\in\mathcal{D}}|N_r-N_r^*|\Big)\Big(2\sum_{r\in\mathcal{D}}|N_r-k_0|+\sum_{r\in\mathcal{D}}|N_r-N_r^*|\Big)$$
$$=O\bigg(d^{2/3}|\mathcal{D}|^{1/3}\frac{n^{1/4}\sqrt{\log n}}{w(n)}+d^2\bigg)=o(n^{1/2}).$$

Similarly we have using also $\delta\leq|\mathcal{D}|+1$ (since $\delta\leq\min_{a\in\mathcal{D}^c}a$ if $\mathcal{D}^c\neq\emptyset$)

$$\bigg|\sum_{r\in\mathcal{D}}(N_r-k_0)^2-\sum_{r\in\mathcal{D}}(N_r^*-k_0)^2\bigg|\leq\sum_{r\in\mathcal{D}}|N_r-N_r^*|(2|N_r-k_0|+|N_r-N_r^*|)$$
$$\ll(\delta+|\mathcal{D}|-1)|\mathcal{D}|^{-2/3}d^{-1/3}\frac{n^{1/4}\sqrt{\log n}}{w(n)}+\delta^2+|\mathcal{D}|$$
$$\ll|\mathcal{D}|^{1/3}d^{-1/3}\frac{n^{1/4}\sqrt{\log n}}{w(n)}+\delta^2+|\mathcal{D}|=o\bigg(\frac{\sqrt{n}}{d}\bigg).\qquad\square$$

This ends the proof of Corollary 11.1.

12. On the normal order of the numbers of parts: proof of Corollary 1.2 of [6]

Let $C_{\mathcal{D}}=\big\lceil\frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2}-\frac{n^{1/4}\sqrt{\log n}}{d^{4/3}|\mathcal{D}|^{2/3}w(n)}\big\rceil d$ and $D_{\mathcal{D}}=\big\lfloor\frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2}+\frac{n^{1/4}\sqrt{\log n}}{d^{4/3}|\mathcal{D}|^{2/3}w(n)}\big\rfloor d$.

To prove Corollary 1.2 of [6], it is sufficient to show that

$$S^*:=\sum_{\substack{N_r\in[C_{\mathcal{D}},D_{\mathcal{D}}[\\ r\in\mathcal{D}\\ \mathcal{R}_{\mathcal{D}}\equiv n \bmod \delta}}\Pi_d^*(n,\mathcal{R}_{\mathcal{D}})=q(n)(1+o(1)).\tag{12.1}$$

As in [5] p. 82, we have to remove the dependence between n and the N_r given by the congruence condition modulo δ. If $1\notin\mathcal{D}$ then $\delta=1$, there are no difficulty and we take $N_r^*=N_r$ for all $r\in\mathcal{D}$. If $1\in\mathcal{D}$ we write $N_1^*=\lfloor\frac{N_1}{\delta}\rfloor\delta$ and $N_r^*=N_r$ for $r\in\mathcal{D}\setminus\{1\}$. Now we will suppose that $1\in\mathcal{D}$, the proof of the other case being similar. Next we apply Corollary 11.1 with these two sets:

$$S^*=(1+o(1))q(n)\frac{\delta}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\bigg(\frac{d}{2\sqrt{3n}}\bigg)^{|\mathcal{D}|/2}$$
$$\times\sum_{\substack{C_{\mathcal{D}}\leq N_1\delta<D_{\mathcal{D}}\\ N_r\in[C_{\mathcal{D}},D_{\mathcal{D}}[\\ r\in\mathcal{D}\setminus\{1\}}}\exp\bigg(-\frac{2\sqrt{3}(\log 2)^2}{\pi\big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\big)\sqrt{n}}\Big(\delta N_1-k_0+\sum_{r\in\mathcal{D}\setminus\{1\}}(N_r-k_0)\Big)^2$$

$$-\frac{\pi d}{2\sqrt{3n}}\Big((\delta N_1-k_0)^2+\sum_{r\in\mathcal{D}\setminus\{1\}}(N_r-k_0)^2\Big)\Big).$$

We apply again Corollary 11.1 with $N_i^*=t_i$ so that $|t_i-N_i|\le 1$ for $i\in\mathcal{D}$, we easily see that

$$|\delta t_1'-\delta N_1|^2+\sum_{r\in\mathcal{D}\setminus\{1\}}(t_r-N_r)^2\le\delta^2+|\mathcal{D}|-1.$$

Thus we have (after replacing $\delta t_1'$ by t_1 in the integral):

$$\begin{aligned}S^*&=(1+o(1))q(n)\frac{1}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\\&\times\Big(\frac{d}{2\sqrt{3n}}\Big)^{|\mathcal{D}|/2}\int_{C_\mathcal{D}}^{D_\mathcal{D}}\cdots\int_{C_\mathcal{D}}^{D_\mathcal{D}}\exp\Big(-\frac{2\sqrt{3}(\log 2)^2}{\pi\big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\big)\sqrt{n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)\Big)^2\\&-\frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)^2\Big)\Big)\prod_{r\in\mathcal{D}}\mathrm{d}t_r.\end{aligned}$$

Lemma 12.1. *We have*

$$\begin{aligned}S^*&=(1+o(1))q(n)\frac{1}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\\&\times\Big(\frac{d}{2\sqrt{3n}}\Big)^{|\mathcal{D}|/2}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\Big(-\frac{2\sqrt{3}(\log 2)^2}{\pi\big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\big)\sqrt{n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)\Big)^2\\&-\frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)^2\Big)\Big)\prod_{r\in\mathcal{D}}\mathrm{d}t_r\\&+O\Big(q(n)\frac{w(n)|\mathcal{D}|^{5/3}}{d^{1/6}\sqrt{\log n}}\exp\Big(-\frac{\pi d^{1/3}\log n}{2\sqrt{3}|\mathcal{D}|^{4/3}w^2(n)}\Big)\Big).\end{aligned}$$

Proof. We have to consider the contribution of terms of type $\int_{D_\mathcal{D}}^{\infty}\int_{C_\mathcal{D}}^{D_\mathcal{D}}\cdots\int_{C_\mathcal{D}}^{D_\mathcal{D}}\ldots$. Clearly it is less than

$$\int_{D_\mathcal{D}}^{\infty}\int_{C_\mathcal{D}}^{D_\mathcal{D}}\cdots\int_{C_\mathcal{D}}^{D_\mathcal{D}}\exp\Big(-\frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)^2\Big)\Big)\prod_{r\in\mathcal{D}}\mathrm{d}t_r.$$

But

$$\int_{C_\mathcal{D}}^{D_\mathcal{D}}\exp\Big(-\frac{\pi d}{2\sqrt{3n}}(t_r-k_0)^2\Big)\mathrm{d}t_r\le\int_{-\infty}^{+\infty}\exp\Big(-\frac{\pi dt_r^2}{2\sqrt{3n}}\Big)\mathrm{d}t_r=\sqrt{\frac{2\sqrt{3n}}{d}},$$

and

$$\int_{D_{\mathcal{D}}}^{\infty} \exp\Big(-\frac{\pi d}{2\sqrt{3n}}(t_r-k_0)^2\Big)\mathrm{d}t_r \leq \int_{D_{\mathcal{D}}}^{\infty} \exp\Big(-\frac{\pi d}{2\sqrt{3n}}(D_{\mathcal{D}}-k_0)(t_r-k_0)\Big)\mathrm{d}t_r$$
$$= \frac{2\sqrt{3n}}{\pi d(D_{\mathcal{D}}-k_0)}\exp\Big(-\frac{\pi d}{2\sqrt{3n}}(D_{\mathcal{D}}-k_0)^2\Big).$$

We have $d^{4/3}|\mathcal{D}|^{2/3} = d^{8/6}|\mathcal{D}|^{1/2}|\mathcal{D}|^{1/6} \leq d^{9/6}|\mathcal{D}|^{1/2} = (d^3|\mathcal{D}|)^{1/2} \leq n^{1/4-\varepsilon}$. Thus the contribution of some term of type $\int_{D_{\mathcal{D}}}^{\infty}\int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}}\dots\int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}}\dots$ to S^* is

$$O\bigg(q(n)\frac{w(n)|\mathcal{D}|^{2/3}}{d^{1/6}\sqrt{\log n}}\exp\Big(-\frac{\pi d^{1/3}\log n}{2\sqrt{3}|\mathcal{D}|^{4/3}w^2(n)}\Big)\bigg).$$

Since there are $2|\mathcal{D}|$ such error terms we obtain Lemma 12.1. It remains to compute the main term to finish the proof of Corollary 1.2 of [6]. We remark that

$$\exp\bigg(-\frac{2\sqrt{3}(\log 2)^2}{\pi\big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\big)\sqrt{n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)\Big)^2-\frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)^2\Big)\bigg)$$
$$= \exp(-\frac{1}{2}T^tMT), \tag{12.2}$$

with $T = \begin{pmatrix} t_1-k_0 \\ t_2-k_0 \\ \vdots \\ t_{\mathcal{D}}-k_0 \end{pmatrix} \in \mathbb{R}^{|\mathcal{D}|}$ and M is the symmetric matrix $M=(m_{ij})_{1\leq i,j\leq|\mathcal{D}|}$ defined by

$$\frac{m_{ii}}{2} = \frac{2\sqrt{3}(\log 2)^2}{\pi\big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\big)\sqrt{n}}+\frac{\pi d}{2\sqrt{3n}} =: V+U \qquad (1\leq i\leq|\mathcal{D}|)$$
$$\frac{m_{ij}}{2} = \frac{m_{ji}}{2} = \frac{2\sqrt{3}(\log 2)^2}{\pi\big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\big)\sqrt{n}} =: V \qquad (1\leq i<j\leq|\mathcal{D}|).$$

A classical result on determinant announces that

$$\det M = 2^{|\mathcal{D}|}U^{|\mathcal{D}|-1}(U+|\mathcal{D}|V) = \Big(\frac{\pi d}{\sqrt{3n}}\Big)^{|\mathcal{D}|}\Big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\Big)^{-1}.$$

Thus the function (12.2) is proportional to the density of the law of a Gaussian vector with covariance matrix M^{-1}. We deduce that

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\bigg(-\frac{2\sqrt{3}(\log 2)^2}{\pi\big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\big)\sqrt{n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)\Big)^2$$
$$-\frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)^2\Big)\bigg)\prod_{r\in\mathcal{D}}d_{t_r} = (2\pi)^{|\mathcal{D}|/2}\sqrt{\det M^{-1}}.$$

This ends the proof of Corollary 1.2 of [6]. □

13. Unequal partitions with equilibrated residue classes: proof of Corollary 1.3 of [6]

For $1 \le a < b \le d$, let $E^*(a,b)$ denote the number of unequal partitions of n such that $N_a = N_b$:

$$E^*(a,b) = \sum_{\substack{N_a=N_b \\ n\equiv aN_a+bN_b \pmod{\delta}}} \Pi_d^*(n, \mathcal{R}_{\{a,b\}}) = \sum_{\substack{N_a=N_b \\ N_a\in[C,D] \\ n\equiv aN_a+bN_b \pmod{\delta}}} \Pi_d^*(n, \mathcal{R}_{\{a,b\}}) + o(q(n)),$$

by Corollary 1.2 of [6] applied with $w(n) = 2^{-2/3}\log\log n$,

$$C = \left\lceil \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} - \frac{n^{1/4}\sqrt{\log n}}{d^{4/3}\log\log n} \right\rceil d \ \text{ and } \ D = \left\lfloor \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} + \frac{n^{1/4}\sqrt{\log n}}{d^{4/3}\log\log n} \right\rfloor d.$$

Next we apply Theorem 1.1 and Corollary 11.1. Here again we have to remove the condition $n \equiv aN_a + bN_b \pmod{\delta}$ otherwise N_a, N_b, n wouldn't be independent. If $d \ge 5$ then $\delta = 1$. For $2 \le d \le 4$, the problematic cases are $(a,b,d) \in \{(1,2,2),\ (1,2,3),\ (1,3,4)\}$. We will handle these cases later. Suppose now that $\delta = 1$.

$$\begin{aligned} E^*(a,b) &= q(n)\frac{1}{\sqrt{1-\frac{24(\log 2)^2}{d\pi^2}}}\left(\frac{d}{2\sqrt{3n}}\right) \\ &\quad\times \int_C^D \exp\left[-\left(\frac{8\sqrt{3}(\log 2)^2}{\pi\left(1-\frac{24(\log 2)^2}{d\pi^2}\right)\sqrt{n}} + \frac{\pi d}{\sqrt{3n}}\right)(t-k_0)^2\right]\mathrm{d}t + o(q(n)) \\ &= \sqrt{\frac{d}{4\sqrt{3n}}}\,q(n) + o(q(n)). \end{aligned}$$

As in the proof of Corollary 1.2 of [6] we show that the contributions of $\int_D^\infty \dots$ and $\int_{-\infty}^C$ are small enough. When $\delta > 1$, as explained in [5] we fixe some congruence conditions on N_a and N_b and next do quite the same computations.

14. Comparison between the number of parts in two residue classes: proof of Corollary 1.4 of [6]

We proceed in the same way as in the previous section. Le $\Delta \in \{0,1\}$. We have to estimate

$$T^*(a,b) = \sum_{\substack{N_a \ge N_b+\Delta \\ n\equiv R_{\{a,b\}} \pmod{\delta}}} \Pi_d^*(n, \mathcal{R}_{\{a,b\}}).$$

We take $\Delta = 0$ to examine the unequal partitions with $N_a \geq N_b$ and $\Delta = 1$ for the unequal partitions with $N_a > N_b$. In the same way as in the proof of Corollary 1.3 of [6], we obtain

$$T^*(a,b) = q(n)\frac{1}{\sqrt{1-\frac{24(\log 2)^2}{d\pi^2}}}\left(\frac{d}{2\sqrt{3n}}\right)$$
$$\times \int_{-\infty}^{\infty}\int_{t_b}^{\infty} \exp\left[-\frac{2\sqrt{3}(\log 2)^2}{\pi\left(1-\frac{24(\log 2)^2}{d\pi^2}\right)\sqrt{n}}(t_a+t_b-2k_0)^2\right.$$
$$\left. -\frac{\pi d}{2\sqrt{3n}}((t_a-k_0)^2+(t_b-k_0)^2\right]\mathrm{d}t_a\mathrm{d}t_b + o(q(n)).$$

In the same way we find a similar formula for $T^*(b,a)$. This proves that $T^*(a,b) = q(n)/2 + o(q(n))$.

15. On the d regularity of the unequal partitions: proof of Corollary 1.5 of [6]

Now we suppose that d is fixed in order to use Corollary 1.3 of [6] uniformly. We now study for $\Delta = 0$ or 1 :

$$W^*(a) := \sum_{\substack{N_1,\ldots,N_d \\ n\equiv R \pmod d \\ N_a\geq \Delta+\max_{b\neq a} N_b}} \Pi_d^*(n,\mathcal{R}).$$

We proceed as [5] Sections 12, 13 and like the previous paragraphs of this present paper:

$$W^*(a) = o(q(n)) + \frac{d}{\sqrt{1-\frac{12(\log 2)^2}{\pi^2}}}\left(\frac{d}{2\sqrt{3n}}\right)^{d/2}\int\cdots\int_{t_a\geq \max_{b\neq a} t_b} f(t_1,\ldots,t_d)\mathrm{d}t_1\ldots\mathrm{d}t_d,$$

with

$$f(t_1,\ldots,t_d) = \exp\left\{-\frac{2\sqrt{3}\log^2 2}{\pi\left(1-\frac{12(\log 2)^2}{\pi^2}\right)\sqrt{n}}\left(\sum_{r=1}^{d}(t_r-k_0)\right)^2 - \frac{\pi d}{2\sqrt{3n}}\sum_{r=1}^{d}(t_r-k_0)^2\right\}.$$

By Corollary 1.3 of [6] applied d times (it is why d is fixed), we have

$$\sum_{a=1}^{d} W^*(a) = q(n) + o(q(n)).$$

Since $f(t_1, \dots, t_d)$ is symmetrical the above terms are asymptotically equal:

$$W^*(a) = \left(\frac{1}{d} + o(1)\right) q(n),$$

as it was conjectured in [4] p. 334 and in the introduction of [3].

The proof of the second assertion of Corollary 1.5 of [6] is similar. We have only to replace the condition $N_a \geq \max_{b \neq a} N_b$ by $N_{\sigma(1)} \geq N_{\sigma(2)} \geq \dots \geq N_{\sigma(d)}$.

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