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On the sign-changing solutions for strong singular one-dimensional p-Laplacian problems with p-superlinearity

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Abstract. We consider the one-dimensional p-Laplacian problem

$$\begin{cases} (\varphi_p(u'(t)))' + h(t)f(u(t)) = 0, \quad \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{P}$$

where $\varphi_p(s) = |s|^{p-2}s,\ p > 1, h(t) \geq 0$ and $0 < \int_I h(t)dt < \infty$ for any compact subinterval $I \subset (0,1)$, and $f \in C(\mathbb{R},\mathbb{R})$ with f p-superlinear at ∞. By applying the global bifurcation argument and nonlinear eigenvalue theory, we establish an existence and multiplicity result of sign-changing solutions for (P). Our result generalizes and improves some recent result from the case $h \in L^1(0,1)$ to a strong singular case $h \in \mathcal{A} \triangleq \{h \in L^1_{loc}(0,1) : \int_0^1 (s(1-s))^{p-1}h(s)ds < \infty\}$.

1. Introduction

In this paper, we present an existence and multiplicity result of sign-changing solutions for the singular boundary value problem

$$\begin{cases} (\varphi_p(u'(t)))' + h(t)f(u(t)) = 0, \quad \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{P}$$

where $\varphi_p(s) = |s|^{p-2}s,\ p > 1, h(t) \geq 0$ and $0 < \int_I h(t)dt < \infty$ for any compact subinterval $I \subset (0,1)$, and $f \in C(\mathbb{R},\mathbb{R})$. We assume that the basic conditions

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on h and f given here are satisfied in this paper without any specific mention.

Recently, many attentions are focused on the study of the existence, nonexistence and multiplicity of positive solutions as well as sign-changing solutions for one-dimensional p-Laplacian problems with Dirichlet boundary condition (see e.g. [1], [4], [5], [9], [8], [12], [11], [20], [22], [23] and references therein).

Since our main concern is the sign-changing solutions for the problem (P) with a nonnegative indefinite weight h in this paper, let us summarize the relative results along this line in the literature briefly. For the continuous weight case $h \in C^1[0,1]$, Naito and Tanaka [17] established the existence of sign-changing solutions to (P) for the case $p = 2$ by employing the shooting method and Sturm's comparison theorem. Then in [18], using similar arguments based on the shooting method together with the qualitative theory for half-linear differential equations, they extended their results to (P). When $h \in C^1([0,1],[0,\infty))$, Ma and Thompson [13] and Ma [14] showed the existence and multiplicity results of sign-changing solutions of (P) for the case $p = 2$ by using the global continuation techniques.

For the singular weight case $h \in L^1(0,1)$, Lee and Sim [11] gave an existence and multiplicity result of sign-changing solutions for (P) under assumptions $f_0 \triangleq \lim_{u\to 0} f(u)/u^{p-1} = 0$, $f_\infty \triangleq \lim_{u\to\infty} f(u)/u^{p-1} = \infty$ and

(F) $sf(s) > 0$ for $s \neq 0$.

Moreover, similar result was presented for the case $f_0 = \infty$, $f_\infty = 0$ with additional assumptions that $h \in C^1(0,1) \cap L^1(0,1)$, and $\lim_{t\to 0+} th(t)$ and $\lim_{t\to 1-}(1-t)h(t)$ exist. Their proofs are based on the global bifurcation theorem and deriving the shape of the unbounded subcontinua of solutions for the auxiliary problem

$$\begin{cases} (\varphi_p(u'(t)))' + \lambda h(t)\varphi_p(u(t)) + h(t)f(u(t)) = 0, & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{AP_1}$$

where $\lambda \in \mathbb{R}$. When $h \in L^1(0,1)$, $0 < f_0 < \infty$, Lee and Sim [12] proved some existence, uniqueness, nonexistence and multiplicity results of positive solutions as well as sign-changing solutions with respect to given positive parameter λ for the following problem

$$\begin{cases} (\varphi_p(u'(t)))' + \lambda h(t)f(u(t)) = 0, & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{P_λ}$$

For the strong singular weight case $h \in \mathcal{A} \triangleq \{h \in L^1_{loc}(0,1) : \int_0^1 (s(1-s))^{p-1}h(s)ds < \infty\}$, Kajikiya, Lee and Sim [8] gave some existence, uniqueness, nonexistence and multiplicity results of positive solutions as well as sign-changing solutions of (P_λ) for the case $0 < f_0 < \infty$.

We note that, from Theorem 2.1 and Proposition 2.6 in Kajikiya, Lee and Sim [7], the assumption $h \in \mathcal{A}$ should be the weakest one to guarantee the existence of nontrivial solutions in $C^1[0,1]$ because it is necessary for the existence of nontrivial solutions having C^1-regularity at the boundary. One may also refer to [2] for more details of the class $\mathcal{A}$ of indefinite weights which is larger than L^1-weight.

The main purpose of this paper is to relax the condition on the indefinite weight h in [11] from $h \in L^1(0,1)$ to the strong singular case $h \in \mathcal{A}$ without losing the existence and multiplicity result for (P) in the case $f_0 = 0$, $f_\infty = \infty$. Proofs of results in the literature mainly based on the unboundedness of continua $\mathcal{C}_k, k \in \mathbb{N}$ of solutions for (AP_1) (see [11], [12], [8]). However, we consider the special part $\mathcal{C}_k^0 \triangleq \{(\lambda, u) \in \mathcal{C}_k : \lambda \geq 0\}$ instead, and then the unboundedness of the continua is not indispensable. By using the Rabinowitz's bifurcation argument and Picone's type identity, we get a result of alternative of the continua (see Theorem 2.1) and some essential properties of $\mathcal{C}_k^0$ (see Proposition 3.1–3.3). Applying these results, we prove the existence and multiplicity of sign-changing solutions for (P). Here we state our main result in this paper.

Theorem 1.1. *Assume $h \in \mathcal{A}, f_0 = 0, f_\infty = \infty$ and* (F). *Then for each $k \in \mathbb{N}$, Problem (P) has two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near $t = 0$, and u_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near $t = 0$.*

We shall set $C_0[0,1] = \{x \in C[0,1] : x(0) = x(1) = 0\}$ with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ and $C_0^1[0,1] = C^1[0,1] \cap C_0[0,1]$ with norm $\|x\|_1 = \max_{t\in[0,1]} |x'(t)|$. By a solution (λ, u) of (AP_1) we mean a pair $(\lambda, u) \in \mathbb{R} \times C_0^1[0,1]$ with $\varphi_p(u') \in W^{1,1}(0,1)$ satisfying (AP_1).

The paper is organized as follows. In Section 2, we transform the problem (AP_1) into operator equation, then show a bifurcation result of solutions for (AP_1) by employing the global bifurcation theorem. In Section 3, we prove Theorem 1.1 by making use of the properties of $\mathcal{C}_k^0$ and the bifurcation result obtained in the previous section.

2. Bifurcation

In this section, we transform the auxiliary problem (AP_1) into operator equation on Banach space $C_0^1[0,1]$, then present a bifurcation results of solutions for

(AP_1). We assume that $f_0 = 0$ and $h \in \mathcal{A}$ without any further mention in the sequel.

Consider the problem

$$\begin{cases} (\varphi_p(u'(t)))' = g, & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{AP}$$

where $g \in L^1(0,1)$ and p is as in (P). Problem (AP) is equivalently written as

$$u(t) = G_p(g)(t) \triangleq \int_0^t \varphi_p^{-1}\left(a(g) + \int_0^s g(\tau)d\tau\right) ds,$$

where $a : L^1(0,1) \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^1 \varphi_p^{-1}\left(a(g) + \int_0^s g(\tau)d\tau\right) dt = 0.$$

It is known that a is homogeneous and sends bounded sets of $L^1(0,1)$ into bounded sets of $\mathbb{R}$, and $G_p : L^1(0,1) \to C_0^1[0,1]$ is continuous and maps equi-integrable sets of $L^1(0,1)$ into relatively compact sets of $C_0^1[0,1]$ (see [3], [15], [16]). Moreover, it is easy to see that

$$cG_p(u) = G_p(\varphi_p(c)u) \quad \text{for } c \in \mathbb{R} \text{ and } u \in L^1(0,1). \tag{2.1}$$

For $u \in C_0^1[0,1]$ we have

$$\|u(t)\| \le 2\|u\|_1 t(1-t) \quad \text{for } t \in [0,1], \tag{2.2}$$

and then $h\varphi_p(u) \in L^1(0,1)$ since $h \in \mathcal{A}$. By $f_0 = 0$, there exists $M_u > 0$ such that

$$|f(s)| \le M_u|\varphi_p(s)| \quad \text{for } t \in [0,1],\ |s| \le 2\|u\|_1. \tag{2.3}$$

So by (2.2) and (2.3),

$$|f(u(t))| \le M_u(2\|u\|_1)^{p-1}(t(1-t))^{p-1} \quad \text{for } t \in [0,1],$$

which implies that $hf(u) \in L^1(0,1)$. Thus we can define the Nemitskii operator $H : \mathbb{R} \times C_0^1[0,1] \to L^1(0,1)$ by

$$H(\lambda, u)(t) \triangleq -\lambda h(t)\varphi_p(u(t)) - h(t)f(u(t)).$$

Furthermore, it is easy to get from (2.2) and (2.3) that H is continuous and sends bounded sets of $\mathbb{R} \times C_0^1[0,1]$ into equi-integrable sets of $L^1(0,1)$. Let

$$F(\lambda, u) = G_p(H(\lambda, u)).$$

Then $F : \mathbb{R} \times C_0^1[0,1] \to C_0^1[0,1]$ is completely continuous and $F(\lambda, 0) = 0, \forall \lambda \in \mathbb{R}$. Now problem (AP_1) can be equivalently written as

$$u = F(\lambda, u). \tag{AP_1}$$

Next, consider the eigenvalue problem

$$\begin{cases} (\varphi_p(u'(t)))' + \lambda h(t)\varphi_p(u(t)) = 0, \quad \text{a.e.} \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{E_λ}$$

Define the operator $T_\lambda : C_0^1[0,1] \to C_0^1[0,1]$ by

$$T_\lambda(u)(t) = G_p(-\lambda h \varphi_p(u))(t).$$

Then (E_λ) can be rewritten as

$$u = T_\lambda(u). \tag{E_λ}$$

From the argument to get the complete continuity of F, we can see easily that T_λ is completely continuous. The properties of eigenvalues and corresponding eigenfunctions for (E_λ) are as follows.

Lemma 2.1 (Theorem 2.1, [7]). *Assume $h \in \mathcal{A}$. Then there exists a countable set of eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ for (E_λ) which satisfies the following:*

(i) *λ_k is strictly increasing on k and diverges to ∞ as $k \to \infty$.*

(ii) *Its corresponding eigenfunctions u belong to $C_0^1[0,1]$ and $\varphi_p(u') \in W^{1,1}(0,1)$.*

(iii) *Each eigenspace is one-dimensional.*

(iv) *Any eigenfunction corresponding to λ_k has exactly $k-1$ simple zeros in $(0,1)$.*

(v) *If λ is an eigenvalue for (E_λ) with $\lambda \neq \lambda_k$, then corresponding eigenfunctions are not of $C_0^1[0,1]$.*

λ_k in Lemma 2.1 is called the kth eigenvalue of (E_λ), and we note that $\lambda_k > 0$ for each $k \in \mathbb{N}$. Let $B_r(0) = \{u \in C_0^1[0,1] : \|u\| < r\}$ with $r > 0$. For the Leray–Schauder degree of $I - T_\lambda$ we have

Lemma 2.2 (Theorem 3.2, [8]). *Assume $h \in \mathcal{A}$, we have*

$$d_{\mathrm{LS}}(I - T_\lambda, B_r(0), 0) = \begin{cases} 1, & \text{if } 0 \le \lambda < \lambda_1, \\ (-1)^k, & \text{if } \lambda \in (\lambda_k, \lambda_{k+1}). \end{cases}$$

To show the bifurcation phenomenon for (AP_1), we will make use of the following well-known global bifurcation theorem.

Lemma 2.3 ([21]). *Let $F : \mathbb{R} \times E \to E$ with E a Banach space be completely continuous such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Suppose that there exist constants $\rho, \eta \in \mathbb{R}$, with $\rho < \eta$, such that $(\rho, 0)$ and $(\eta, 0)$ are not bifurcation points for the equation*

$$u - F(\lambda, u) = 0. \tag{2.4}$$

Furthermore, assume that

$$d_{\mathrm{LS}}(I - F(\rho, \cdot), B_r(0), 0) \neq d_{\mathrm{LS}}(I - F(\eta, \cdot), B_r(0), 0),$$

where $B_r(0) = \{u \in E : \|u\|_E < r\}$ is an isolating neighborhood of the trivial solution for both constants ρ and η. Let

$$\mathcal{S} = \overline{\{(\lambda, u) : (\lambda, u) \text{ is a solution of } (2.4) \text{ with } u \neq 0\}} \cup ([\rho, \eta] \times 0),$$

and let $\mathcal{C}$ be the component of $\mathcal{S}$ containing $[\rho, \eta] \times 0$. Then either

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times E$, or*

(ii) *$\mathcal{C} \cap [(\mathbb{R} \backslash [\rho, \eta]) \times \{0\}] \neq \emptyset$.*

The following lemma will be useful in the proof of the main result in this section.

Lemma 2.4. *Assume $h \in \mathcal{A}$ and $f_0 = 0$. Then there is no bifurcation point of (AP_1) except for $\{(\lambda_k, 0) : k \in \mathbb{N}\}$.*

Proof. Suppose $(\lambda, 0)$ is a bifurcation point of (AP_1). Then there exists $\{(\gamma_n, u_n)\} \subset \mathbb{R} \times C_0^1[0,1]$ such that (γ_n, u_n) is a solution of (AP_1) with $(\gamma_n, u_n) \to (\lambda, 0)$ in $\mathbb{R} \times C_0^1[0,1]$ and $u_n \not\equiv 0$. Let $v_n \triangleq \frac{u_n}{\|u_n\|_1}$. Then $\|v_n\|_1 = 1$, and by (2.1),

$$v_n = \frac{F(\gamma_n, u_n)}{\|u_n\|_1} = G_p\left(\frac{H(\gamma_n, u_n)}{\|u_n\|_1^{p-1}}\right) = G_p\left(-\gamma_n h \varphi_p(v_n) - h \frac{f(u_n)}{\|u_n\|_1^{p-1}}\right) \tag{2.5}$$

Notice that $\|u_n\|_1 \to 0$ implies $\|u_n\| \to 0$. By $f_0 = 0$, for any $\varepsilon > 0$, there exists $N > 0$ such that $|f(u_n(t))| \le \varepsilon|\varphi_p(u_n(t))|$ for $t \in [0,1]$ and $n > N$, and by (2.2) we have

$$h(t)\frac{|f(u_n(t))|}{\|u_n\|_1^{p-1}} \le h(t)\frac{\varepsilon|\varphi_p(u_n(t))|}{\|u_n\|_1^{p-1}} = \varepsilon h(t)|\varphi_p(v_n(t))| \le \varepsilon h(t)(2t(1-t))^{p-1}.$$

This implies that $hf(u_n)/\|u_n\|_1^{p-1} \to 0$ in $L^1(0,1)$ and $\{\gamma_n h\varphi_p(v_n)+hf(u_n)/\|u_n\|_1^{p-1}\}$ is an equi-integrable set of $L^1(0,1)$. Thus by (2.5), $\{v_n\}$ is relatively compact in $C_0^1[0,1]$ since G_p sends equi-integrable sets of $L^1(0,1)$ into relatively compact sets of $C_0^1[0,1]$. So $\{v_n\}$ has a subsequence (denoted again by $\{v_n\}$) converging to some $v \in C_0^1[0,1]$, and then

$$\gamma_n h\varphi_p(v_n) + hf(u_n)/\|u_n\|_1^{p-1} \to \lambda h\varphi_p(v) \quad \text{in } L^1(0,1).$$

By (2.5) we have,

$$v = G_p(-\lambda h\varphi_p(v)) = T_\lambda(v),$$

which yields that λ is an eigenvalue of (E_λ) with an eigenfunction $v \in C_0^1[0,1]$. Then it follows from Lemma 2.1 that $\lambda \in \{\lambda_k : k \in \mathbb{N}\}$. This completes the proof. □

Now we have the following result of bifurcation.

Theorem 2.1. *Assume $h \in \mathcal{A}$ and $f_0 = 0$. Each $(\lambda_k, 0)$ is a bifurcation point of (AP_1) and the associated bifurcation branch $\mathcal{C}_k$ of solutions of (AP_1) satisfies the alternatives in Lemma 2.3.*

Proof. Let $\rho = \lambda_k - \delta_k$ and $\eta = \lambda_k + \delta_k$ with such a small $\delta_k > 0$ that $\lambda_{k-1} < \rho < \eta < \lambda_{k+1}$ for $k > 1$ and $0 < \rho < \lambda_1 < \eta < \lambda_2$ for $k = 1$. If there exists $r > 0$ such that

$$d_{LS}(I - F(\rho,\cdot), B_r(0), 0) = (-1)^{k-1} \tag{2.6}$$

and

$$d_{LS}(I - F(\eta,\cdot), B_r(0), 0) = (-1)^{k}, \tag{2.7}$$

then we have

$$d_{LS}(I - F(\rho,\cdot), B_r(0), 0) \ne d_{LS}(I - F(\eta,\cdot), B_r(0), 0),$$

and the conclusion is a consequence of Lemma 2.3 and 2.4. So it is enough to prove (2.6) and (2.7).

Here we prove (2.6). The proof of (2.7) is similar and we omit the details. For this purpose, consider the following statement:

(C) There exists $r > 0$ such that the equation $u = J(\tau, u) \triangleq \tau T_\rho(u) + (1 - \tau)F(\rho, u)$ has only trivial solution 0 in $\overline{B_r(0)}$ for all $\tau \in [0, 1]$.

If statement (C) is true, $d_{LS}(I - J(\tau, \cdot), B_r(0), 0)$ is well defined for all $\tau \in [0, 1]$ and by the property of homotopy invariance, we have $d_{LS}(I - J(1, \cdot), B_r(0), 0) = d_{LS}(I - J(0, \cdot), B_r(0), 0)$, that is

$$d_{LS}(I - T_\rho, B_r(0), 0) = d_{LS}(I - F(\rho, \cdot), B_r(0), 0).$$

Meanwhile, it follows from Lemma 2.2 that $d_{LS}(I - T_\rho, B_r(0), 0) = (-1)^{k-1}$ since $\rho \in (\lambda_{k-1}, \lambda_k)$ for $k > 1$ and $0 < \rho < \lambda_1$ for $k = 1$, and then (2.6) holds. So we only need to prove that statement (C) holds.

Suppose on the contrary that there exist sequences $\{u_n\} \subset C_0^1[0, 1]$ and $\{\tau_n\} \subset [0, 1]$ such that $u_n = J(\tau_n, u_n) \not\equiv 0$ and $\|u_n\|_1 \to 0$ as $n \to \infty$. Assume $\tau_n \to \tau_0 \in [0, 1]$ and let $v_n \triangleq \frac{u_n}{\|u_n\|_1}$, then $\|v_n\|_1 = 1$ and by (2.1),

$$\begin{aligned} v_n = \frac{J(\tau_n, u_n)}{\|u_n\|_1} &= \tau_n G_p(-\rho h \varphi_p(v_n)) \\ &\quad + (1 - \tau_n) G_p\bigg(- \rho h \varphi_p(v_n) - h \frac{f(u_n)}{\|u_n\|_1^{p-1}} \bigg). \end{aligned} \tag{2.8}$$

By the same argument of the proof of Lemma 2.4 we can get that $\{v_n\}$ has a subsequence converging to some $v \in C_0^1[0, 1]$ and $hf(u_n)/\|u_n\|_1^{p-1} \to 0$ in $L^1(0, 1)$. Then (2.8) implies that

$$v = G_p(-\rho h \varphi_p(v)) = T_\rho(v),$$

which yields that ρ is an eigenvalue of (E_λ) with an eigenfunction v. This contradicts Lemma 2.1 and the proof is complete. □

3. Proof of Theorem 1.1

For each $k \in \mathbb{N}$, let us denote $N_k^+ = \{u \in C_0^1[0, 1] : u$ has exactly $k - 1$ simple zeros in $(0, 1), u > 0$ near $0\}$, $N_k^- = -N_k^+$ and $N_k = N_k^- \cup N_k^+$. It is clear that N_k is open in $C_0^1[0, 1]$, $N_k^+ \cap N_k^- = \emptyset$, $N_k \cap N_j = \emptyset$ for $k \neq j$ and u has a double zero $t^* \in [0, 1]$ for $u \in \partial N_k$ (i.e., $u(t^*) = 0 = u'(t^*)$). Let $\mathcal{T}_k^+ = \mathbb{R} \times N_k^+$, $\mathcal{T}_k^- = \mathbb{R} \times N_k^-$, $\mathcal{T}_k = \mathbb{R} \times N_k$ and $\mathcal{C}_k^0 \triangleq \{(\lambda, u) \in \mathcal{C}_k : \lambda \geq 0\}$, where $\mathcal{C}_k$ is as in Theorem 2.1.

Notice that (AP_1) becomes (P) if $\lambda = 0$. Then Theorem 1.1 can be gotten immediately from the following theorem.

Theorem 3.1. *Assume* $h \in \mathcal{A}, f_0 = 0, f_\infty = \infty$ *and* (F). *For each* $k \in \mathbb{N}$ *and* $\nu \in \{+, -, \}$, *there exists a continuum* $\mathcal{C}_k^\nu$ *of solutions for* (AP_1) *such that*

$$\mathcal{C}_k^\nu \cap (\{\lambda\} \times N_k^\nu) \neq \emptyset \quad \text{for } \lambda \in [0, \lambda_k). \tag{3.1}$$

So, to prove Theorem 1.1, it is sufficient to prove Theorem 3.1. For this purpose, we need to study the properties of $\mathcal{C}_k^0$. Let us start with the following lemma.

Lemma 3.1. *Assume* $h \in \mathcal{A}$ *and* $f_0 = 0$. *For* $\Lambda > 0$, *there exists* $\delta > 0$ *such that if* (λ, u) *is a solution of* (AP_1) *in* (t_1, t_2) *with* $|\lambda| \leq \Lambda, u(t_1) = u(t_2) = 0$, $0 \leq t_1 \leq t_2 \leq 1$ *and* $t_2 - t_1 < \delta$, *then* $u \equiv 0$ *in* $[t_1, t_2]$.

Proof. Let (λ, u) be as in the lemma. By $f_0 = 0$, we have $|f(u(t))| \leq C_u|u(t)|^{p-1}$ for $t \in [t_1, t_2]$ and some $C_u > 0$. Since $h \in \mathcal{A}$, we may choose $\delta > 0$ sufficiently small that

$$2^{p-1}(\Lambda + C_u) \int_{t_1}^{t_2} (t(1-t))^{p-1} h(t) dt \leq 1/2.$$

Multiplying (AP_1) by u and integrating over (t_1, t_2), then we get

$$\int_{t_1}^{t_2} |u'|^p dt = \int_{t_1}^{t_2} (\lambda h \varphi_p(u) + h f(u)) u dt \leq (|\lambda| + C_u) \int_{t_1}^{t_2} h|u|^p dt.$$

By Lemma 3.1 in [7] we have

$$|u(t)|^p \leq (2t(1-t))^{p-1} \int_{t_1}^{t_2} |u'(s)|^p ds, \quad t \in [t_1, t_2].$$

Combining the three inequalities above, we get

$$\int_{t_1}^{t_2} |u'|^p dt \leq 2^{p-1}(|\lambda| + C_u) \int_{t_1}^{t_2} (t(1-t))^{p-1} h(t) dt \int_{t_1}^{t_2} |u'|^p dt \leq \frac{1}{2} \int_{t_1}^{t_2} |u'|^p dt,$$

which implies that $u \equiv 0$ in $[t_1, t_2]$ and the proof is complete. □

Lemma 3.2. *Assume* $h \in \mathcal{A}$ *and* $f_0 = 0$. *For each* $k \in \mathbb{N}$, *there is a neighborhood* $\mathcal{O}_k$ *of* $(\lambda_k, 0)$ *such that* $u \in N_k$ *if* $(\lambda, u) \in \mathcal{O}_k$ *is a nontrivial solution of* (AP_1).

PROOF. Suppose on the contrary that there is a sequence $\{(\mu_n, u_n)\}$ of nontrivial solutions for (AP_1) such that $u_n \notin N_k$ and $(\mu_n, u_n) \to (\lambda_k, 0)$ in $\mathbb{R}\times C_0^1[0,1]$. Then by the same argument of the proof of Lemma 2.4, we can get a subsequence of $\{u_n/\|u_n\|_1\}$ converging to an eigenfunction $v \in C_0^1[0,1]$ corresponding to eigenvalue λ_k for the problem (E_λ). Thus $v \in N_k$ by Lemma 2.1, and then $u_n/\|u_n\|_1 \in N_k$ for sufficiently large n since N_k is open. This contradicts $u_n \notin N_k$ and the proof is complete. □

Proposition 3.1. *Assume $h \in \mathcal{A}$, $f_0 = 0$ and* (F). *Then $\mathcal{C}_k^0 \subset \mathcal{T}_k \cup \{(\lambda_k, 0)\}$.*

PROOF. Suppose on the contrary that there exists $(\lambda, u) \in \mathcal{C}_k^0 \backslash (\mathcal{T}_k \cup \{(\lambda_k, 0)\})$. Without loss of generality, we may assume that $\lambda < \lambda_k$. It is easy to get from Lemma 3.2 that there exists $(\mu, u_1) \in \partial\mathcal{T}_k \cap \mathcal{C}_k^0$ with $\lambda \le \mu < \lambda_k$. Then $u_1 \in \partial N_k$ and u_1 has a double zero $t^* \in [0,1]$. If $u_1 \equiv 0$, $\mu = \lambda_j$ for some $j \neq k$ by Lemma 2.4. Let $\{(\mu_n, u_n)\} \subset \mathcal{T}_k \cap \mathcal{C}_k^0$ such that $(\mu_n, u_n) \to (\mu, u_1) = (\lambda_j, 0)$ in $\mathbb{R}\times C_0^1[0,1]$. Then $u_n \in N_j$ for sufficiently large n by Lemma 3.2. This contradicts the fact that $N_j \cap N_k = \emptyset$ for $j \neq k$. So $u_1 \not\equiv 0$. By Lemma 3.1, there exists $\delta > 0$ such that $u_1(t) \neq 0$, say, $u_1(t) > 0$ for $t \in (t^*, t^* + \delta)$ (consider $t \in (t^* - \delta, t^*)$ if $t^* = 1$ and we can analyze similarly). By (F) and $\mu \ge 0$ we have $(\varphi_p(u_1'))' < 0$ in $(t^*, t^* + \delta)$, and $u_1'(t) < 0$ for $t \in (t^*, t^* + \delta)$ since $u_1'(t^*) = 0$. Thus $u_1(t) < 0$ for $t \in (t^*, t^* + \delta)$ because $u_1(t^*) = 0$. This is a contradiction and the proof is complete. □

The following lemma is known as the generalized Picone identity [6], [7], [10].

Lemma 3.3. *Let $b_1(t)$ and $b_2(t)$ be measurable functions on an interval I. If $y, z, \varphi_p(y')$ and $\varphi_p(z')$ are differentiable a.e. in I and $z(t) \neq 0$ in I, then the following identity holds*

$$\frac{d}{dt}\left\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\right\} = (b_1 - b_2)|y|^p$$
$$- \left[|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right)\right] - yl_p(y) + \frac{|y|^p}{\varphi_p(z)}L_p(z), \quad (3.2)$$

where $l_p(y) = (\varphi_p(y'))' + b_1(t)\varphi_p(y)$ and $L_p(z) = (\varphi_p(z'))' + b_2(t)\varphi_p(z)$.

We note that, by Young's inequality, we get

$$|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right) \ge 0.$$

Proposition 3.2. *Assume $h \in \mathcal{A}$, $f_0 = 0$ and* (F). *Then $\lambda \le \lambda_k$ for all $(\lambda, u) \in \mathcal{C}_k^0$.*

Proof. Let $(\lambda, u) \in \mathcal{C}_k^0$ and $\phi_k \in N_k$ be an eigenfunction corresponding to the kth eigenvalue λ_k of (E_λ). If $(\lambda, u) = (\lambda_k, 0)$, the proof is done. Otherwise, we have $u \in N_k$ by Proposition 3.1. Let $t_0^*, t_1^*, \ldots, t_k^*$ and $t_0, t_1, \ldots, t_k$ be the zeroes of u and ϕ_k in $[0,1]$, respectively. We note that $t_0^* = t_0 = 0, t_k^* = t_k = 1$. Then it is easy to see that there exists some $i \in \{0, 1, \ldots, k-1\}$ such that $(t_i, t_{i+1}) \subset (t_i^*, t_{i+1}^*)$. We claim that

$$\int_{t_i}^{t_{i+1}} \left\{ \frac{|\phi_k|^p \varphi_p(u')}{\varphi_p(u)} - \phi_k \varphi_p(\phi_k') \right\}' dt = 0. \tag{3.3}$$

In fact, if $t_i^* < t_i < t_{i+1} < t_{i+1}^*$, it is clear that that (3.3) is true. Suppose $t_{i+1}^* = t_{i+1}$. We prove (3.3) only for the case $u(t) > 0$ and $\phi_k(t) > 0$ for $t \in (t_i, t_{i+1})$. The proof for the other cases is similar. Noticing that $u'(t_{i+1}) \neq 0$ and $\phi_k'(t_{i+1}) \neq 0$, if $1 < p \leq 2$, by L'Hospital's rule, we have

$$\begin{aligned}\lim_{t \to t_{i+1}-} \frac{|\phi_k(t)|^p}{\varphi_p(u(t))} &= \lim_{t \to t_{i+1}-} \frac{p(\phi_k(t))^{p-1} {\phi_k}'(t)}{(p-1)(u(t))^{p-2} u'(t)} \\ &= \frac{p {\phi_k}'(t_{i+1})}{(p-1) u'(t_{i+1})} \lim_{t \to t_{i+1}-} \frac{|\phi_k(t)|^{p-1}}{(u(t))^{p-2}} = 0.\end{aligned}$$

If $k < p \leq k+1, k \geq 2$, then applying the L'Hospital's rule k times, we get

$$\lim_{t \to t_{i+1}-} \frac{|\phi_k(t)|^p}{\varphi_p(u(t))} = \frac{p({\phi_k}'(t_{i+1}))^k}{(p-k)(u'(t_{i+1}))^k} \lim_{t \to t_{i+1}-} \frac{|\phi_k(t)|^{p-k}}{(u(t))^{p-k-1}} = 0.$$

So, for all $p > 1$,

$$\lim_{t \to t_{i+1}-} \frac{|\phi_k|^p \varphi_p(u'(t))}{\varphi_p(u(t))} = \varphi_p(u'(t_{i+1})) \lim_{t \to t_{i+1}-} \frac{|\phi_k(t)|^p}{\varphi_p(u(t))} = 0.$$

Similarly, if $t_i = t_i^*$, we can prove that

$$\lim_{t \to t_i+} \frac{|\phi_k|^p \varphi_p(u'(t))}{\varphi_p(u(t))} = 0.$$

Therefore, if $t_i = t_i^*$ or $t_{i+1} = t_{i+1}^*$, we always have

$$\int_{t_i}^{t_{i+1}} \left\{ \frac{|\phi_k|^p \varphi_p(u'(t))}{\varphi_p(u(t))} \right\}' dt = \lim_{t \to t_{i+1}-} \frac{|\phi_k|^p \varphi_p(u'(t))}{\varphi_p(u(t))} - \lim_{t \to t_i+} \frac{|\phi_k|^p \varphi_p(u'(t))}{\varphi_p(u(t))} = 0.$$

This implies that (3.3) also holds if $t_i = t_i^*$ or $t_{i+1} = t_{i+1}^*$.

Meanwhile, either $u > 0$ or $u < 0$ in (t_i, t_{i+1}), by (F) we have

$$\begin{aligned} 0 &= \frac{1}{\varphi_p(u(t))}\left[(\varphi_p(u'(t)))' + \lambda h(t)\varphi_p(u(t)) + h(t)f(u(t))\right] \\ &\geq \frac{1}{\varphi_p(u(t))}\left[(\varphi_p(u'(t)))' + \lambda h(t)\varphi_p(u(t))\right] \end{aligned}$$

and

$$0 = (\varphi_p(\phi_k'(t)))' + \lambda_k h(t)\varphi_p(\phi_k(t))$$

If we take $y = \phi_k, b_1(t) = \lambda_k h(t)$ and $z = u, b_2(t) = \lambda h(t)$ and integrate (3.2) from t_i to t_{i+1}, we obtain

$$\int_{t_i}^{t_{i+1}} (\lambda_k h(t) - \lambda h(t))|u(t)|^p dt \geq 0,$$

which implies that $\lambda \leq \lambda_k$. The proof is complete. □

We note that $\mathcal{C}_k^0 = \{(\lambda, u) \in \mathcal{C}_k : 0 \leq \lambda \leq \lambda_k\}$ by Proposition 3.2. Let $(t_1, t_2) \subset (0, 1)$. Then we have the following Lemma.

Lemma 3.4. *Suppose $h \in \mathcal{A}, f_0 = 0$ and (F). Let $(\lambda, u) \in \mathcal{C}_k^0$ such that $u(t_1) = u(t_2) = 0$ and $|u(t)| \leq M_0$ for $t \in [t_1, t_2]$ and some $M_0 > 0$. Then $|u'(t)| \leq M_1$ for $t \in [t_1, t_2]$ and some $M_1 > 0$. Here M_1 depends only on M_0 but does not on t_1, t_2.*

Proof. By $f_0 = 0$, there exists $C_0 > 0$ such $|f(u)| \leq C_0|u|^{p-1}$ for $u \in [0, M_0]$. Since $h \in \mathcal{A}$, it is easy to see that there exists $\delta \in (0, 1/2)$ such that

$$2^p(\lambda_k + C_0)\int_\alpha^\beta h(s)(s(1-s))^{p-1}ds \leq 1 \tag{3.4}$$

for any interval $[\alpha, \beta] \subset [0, 1]$ with $\beta - \alpha \leq \delta$. Clearly, δ depends only on C_0, and thus only on M_0. Let $(\lambda, u) \in \mathcal{C}_k^0$ be as in the Lemma. Then $|f(u(t))| \leq C_0|u(t)|^{p-1}$ for $t \in [t_1, t_2]$. We may assume that $u(t) > 0$ for $t \in (t_1, t_2)$. Let $u(\bar{t}) = \max_{t\in[t_1,t_2]} u(t)$. We prove that

$$\bar{t} - t_1 > \delta \text{ if } \bar{t} \leq 1/2; \quad t_2 - \bar{t} > \delta \text{ if } \bar{t} \geq 1/2. \tag{3.5}$$

If $\bar{t} \leq 1/2$, suppose on the contrary that $\bar{t} - t_1 \leq \delta$. By (F), u is concave in (t_1, t_2), and then $u(t)/(t - t_1)$ is decreasing in (t_1, t_2). By (3.4) we have, for $t \in [t_1, \bar{t}]$,

$$\varphi_p(u'(t)) = \int_t^{\bar{t}} [\lambda h(s)\varphi_p(u(s)) + h(s)f(u(s))]ds$$

$$
\begin{aligned}
&\le (\lambda + C_0) \int_t^{\bar{t}} h(s)\varphi_p(u(s))ds \\
&= (\lambda + C_0) \int_t^{\bar{t}} h(s) \left(\frac{u(s)}{(s-t_1)(1-s)} \right)^{p-1} ((s-t_1)(1-s))^{p-1} ds \\
&\le \left(\frac{u(s)}{t-t_1} \right)^{p-1} 2^{p-1}(\lambda + C_0) \int_t^{\bar{t}} h(s)(s(1-s))^{p-1} ds \le \frac{1}{2} \left(\frac{u(s)}{t-t_1} \right)^{p-1}.
\end{aligned}
$$

Letting $t \to t_1+$, we get $\varphi_p(u'(t_1)) \le (1/2)(u'(t_1))^{p-1}$. That is $u'(t_1) = 0$, and t_1 is a double zero of u. This contradicts Proposition 3.1 and then $\bar{t} - t_1 > \delta$. By an argument symmetric to that for the case $\bar{t} \le 1/2$, we can also prove that $t_2 - \bar{t} > \delta$ and then (3.5) is true.

Since $h \in \mathcal{A}$, there exists $\eta \in (0, \delta)$ such that

$$
(\lambda_k + C_0) \max \left\{ \int_{\alpha_1}^{\beta_1} h(s)s^{p-1} ds, \int_{\alpha_2}^{\beta_2} h(s)(1-s)^{p-1} ds \right\} \le \frac{1}{2} \tag{3.6}
$$

for all $\alpha_1, \beta_1 \in [0, 1-\delta]$, $\alpha_2, \beta_2 \in [\delta, 1]$ satisfying that $0 \le \beta_i - \alpha_i \le \eta$, $i = 1, 2$. It is clear that η depends only on δ and C_0, and consequently only on M_0.

If $t_1 + \eta \le \bar{t}$, then $0 \le u'(t_1 + \eta) \le u(t_1 + \eta)/\eta \le M_0/\eta$ since u is concave in (t_1, t_2). Let $\tilde{t} = \min\{\bar{t}, t_1 + \eta\}$, then $0 \le u'(\tilde{t}) \le M_0/\eta$. For $t \in (t_1, \tilde{t})$, by (3.5) and the fact that $0 < \eta < \delta < 1/2$ we can get easily that

$$
[t_1, \tilde{t}] \subset [t_1, \bar{t}] \subset [0, 1-\delta].
$$

Noticing that $u(t)/(t - t_1)$ is decreasing in (t_1, t_2), by (3.6)

$$
\begin{aligned}
\varphi_p(u'(t)) &= \varphi_p(u'(\tilde{t})) + \int_t^{\tilde{t}} [\lambda h(s)\varphi_p(u(s)) + h(s)f(u(s))] ds \\
&\le (M_0/\eta)^{p-1} + (\lambda_k + C_0) \int_t^{\tilde{t}} h(s) \left(\frac{u(s)}{s-t_1} \right)^{p-1} (s-t_1)^{p-1} ds \\
&\le (M_0/\eta)^{p-1} + (\lambda_k + C_0) \left(\frac{u(t)}{t-t_1} \right)^{p-1} \int_t^{\tilde{t}} h(s)s^{p-1} ds \\
&\le (M_0/\eta)^{p-1} + \frac{1}{2} \left(\frac{u(t)}{t-t_1} \right)^{p-1}.
\end{aligned}
$$

Letting $t \to t_1+$, we get $\varphi_p(u'(t_1)) \le (M_0/\eta)^{p-1} + (1/2)(u'(t_1))^{p-1}$. This implies that

$$
u'(t_1) \le 2^{1/(p-1)} M_0/\eta.
$$

Similarly, we can also prove that $|u'(t_2)| \le 2^{1/(p-1)} M_0/\eta$. Let $M_1 = 2^{1/(p-1)} M_0/\eta$. Then $|u'(t)| \le M_1$ for $t \in [t_1, t_2]$ since u is concave in $[t_1, t_2]$. This completes the proof. □

Proposition 3.3. *Assume $f_0 = 0, f_\infty = \infty$, $h \in \mathcal{A}$ and* (F). *Then there exists $b_k > 0$ such that $\|u\|_1 \le b_k$ for all $(\lambda, u) \in \mathcal{C}_k^0$.*

Proof. Set

$$C_h = \min \left\{ \begin{array}{r} \displaystyle\int_\alpha^\beta \varphi_p^{-1}\Big(\int_s^\beta h(\tau)d\tau\Big)ds + \int_\alpha^\beta \varphi_p^{-1}\Big(\int_\alpha^s h(\tau)d\tau\Big)ds : \\ \alpha, \beta \in [1/(4k), 1-1/(4k)], \beta - \alpha = 1/4k \end{array} \right\}.$$

Then $0 < C_h < \infty$ since $0 < \int_I h(t)dt < \infty$ for any compact interval $I \subset (0,1)$. We may choose $\eta > 0$ so large that

$$\frac{\varphi_p^{-1}(\eta)C_h}{16k^2} > 1. \tag{3.7}$$

Clearly, η depends only on k. Since $f_\infty = \infty$, there exists $M_0 > 0$ such that

$$f(u) \ge \eta\varphi_p(u) \text{ for } u > \frac{M_0}{16k^2} \quad \text{and} \quad f(u) \le \eta\varphi_p(u) \text{ for } u < -\frac{M_0}{16k^2}. \tag{3.8}$$

Notice that M_0 depends only on k and η, and consequently only on k.

Let $(\lambda, u) \in \mathcal{C}_k^0$ with $u \not\equiv 0$. Then $u \in N_k$ by Proposition 3.1. Denote by $t_0, t_1, \ldots, t_k$ the zeroes of u in $[0,1]$ and $I_i = [t_i, t_{i+1}]$ for $0 \le i \le k-1$. Then there exists some $j \in \{0, 1, \ldots, k-1\}$ such that $t_{j+1} - t_j \ge 1/k$. We may assume that $u(t) > 0$ for $t \in (t_j, t_{j+1})$. Noticing that condition (F) implies that u is concave in I_j, by the same argument of the proof of Lemma 1 in [20], we can get that, for any $0 < \epsilon < 1/(2k)$, $u(t) \ge \epsilon^2\|u\|_{I_j}$ for all $t \in [t_j + \epsilon, t_{j+1} - \epsilon]$, where $\|u\|_{I_j} \triangleq \max_{t \in I_j} |u(t)|$. Choosing $\epsilon = 1/(4k)$, we have

$$u(t) \ge \frac{\|u\|_{I_j}}{16k^2} \quad \text{for } t \in [t_j + 1/(4k), t_{j+1} - 1/(4k)], \tag{3.9}$$

We assert that

$$\|u\|_{I_j} \le M_0. \tag{3.10}$$

In fact, suppose on the contrary that $\|u\|_{I_j} > M_0$. Then by (3.8) and (3.9),

$$f(u(t)) \ge \eta\varphi_p(u(t)) \quad \text{for } t \in [t_j + 1/(4k), t_{j+1} - 1/(4k)]. \tag{3.11}$$

Let $u(\delta) = \max_{t \in I_j} u(t)$, and suppose that $\delta \ge (t_{j+1} + t_j)/2$ (for the case $\delta \le (t_{j+1} + t_j)/2$, we can analyze exactly the same way on $[\delta, t_{j+1}]$ and we omit the details). Then

$$[t_j + 1/(4k), t_j + 1/(2k)] \subset [t_j + 1/(4k), t_{j+1} - 1/(4k)] \cap [t_j, \delta].$$

Hence by (3.7), (3.9) and (3.11) we have

$$\begin{aligned}
\|u\|_{I_j} = u(\delta) &= \int_{t_j}^{\delta} \varphi_p^{-1}\left(\int_s^{\delta} h(\tau)(\lambda\varphi_p(u(t)) + f(u(\tau)))d\tau\right) ds \\
&\geq \int_{t_j+1/(4k)}^{t_j+1/(2k)} \varphi_p^{-1}\left(\int_s^{t_j+1/(2k)} h(\tau)\eta\varphi_p(u(t))d\tau\right) ds \\
&\geq \int_{t_j+1/(4k)}^{t_j+1/(2k)} \varphi_p^{-1}\left(\int_s^{t_j+1/(2k)} h(\tau)\eta\varphi_p\left(\frac{\|u\|_{I_j}}{16k^2}\right) d\tau\right) ds \\
&= \varphi_p^{-1}(\eta)\frac{\|u\|_{I_j}}{16k^2}\int_{t_j+1/(4k)}^{t_j+1/(2k)} \varphi_p^{-1}\left(\int_s^{t_j+1/(2k)} h(\tau)d\tau\right) ds \\
&\geq \frac{\varphi_p^{-1}(\eta)C_h}{16k^2}\|u\|_{I_j} > \|u\|_{I_j}.
\end{aligned}$$

This contradiction shows that (3.10) holds.

By (3.10) and Lemma 3.4, there exists $M_1 > 0$ such that $|u'(t)| \leq M_1$ for all $t \in [t_j, t_{j+1}]$. Here M_1 depends only on M_0 but does not on I_j, and thus only on k. Consider I_{j-1} or I_{j+1}, since u is convex (or concave) on I_{j-1} or I_{j+1} because of (F), we have $|u(t)| \leq M_1$ for $t \in I_{j-1}$ or $t \in I_{j+1}$. Then again by Lemma 3.4, we get $|u'(t)| < M_2$ for $t \in I_{j-1}$ or $t \in I_{j+1}$ and some $M_2 > 0$. Here M_2 depends only on M_1, and thus only on k. In $k-1$ steps, this procedure shows that there exists some constant $b_k > 0$ such that $|u'(t)| \leq b_k$ for $t \in [0,1]$. Here b_k depends only on k. This completes the proof. $\square$

Proof of Theorem 3.1. Fix $k \in \mathbb{N}$. We first prove that

$$\mathcal{C}_k \cap (\{\lambda\} \times N_k) \neq \emptyset \quad \text{for } \lambda \in [0, \lambda_k). \tag{3.12}$$

In fact, if $\mathcal{C}_k \subset \mathcal{T}_k \cup \{(\lambda_k, 0)\}$. Then $\mathcal{C}_k$ is unbounded by Theorem 2.1, and it follows from Propositions 3.1-3.3 that (3.12) is true. If $\mathcal{C}_k \not\subset \mathcal{T}_k \cup \{(\lambda_k, 0)\}$, there exists $(\mu, u) \in \mathcal{C}_k \cap \partial\mathcal{T}_k$ such that $(\mu, u) \neq (\lambda_k, 0)$ and $u \in \partial N_k$. Then $\mu < 0$ by Proposition 3.1. Thus $\mathcal{C}_k \cap (\{\lambda\} \times C_0^1[0,1]) \neq \emptyset$ for $\lambda \in [\mu, \lambda_k]$, and again by Proposition 3.1 we get (3.12).

By (3.12) and Lemma 2.4, the result is true for at least one set, say, there exists a continuum $\mathcal{C}_k^+$ of solutions for (AP_1) such that (3.1) holds for $\nu = +$. To prove that (3.1) holds for $\nu = -$, we employ the idea of the reflection argument as in Rabinowitz [19].

Let $\mathcal{T}_k^0 = (\mathbb{R} \times C_0^1[0,1]) \setminus (\mathcal{T}_k^+ \cup \mathcal{T}_k^-)$ and defined $\tilde{F}_k : \mathbb{R} \times C_0^1[0,1] \to C_0^1[0,1]$ by

$$\tilde{F}_k(u)(t) = \begin{cases} F(\lambda, u), & (\lambda, u) \in \mathcal{T}_k^-, \\ 0, & (\lambda, u) \in \mathcal{T}_k^0, \\ -F(\lambda, -u), & (\lambda, u) \in \mathcal{T}_k^+, \end{cases}$$

where F is as in the operator equation (AP_1). Then it is not hard for us to verify that $\tilde{F}_k$ possesses the same properties as does F such that (3.12) is also valid to system $u = \tilde{F}_k(\lambda, u)$, i.e. there exists a continuum $\tilde{\mathcal{C}}_k$ of solutions for $u = \tilde{F}_k(\lambda, u)$ such that

$$\tilde{\mathcal{C}}_k \cap (\{\lambda\} \times N_k) \neq \emptyset \quad \text{for } \lambda \in [0, \lambda_k). \tag{3.13}$$

Suppose (3.1) does not hold for $\nu = -$. Then for any continuum $\mathcal{C}_k^-$ of solutions for (AP_1), there exists some $\lambda^- \in [0, \lambda_k)$ such that $\mathcal{C}_k^- \cap (\{\lambda^-\} \times N_k^-) = \emptyset$. So we have $\tilde{\mathcal{C}}_k \cap (\{\tilde{\lambda}\} \times N_k^-) = \emptyset$ for some $\tilde{\lambda} \in [0, \lambda_k)$ since $\tilde{F}(\lambda, u) = F(\lambda, u)$ for $(\lambda, u) \in \mathcal{T}_k^-$. The oddness of $\tilde{F}$ implies that $\tilde{\mathcal{C}}_k$ is symmetric, namely, $(\lambda, -u) \in \tilde{\mathcal{C}}_k$ if $(\lambda, u) \in \tilde{\mathcal{C}}_k$. Thus $\tilde{\mathcal{C}}_k \cap (\{\tilde{\lambda}\} \times N_k^+) = \emptyset$. Therefore $\tilde{\mathcal{C}}_k \cap (\{\tilde{\lambda}\} \times N_k) = \emptyset$, this contradicts (3.13), and then (3.1) holds for $\nu = -$. The proof is complete. $\square$

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References

[1] R. P. Agarwal, H. Lü and D. O'Regan, Eigenvalues and the one-dimensional p-Laplacian, *J. Math. Anal. Appl.* **266** (2002), 383–400.

[2] J. Byun and I. Sim, A relation between two classes of indefinite weights in singular one–dimensional p-Laplacian problems, *Math. Inequal. Appl.* **10** (2007), 889–894.

[3] M. García-Huidobro, R. Manásevich and J. R. Ward, A homotopy along p for systems with a vector p-Laplace operator, *Adv. Differential Equations* **8** (2003), 337–356.

[4] X. He and W. Ge, Twin positive solutions for the one-dimensional p-Laplacian boundary value problems, *Nonlinear Anal.* **56** (2004), 975–984.

[5] N. Huy and T. Thanh, On an eigenvalue problem involving the one-dimensional p-Laplacian, *J. Math. Anal. Appl.* **290** (2004), 123–131.

[6] J. Jaroš and T. Kusano, A Picone type identity for second order half-linear differential equations, *Acta Math. Univ. Comenian.* **68** (1999), 117–121.

[7] R. Kajikiya, Y. H. Lee and I. Sim, One-dimensional p-Laplacian with a strong singular indefinite weight, I. Eigenvalue, *J. Differential Equations vol 244* (2008), 1995–2019.

[8] R. Kajikiya, Y. H. Lee and I. Sim, Bifurcation of sign-changing solutions for one-dimensional p-Laplacian with a strong singular weight; p-sublinear at ∞, *Nonlinear Anal.* **71** (2009), 1235–1249.

[9] L. Kong and J. Wang, Multiple positive solutions for the one-dimensional p-Laplacian, *Nonlinear Anal.* **42** (2000), 1327–1333.

[10] T. Kusano, T. Jaroš and N. Yoshida, *Nonlinear Anal.* **40** (2000), 381–395.

[11] Y. H. Lee and I. Sim, Existence results of sign-changing solutions for singular one-dimensional p-Laplacian problems, *Nonlinear Anal.* **68** (2008), 1195–1209.

[12] Y. H. Lee and I. Sim, Global bifurcation phenomena for singular one-dimensional p-Laplacian, *J. Differential Equations* **229** (2006), 229–256.

[13] R. Ma and B. Thompson, Multiplicity results for second-order two-point boundary value problems with superlinear or sublinear nonlinearities, *J. Math. Anal. Appl.* **303** (2005), 726–735.

[14] R. Ma, Nodal solutions second-order two-point boundary value problems with superlinear or sublinear nonlinearities, *J. Math. Anal. Appl.* **66** (2007), 950–961.

[15] R. Manásevich and J. Mawhin, Periodic solutions of nonlinear systems with p-Laplacian-like operators, *J. Differential Equations* **145** (1998), 367–393.

[16] R. Manásevich and J. Mawhin, Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators, *J. Korean Math. Soc.* **37** (2000), 665–685.

[17] Y. Naito and S. Tanaka, On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations, *Nonlinear Anal.* **56** (2004), 919–935.

[18] Y. Naito and S. Tanaka, Sharp conditions for the existence of sign-changing solutions to equations involving the one-dimensional p-Laplacian, *Nonlinear Anal.* **69** (2008), 3070–3083.

[19] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7** (1971), 487–513.

[20] J. Sánchez, Multiple positive solutions of singular eigenvalue type problems involving the one-dimensional p-Laplacian, *J. Math. Anal. Appl.* **292** (2004), 401–414.

[21] K. Schmitt and R. Thompson, Nonlinear Analysis and Differential Equations, An Introduction, Univ. of Utah Lecture Notes, Univ. of Utah Press, Salt Lake City, 2004.

[22] Z. Wang and J. Zhang, Positive solutions for one-dimensional p-Laplacian boundary value problems with dependence on the first order derivatives, *J. Math. Anal. Appl.* **314** (2006), 618–630.

[23] X. Yang, Sturm type problems for singular p-Laplacian boundary value problems, *Appl. Math. Comput.* **136** (2003), 181–193.

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