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Weak convergence of vector measures

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Abstract. We consider a notion of weak convergence for measures taking values in a Banach space. A version of Prokhoroff's Theorem is proved for such measures, and applications are given to the existence of products of measures with values in a Banach algebra and to a Strassen's Theorem for measures taking values in the positive cone of a Banach lattice.

0. Introduction

We treat weak convergence of measures taking values in a Banach space. This notion of weak convergence is due to M. Dekiert [2] and generalises the notion of stochastic convergence used in probability theory. The focal point of the latter is the theorem of Prokhoroff regarding the weak sequential compactness of a uniformly tight family of probabilities [10]. After the exposition of some elementary concepts, our first chapter proves a type of Prokhoroff Theorem for families of measures that are 1) bounded in semivariation, 2) are uniformly tight, and 3) are uniformly weakly compact as a family of operators (i.e. they jointly map the unit hall of $C(X)$ into a weakly compact subset of the Banach spaces.) Such families are shown (Theorem 1.3) to be weakly sequentially compact.

In chapter 2, we present an application of this result to measures taking values in a Banach algebra. We show how, under suitable hypotheses on measures V and W, to construct a product measure $V \otimes W$ (Theorem 2.8). A corresponding analogue of Fubini's Theorem (Theorem 2.10) is also developed.

Chapter 3 considers measures with values in the positive cone of a Banach lattice. Convergence theorems more akin to the classical are possible

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in this setting (Theorem 3.4), and the same may be said of applications. We prove a version of Strassen's Theorem (Theorem 3.7) in this context.

Some of our results appear in the dissertation of the first-named author (in preparation).

1. Weak convergence of vector measures

Let $\mathcal{F}$ be a field of subsets of a set X and let $(B, \|\cdot\|)$ be a Banach space. (All vector spaces considered will be taken to have real scalars.) Then $\mathrm{ca}(\mathcal{F}, B)$ is the set of all countably additive functions $V : \mathcal{F} \to B$, i.e. $V(E_1 \cup E_2 \cup \dots) = V(E_1) + V(E_2) + \dots$ for all disjoint sequences (E_n) drawn from $\mathcal{F}$. The elements of $\mathrm{ca}(\mathcal{F}, B)$ we term *vector measures*; $\mathrm{ca}(\mathcal{F}, B)$ is a vector space. Generally, we follow the conventions of Dunford and Schwartz [5] or Diestel and Uhl [3], to whose treatises we refer the reader.

Let $V : \mathcal{F} \to B$ be a vector measure. The *variation* of V is the set function $|V| : \mathcal{F} \to [0, \infty]$ defined by

$$|V|(E) = \sup_{\pi} \sum_{A \in \pi} \|V(A)\|,$$

where the supremum is taken over all finite partitions π of E into sets $A \in \mathcal{F}$. The set function $|V|$ is a (countably additive) real measure on $\mathcal{F}$: see the *proof* of Proposition 9 in [3; p. 3]. We also write, with abuse of notation, $|V| = |V|(X)$; if $|V| < \infty$, we say that V is of *bounded variation*.

Let B^* be the (continuous) dual of the Banach space B. If $V \in \mathrm{ca}(\mathcal{F}, B)$ and $\varphi \in B^*$, then $\varphi(V) = \varphi \circ V$ is a finite signed measure on $\mathcal{F}$. The *semivariation* of V is the set function $\|V\| : \mathcal{F} \to [0, \infty]$ defined by

$$\|V\|(E) = \sup\bigl\{|\varphi(V)|(E) : \varphi \in B^*,\ \|\varphi\| \le 1\bigr\}.$$

The countable additivity of V implies that $\|V\|$ is actually real-valued: see [3; pp. 6–7 and Corollary 19, p. 9]. We have $\|V(E)\| \le \|V\|(E) \le |V|(E)$. The set function $\|\cdot\|$ is finitely sub-additive. We also write $\|V\| = \|V\|(X)$. The association $V \to \|V\|$ is a norm on $\mathrm{ca}(\mathcal{F}, B)$. A family $\mathcal{V} \subseteq \mathrm{ca}(\mathcal{F}, B)$ is said to be *uniformly bounded* if $\sup\{\|V\| : V \in \mathcal{V}\} < \infty$.

Let $f : X \to \mathbb{R}$ be an $\mathcal{F}$-measurable simple function $f = \sum a_n I_{E_n}$ for $E_n \in \mathcal{F}$. For $V \in \mathrm{ca}(\mathcal{F}, B)$, we define

$$\int f dV = \sum a_n V(E_n).$$

This integral is well-defined and linear on simple functions, and $\| \int f dV\| \le \|f\|_\infty \|V\|$, where $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Now suppose that $f : X \to$

$\mathbb{R}$ is a function that is the uniform limit of a sequence of $\mathcal{F}$-measurable simple functions (f_n). Then $\| \int f_n - f_m dV\| \le \|f_n - f_m\|_\infty \|V\|$, so that $(\int f_n dV)$ is a Cauchy sequence in B. Define

$$\int f dV = \lim \int f_n dV.$$

Again, this integral is well-defined and linear, and $\| \int f dV\| \le \|f\|_\infty \|V\|$.

Now suppose that X is a topological space with Borel σ-field $\mathcal{B}(X)$. A family $\mathcal{V} \subseteq \mathrm{ca}(\mathcal{B}(X), B)$ is *uniformly tight* if, for each $\varepsilon > 0$, there is some compact $K \subseteq X$ with $\|V\|(X - K) < \varepsilon$ for all $V \in \mathcal{V}$. A single measure $V \in \mathrm{ca}(\mathcal{B}(X), B)$ is *tight* if the family $\{V\}$ is uniformly tight. A measure $V \in \mathrm{ca}(\mathcal{B}(X), B)$ is *regular* if for each $E \in \mathcal{B}(X)$ and $\varepsilon > 0$, there is an open set $G \supseteq E$ and a closed set $F \subseteq E$ such that $\|V\|(G - F) < \varepsilon$. We say that V is *weakly regular* if $\varphi(V)$ is regular for each $\varphi \in B^*$. Let $C(X)$ be the space of all continuous $f : X \to \mathbb{R}$. The following result is crucial:

1.1 Theorem (Bartle, Dunford, Schwartz). *Let X be a compact Hausdorff space and let B be a Banach space. If $T : C(X) \to B$ is a weakly compact operator (i.e. one that sends bounded sets to weakly compact sets), then there is a unique regular measure $V \in \mathrm{ca}(\mathcal{B}(X), B)$ such that $T(f) = \int f dV$ for every $f \in C(X)$ and $\|T\| = \|V\|$.*

Conversely, if $V \in \mathrm{ca}(\mathcal{B}(X), B)$ is a weakly regular measure, then the operator $T : C(X) \to B$ defined by $T(f) = \mathrm{int}\ f dV$ is weakly compact.

Indication. For the first paragraph, see [3; p. 159, Corollary 14]; for the second, see [5, VI. 7.3].

For metric spaces, the supposition of regularity is nugatory:

1.2 Lemma. *If X is metrisable, then every $V \in \mathrm{ca}(\mathcal{B}(X), B)$ is regular.*

Indiaction. This is Satz 1.4 on p. 99 of [2].

Suppose that X is a topological space with Borel σ-field $\mathcal{B}(X)$. Let V and (V_n) be measures in $\mathrm{ca}(\mathcal{B}(X), B)$. We say that the sequence (V_n) *converges weakly to* V if for each continuous bounded $f : X \to \mathbb{R}$ and $\varphi \in B^*$, we have

$$\lim \varphi\left(\int f dV_n\right) = \varphi\left(\int f dV\right).$$

A family $\mathcal{V} \subseteq \mathrm{ca}(\mathcal{B}(X), B)$ is *weakly sequentially compact* if every sequence (V_n) drawn from $\mathcal{V}$ has a subsequence converging weakly to some $V \in \mathrm{ca}(\mathcal{B}(X), B)$.

The following theorem proceeds along the lines of a well-known result of Prokhoroff [10]; also see [4; 11.5.4].

1.3 Theorem. *Let X be a compact metrisable space with Borel σ-field $\mathcal{B}(X)$ and suppose that B is a Banach space. Suppose that $\mathcal{V}$ is a uniformly bounded subset of* $\mathrm{ca}(\mathcal{B}(X), B)$ *such that*

$$\Big\{\int f dV : f \in C(X),\ \|f\|_\infty \leq 1,\ V \in \mathcal{V}\Big\} \subseteq A,$$

where A is a weakly compact subset of B. Then $\mathcal{V}$ is weakly sequentially compact.

Proof. Let (V_n) be a sequence in $\mathcal{V}$ and let $D = \{g_1, g_2, \ldots\}$ be a dense sequence in $C(X)$. For each fixed m, we consider the sequence $(\int g_m dV_n)_n$ in B. This sequence lies in a weakly compact subset of B, so there is a subsequence converging weakly to some $b_m \in B$. The index m is arbitrary, so a standard diagonal argument produces a subsequence $(V_{n(k)})$ of (V_n) such that $\int g_m dV_{n(k)} \to b_m$ weakly for each m. If $\|g_m\|_\infty \leq 1$, then $b_m \in A$.

We now set about defining a map $T : C(X) \to B$. For each m, define $T(g_m) = b_m$. Note that

$$\left\|\int g_m dV_n\right\| \leq \|g_m\|_\infty \|V_n\| \leq C\|g_m\|,$$

where C is a uniform bound on the $\|V_n\|$. Then it is relatively easy to check that $\|T(g_m) - T(g_n)\| = \|b_m - b_n\| \leq C\|g_m - g_n\|_\infty$. Thus T is uniformly continuous on D and so extends uniquely to a continuous function on $C(X)$ with $T(f) = \lim T(f_n)$, where (f_n) is a sequence in D converging uniformly to f. It is routine to check that T is linear and that $\|T(f)\| \leq C\|f\|_\infty$. If $\|f\|_\infty \leq 1$, we choose $f_n \to f$ uniformly with $f_n \in D$ and $\|f_n\|_\infty \leq 1$. Each $T(f_n) \in A$, so $T(f) \in A$. This shows that T is a weakly compact operator.

By Theorem 1.1, there is a measure $V \in \mathrm{ca}(\mathcal{B}(X), B)$ such that for each $f \in C(X)$, we have

$$T(f) = \int f(s) dV(s) \quad \text{and} \quad \|V\| = \|T\| \leq C.$$

It remains only to check that $T(f) = w - \lim \int f dV_{n(k)}$. For $f \in D$, this is automatic. Given $f \in C(X)$ and $\varphi \in B^*$ with $\|\varphi\| \leq 1$, we have for

$g \in C(X)$

$$\begin{aligned}
&\left|\varphi\left(\int f dV_{n(k)}\right) - \varphi(T(f))\right| \\
&\leq \left|\varphi\left(\int f dV_{n(k)}\right) - \varphi\left(\int g\, dV_{n(k)}\right)\right| + \left|\varphi\left(\int g\, dV_{n(k)}\right) - \varphi(T(g))\right| \\
&\qquad + |\varphi(T(g)) - \varphi(T(f))|.
\end{aligned}$$

The last term is bounded by $C \cdot \|f - q\|_\infty$. We choose $g \in D$ with $\|f - g\|_\infty$ small; the first two terms then go to zero as $k \to \infty$. □

1.4 Corollary. *Let X be a compact metric space with Borel σ-field $\mathcal{F}$ and suppose that B is a reflexive Banach space. Suppose that $\mathcal{V}$ is a uniformly bounded subset of $\mathrm{ca}(\mathcal{F}, B)$. Then $\mathcal{V}$ is weakly sequentially compact.*

Proof. Apply the theorem, noting that $\|\int f dV\| \leq \|f\|_\infty \|V\| \leq C \cdot \|f\|_\infty$. Thus, the integrals $\int f dV$, $f \in C(X)$, $\|f\|_\infty \leq 1$, $V \in \mathcal{V}$, belong to the ball of radius C in B. In a reflexive Banach space, every ball is weakly compact. □

We now generalise these results to the case where the base space is not compact, but the family of measures is uniformly tight. The thereom now resembles more exactly the fabled Prokhoroff Theorem.

1.5 Theorem. *Let X be a metric space with Borel σ-field $\mathcal{B}(X)$ and let B be a Banach space. Suppose that $\mathcal{V}$ is a uniformly bounded, uniformly tight subset of $\mathrm{ca}(\mathcal{B}(X), B)$ such that for each compact $S \subseteq X$,*

$$\left\{\int_S f dV : f \in C(X),\ \|f\|_\infty \leq 1,\ V \in \mathcal{V}\right\} \subseteq A,$$

where A is a weakly compact subset of B. (The choice of A may depend on S.) Then $\mathcal{V}$ is weakly sequentially compact.

Proof. Let (V_n) be a sequence in $\mathcal{V}$. From uniform tightness, we can choose an increasing sequence (S_r) of compact sets such that $\sup_n \|V_n\| \cdot (X - S_r) < \frac{1}{r}$ for $r \geq 1$. Let $(V_n | S_r)$ be the restriction of V_n to S_r. By Theorem 1.3, we get a subsequence $(V_{n_k^r} | S_r)$ and some $W^r \in \mathrm{ca}(\mathcal{B}(X), B)$ with $\|W^r\|(X - S_r) = 0$ such that, for every $f \in C(S_r)$ and $\varphi \in B^*$,

$$\varphi\left(\int f dW^r\right) = \lim_k \varphi\left(\int f d(V_{n_k^r} | S_r)\right).$$

Since the index r was arbitrary, a standard diagonal argument shows that for the sub-sequence $(V_{n(k)}) = (V_{n_k^k})$ we have

$$\varphi(\mathrm{int} f dW^r) = \lim_k \varphi\left(\int f d(V_{n(k)}|S_r)\right)$$

for each $r \geq 1$, $f \in C(X)$, and $\varphi \in B^*$.

Claim. *For $s \geq r$, we have $\|W^s - W^r\| \leq \frac{1}{r}$.*

Proof of claim. Given $\varphi \in B^\star$ with $\|\varphi\| \leq 1$, we observe that $|\varphi(W^s - W^r)| = \sup\{\int f d\varphi(W^s - W^r) : f \in C(X),\ \|f\|_\infty \leq 1\}$. But for such f, we have

$$\begin{aligned}\left|\int f d\varphi(W^s - W^r)\right| &= \left|\varphi\left(\int f d(W^s - W^r)\right)\right| \\ &= \left|\lim_k \varphi\left(\int f d(V_{n(k)}|S_s) - \int f d(V_{n(k)}|S_r)\right)\right| \\ &= \left|\lim_k \varphi\left(\int f d(V_{n(k)}|S_s - S_r)\right)\right| \leq \limsup_k \|V_{n(k)}\|(S_s - S_r) \\ &\leq \limsup_k \|V_{n(k)}\|(X - S_r) \leq \frac{1}{r}.\end{aligned}$$

The claim is established.

Since $\|W^s(E) - W^r(E)\| \leq \|W^s - W^r\| \leq \frac{1}{r}$, we have that $(W^r(E))$ is a Cauchy sequence for each $E \in \mathcal{B}(X)$. Define $V(E) = \lim_r W^r(E)$. Also, we see that this convergence is uniform in E : for each $E \in \mathcal{F}$, choose $N \geq r$ so that $\|V(E) - W^N(E)\| < \frac{1}{r}$; then

$$\|V(E) - W^r(E)\| \leq \|V(E) - W^N(E)\| + \|W^N(E) - W^r(E)\| < \frac{2}{r}.$$

The set function V is countably additive: given $E_n \downarrow \emptyset$ in $\mathcal{B}(X)$ and $\varepsilon > 0$, choose $r > \frac{2}{\varepsilon}$ and $N \geq r$ so that $\|W^r(E_n)\| < \frac{\varepsilon}{2}$ for all $n \geq N$. Then $\|V(E_n)\| \leq \|V(E_n) - W^r(E_n)\| + \|W^r(E_n)\| < \frac{2}{m} + \frac{\varepsilon}{2} < \varepsilon$ for all $n \geq N$.

If $g : X \to \mathbb{R}$ is a simple function, then it follows that $\int g\, dW^r \to \int g\, dV$ as $r \to \infty$. If $f \in C(X)$, we may choose a simple function g with $\|f - g\|_\infty$ as small as desired. Then

$$\left\|\int f dW^r - \int f dV\right\| \leq \left\|\int f dW^r - g\, dW^r\right\| + \left\|\int g\, dW^r - \int g\, dV\right\|$$

$$+\left\| \int g\, dV - \int f dV \right\|$$

$$\leq \|f-g\|_\infty \|W^r\| + \left\| \int g\, dW^r - \int g\, dV \right\| + \|f-g\|\ \|V\|\,.$$

Since $\|V\| < \infty$, we see that (W^r) is a Cauchy sequence for the norm $\|\cdot\|$; it is uniformly bounded. Given $\varepsilon > 0$, choose g simple with $\|f-g\|_\infty < \frac{\varepsilon}{3} \cdot \sup_r\{\|W^r\|, \|V\|\}$ and N large so that for $r \geq N$, $\| \int g\, dW^r - \int g\, dV\| < \frac{\varepsilon}{3}$. Then for all $r \geq N$, $\| \int f dW^r - \int f dV\| < \varepsilon$, so that $\int f dW^r \to \int f dV$ for all $f \in C(X)$. In particular, (W^r) converges weakly to V.

We now verify that the subsequence $(V_{n(k)})$ converges weakly to V. Given $\varphi \in B^*$ and $f \in C(X)$, we have

$$\left| \varphi\left(\int f dV_{n(k)}\right) - \varphi\left(\int f dV\right) \right| \leq \left| \varphi\left(\int_{S_r} f dV_{n(k)}\right) - \varphi\left(\int f dW^r\right) \right|$$

$$+\left| \varphi\left(\int f dW^r\right) - \varphi\left(\int f dV\right) \right| + \left| \varphi\left(\int_{X-S_r} f dV_{n(k)}\right) \right|$$

$$\leq \left| \varphi\left(\int_{S_r} f dV_{n(k)}\right) - \varphi\left(\int f dW^r\right) \right| + \left| \varphi\left(\int f dW^r\right) - \varphi\left(\int f dV\right) \right|$$

$$+\|\varphi\|\ \|f\|_\infty\ \|V_{n(k)}\|(X - S_r)\,.$$

Given $\varepsilon > 0$, choose r large so that the last two terms are less than $\frac{\varepsilon}{3}$. Then, for all k large, the first term is majorised by $\frac{\varepsilon}{3}$. This shows that

$$\varphi\left(\int f dV\right) = \lim_k \varphi\left(\int f dV_{n(k)}\right). \qquad \square$$

1.6 Corollary. *Let X be a metric space with Borel σ-field $\mathcal{B}(X)$ and let B be a reflexive Banach space. Suppose that $\mathcal{V}$ is a uniformly bounded, uniformly tight subset of* $\mathrm{ca}(\mathcal{F}, B)$. *Then $\mathcal{V}$ is weakly sequentially compact.*

Proof. As in Corollary 1.4, this follows from the inequality

$$\left\| \int f dV \right\| \leq \|f\|\ \|V\|$$

and the weak sequential compactness of balls in B. $\square$

2. Banach algebras; product measure

We now consider measures taking values in a Banch algebra. Let $(X, \mathcal{F})$ be a measurable space and let B be a (real) Banach algebra. Define the *spectrum* of B as

$$\operatorname{spec}(B) = \{\varphi \in B^* : \varphi(ab) = \varphi(a)\varphi(b) \text{ for all } a, b \in B\}.$$

Given $V \in \operatorname{ca}(\mathcal{F}, B)$, we define the *spectral semivariation* of V to be the set function $\|V\|_S$ on $\mathcal{F}$ specified by

$$\|V\|_S(E) = \sup\{|\varphi(V)| : \varphi \in \operatorname{spec}(B)\}.$$

Much as with the variation and semivariation, we write $\|V\|_S = \|V\|_S(X)$ and have $\|V\|_S \leq \|V\| \leq |V|$.

We say that a Banach algebra B has *full spectrum* if for each $a \in B$, there is some $\varphi \in \operatorname{spec}(B)$ such that $\varphi(a) = \|a\|$. This condition implies that B is isomorphic to a subalgebra of $C(\operatorname{spec}(B))$, in particular, that B is commutative.

2.1 Corollary (to Theorem 1.3). *If B is a Banach algebra with full spectrum, then Theorem 1.3 still holds when the collection $\mathcal{V}$ is assumed only to be uniformly bounded with respect to the spectral semivariation $\|\cdot\|_S$.*

Indication. In the proof of Theorem 1.3, uniform bounded is used only to obtain a corresponding uniform bound on

$$\left\| \int g_m dV_n \right\| \leq \|g_m\|_\infty \ \|V_n\|.$$

But, since B has full spectrum, we have

$$\left\| \int g_m dV_n \right\| \leq \|g_m\| \ \|V_n\|_S \leq \|g_m\| \cdot C,$$

where C is a uniform bound on $\|V_n\|_S$. The rest of the proof may be retained intact. $\square$

We are now ready to begin a construction of product measure for measures taking values in Banach algebras. For the rest of this section, we assume that B is a Banach algebra with full spectrum. Let $K = \{0, 1\}^\omega$ be the Cantor space and let $\mathcal{B}(K)$ be its Borel σ-field. For any string $(k_1, \ldots, k_n)$ of 0's and 1's let $A(k_1, \ldots, k_n)$ be the set of all $x \in K$ such that $x(i) = k_i$ for $i = 1, \ldots, n$. Let π_n be the partition of K into all sets of the form $A(k_1, \ldots, k_n)$. For each $A \in \pi_n$, we distinguish a point $x(A) \in A$.

Now let V and W be measures in $\mathrm{ca}(\mathcal{B}(K), B)$. For each n, we define a measures ρ_n on $\mathcal{B}(K\times K)$, the Borel σ-field of $K \times K$, taking values in B :

$$\rho_n = \sum_{E,F\in\pi_n} V(E)W(F)\delta(x(E), x(F)),$$

where $\delta(x(E), x(F))$ is the Dirac point mass at the point $(x(E), x(F)) \in K \times K$. Clearly, these ρ_n are countably additive.

2.2 Lemma. *The sequence (ρ_n) is uniformly bounded for the spectral semivariation.*

PROOF. Fix $n \geq 1$ and $\varphi \in \mathrm{spec}(B)$. Then we have

$$\begin{aligned} |\varphi(\rho_n)| &= \sum_{E,F\in\pi_n} |\varphi(V(E))\varphi(W(F))| \\ &= \sum_{E\in\pi_n} |\varphi(V(E))| \sum_{F\in\pi_n} |\varphi(W(F))| \\ &\leq \|V\|_S\ \|W\|_S \leq \|V\|\ \|W\|, \end{aligned}$$

so that $\|\rho_n\|_S \leq \|V\|\ \|W\|$. □

2.3 Lemma. *Let V, W and B be as above. Suppose that the set of all sums $\sum c(E\times F)V(E)W(F)$ taken over finite partitions of $K\times K$ into measurable rectangles $E \times F$ with $|c(E \times F)| \leq 1$ is contained in a weakly compact subset of B. Then the set $\{\int f d\rho_n : f \in C(K \times K),\ \|f\|_\infty \leq 1,$ $n \geq 1\}$ is contained in this same weakly compact subset.*

PROOF. Trivial, noting that

$$\int f d\rho_n = \sum_{E,F\in\pi_n} f(x(E), x(F))V(E)W(F). \qquad \square$$

The combination of Lemmas 2.2 and 2.3 with Corollary 2.1 shows the existence of some $\rho \in \mathrm{ca}(\mathcal{B}(K \times K), B)$ and a subsequence $(\rho_{n(k)})$ of (ρ_n) converging weakly to ρ. Since the elements of each partition π_n are clopen, we have

$$\rho(E \times F) = \lim \rho_{n(k)}(E \times F) = V(E)W(F)$$

for $E, F \in \pi_n$. It is not hard to see that the same equation holds for sets E and F in the field generated by $\bigcup_n \pi_n$. A relatively simple monotone class argument establishes that $\rho(E \times F) = V(E)W(F)$ for all $E, F \in \mathcal{B}(K)$. Only one measure can have this property (the field generated by measurable rectangles generates the Borel σ-field: the Carathéodory–Hahn–Kluvanek Theorem [3; p. 27] applies). We write $\rho = V \otimes W$.

A topological space X is *Polish* if its topology is generated by some complete separable metric. A space X is *absolute Borel* (or *standard*) if it is homeomorphic to a Borel subset of some Polish space. The Kuratowski Isomorphism Theorem [4; 13.1.1] asserts that any two absolute Borel spaces are Borel isomorphic. Thus, in the above construction, we may substitute for the Cantor space K an absolute Borel space. We have proved

2.4 Theorem. *Let X and Y be absolute Borel spaces whose Borel σ-fields are $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ and let B be a Banach algebra with full spectrum. Suppose that $V \in \mathrm{ca}(\mathcal{B}(X), B)$ and $W \in \mathrm{ca}(\mathcal{B}(X), B)$ are such that the set of all sums of the form*

$$\sum_{E\times F\in\Pi} c(E\times F)V(E)W(F),$$

where $|c(E\times F)| \leq 1$ and Π is a finite partition of $X\times Y$ into Borel rectangles, is contained in a weakly compact subset of B. Then there is a unique measure $V\otimes W \in \mathrm{ca}(\mathcal{B}(X)\otimes\mathcal{B}(Y), B)$ such that $(V\otimes W)(E\times F) = V(E)W(F)$ for all $E\in\mathcal{B}(X)$ and $F\in\mathcal{B}(Y)$.

We now developed an integration theory for functions and measures taking values in a Banach algebra B. Let $(X,\mathcal{F})$ be a measurable space and let $f : X \to B$ be an $\mathcal{F}$-measurable simple function $f = \sum a_i I_{E_i}$. If $V \in \mathrm{ca}(X, B)$, we define the integral

$$\int f dV = \sum a_i V(E_i).$$

This integral is well-defined on simple functions, and $\|\int f dV\| \leq \int \|f\| d|V|$. If B has full spectrum, then

$$\left\|\int f dV\right\| \leq \|f\|_\infty \|V\|_S$$

where $\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$.

Given an $\mathcal{F}$-measurable function $f : X \to B$, we say that f is V-*integrable* if there is some sequence (f_n) of simple functions $f_n : X \to B$ such that

$$\int \|f - f_n\|\, d|V| \to 0 \text{ as } n \to \infty.$$

We see then that $(\int f_n dV)$ is Cauchy sequence in B :

$$\left\|\int f_n - f_m dV\right\| \leq \int \|f_n - f\|\, d|V| + \int \|f - f_m\|\, d|V|.$$

Define $\int f dV = \lim \int f_n dV$. Clearly, the integral is well-defined, and $\| \int f dV \| \leq \int \|f\| d|V|$. If $\varphi \in \operatorname{spec}(B)$, then

$$\varphi\left(\int f dV\right) = \int \varphi \circ f d\varphi(V).$$

So, if B has full spectrum,

$$\left\| \int f dV \right\| \leq \|f\|_\infty \|V\|_S.$$

Our definition of V-integrability coincides, of course, with Bochner integrability with respect to $|V|$. We do not insist that $|V| < \infty$. Nonetheless, the arguments of [3; p. 45] still apply to show

2.5 Lemma. *Let $f : X \to B$ be a measurable function. If $\int \|f\| d|V| < \infty$, then f is V-integrable.*

The converse of the lemma does not hold unless $|V| < \infty$, e.g. in the case of simple functions. If $\int \|f\| d|V| < \infty$, we say that f is *absolutely V-integrable*.

2.6 Lemma. *Let $(X, \mathcal{F})$ be a measurable space and let B be a Banach algebra. If $f : X \to B$ and $f_n : X \to B$ $(n = 1, 2, \dots)$ are measurable functions such that $f_n \to f$ $|V|$-a.e. (or in measure), and $\|f_n(x)\| \leq g$ for some $|V|$-integrable real function g, then f is absolutely V-integrable, and for each $E \in \mathcal{F}$, $\int_E f dV = \lim \int_E f_n dV$.*

Proof. We apply the usual Lebesgue Dominated Convergence Theorem to the function: $\|f_n - f\|.I_E \leq 2g$ and obtain

$$\left\| \int_E f dV - \int_E f_n dV \right\| \leq \int_E \|f - f_n\| d|V| \to 0$$

as $n \to \infty$. $\square$

We now continue the development of product measures for general measurable spaces. Let X, Y, V, W and B be as in Theorem 2.4 and let $V \otimes W$ be the resultant product measure. For $E \subseteq X \times Y$, $x \in X$ and $y \in Y$, we defined the vertical and horizontal sections

$$E_x = \{y : (x, y) \in E\} \text{ and } E^y = \{x : (x, y) \in E\}.$$

2.7 Lemma. *Suppose that $|V| < \infty$ and that $E \subseteq X \times Y$ is a Borel set. Then the function $g : X \to B$ defined by the rule $g(x) = W(E_x)$ is a Borel function and is absolutely integrable. Also, $\int g dV = (V \otimes W)(E)$.*

Proof. We consider the class C of Borel sets for which the lemma holds. If $E = F \times G$, then $g(x) = W((F \times G)_x) = W(G)I_F(x)$, so $\int \|g\| d|V| = \|W(G)\||V|(F) < \infty$ and $\int g dV = W(G)V(F) = (V \otimes W)(E)$. So this class includes all Borel rectangles.

The class C is closed under countable increasing unions: if $(E_n) \in \mathcal{C}$ with $E_n \uparrow E$, define $g_n(x) = W((E_n)_x)$. By σ-additivity for W, we have $g_n(x) \to g(x)$ for $x \in X$. Also, $\|g_n(x)\| \leq \|W\|$, so that by Lemma 2.6,

$$\int g\, dV = \lim_n \int g_n(x) dV(x) = \lim_n (V \otimes W)(E_n) = (V \otimes W)(E),$$

as desired.

The class $\mathcal{C}$ is also closed under proper differences: if $F \subseteq G$ are sets in $\mathcal{C}$, and $E = G - F$, then $g(x) = W((G-F)_x) = W(G_x - F_x) = W(G_x) - W(F_x)$, and $\int g dV = (V \otimes W)(G) - (V \otimes W)(F) = (V \otimes W)(E)$, as required. An application of the Dynkin $\pi - \lambda$ Theorem [1; 1.6.1] shows that $\mathcal{C}$ contains all Borel subsets of $X \times Y$. $\square$

2.8 Theorem. *Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be measurable spaces and let B be a Banach algebra with full spectrum. Suppose that $V \in \mathrm{ca}(\mathcal{B}(X), B)$ and $W \in \mathrm{ca}(\mathcal{B}(Y), B)$ are measures such that $|V| < \infty$ and such that the set of all sums*

$$\sum_{E \times F \in \Pi} c(E \times F) V(E) W(F)$$

taken over finite partitions Π of $X \times Y$ into measurable rectangles $E \times F$ and functions $c : \Pi \to [-1, 1]$ is contained in a weakly compact subset of B. Then there is a unique measure $V \otimes W \in \mathrm{ca}(\mathcal{B}(X) \otimes \mathcal{B}(Y), B)$ such that $(V \otimes W)(E \times F) = V(E)W(F)$ for all $E \in \mathcal{B}(X)$ and $F \in \mathcal{B}(Y)$.

Proof. We have already established the theorem for the case where X and Y are absolute Borel sets.

Case 1: The sets X and Y are separable metric spaces with Borel structures $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. Let $\bar{X}$ and $\bar{Y}$ be metric compactifications of X and Y with Borel structures $\mathcal{B}(\bar{X})$ and $\mathcal{B}(\bar{Y})$. Define $\bar{V} : \mathcal{B}(\bar{X}) \to B$ and $\bar{W} : \mathcal{B}(\bar{Y}) \to B$ by $\bar{V}(E) = V(E \cap X)$ and $\bar{W}(E) = W(E \cap Y)$. Then $\bar{V} \in \mathrm{ca}(\mathcal{B}(\bar{X}), B)$ and $\bar{W} \in \mathrm{ca}(\mathcal{B}(\bar{Y}), B)$, so that Theorem 2.4 applies to produce product measure $\bar{V} \otimes \bar{W}$ on $\bar{X} \times \bar{Y}$. (We note the partitions Π of $\bar{X} \times \bar{Y}$ into Borel rectangles induce corresponding partitions on $X \times Y$). Also, we have $|\bar{V}| < \infty$.

Now let E be a Borel subset of $\bar{X} \times \bar{Y}$ such that $E \cap (X \times Y) = \emptyset$. We apply Lemma 2.7 to see that

$$(\bar{V} \otimes \bar{W})(E) = \int \bar{W}(E_x) d\bar{V}(x).$$

But $E_x \cap Y = \emptyset$ for $x \in X$, so that $\bar{W}(E_x) = 0$ on some Borel set $C \supseteq X$. But $\bar{V}(\bar{X} - C) = 0$, so $(\bar{V} \otimes \bar{W})(E) = 0$. We define a measure ρ on $X \times Y$ by putting $\rho(E \cap (X \times Y)) = (\bar{V} \otimes \bar{W})(E)$ for $E \subseteq \bar{X} \times \bar{Y}$ Borel. Then ρ is a well-defined measure, and for Borel sets $E \subseteq \bar{X}$ and $F \subseteq \bar{Y}$, we have

$$\begin{aligned} \rho((E \cap X) \times (F \cap Y)) &= (\bar{V} \otimes \bar{W})(E \times F) = \bar{V}(E)\bar{W}(F) \\ &= V(E \cap X) W(F \cap Y). \end{aligned}$$

So $\rho = V \otimes W$ as required. For uniqueness, see [3; p. 27].

Case 2: The measurable spaces $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ have countably generated σ-fields. By a technique of Marczewski [8], we find functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ such that $\mathcal{B}(X) = \{f^{-1}(E) : E \subseteq \mathbb{R}$ Borel$\}$ and $\mathcal{B}(Y) = \{g^{-1}(E) : E \subseteq \mathbb{R}$ Borel$\}$. The level sets $f^{-1}(p)$ and $g^{-1}(p)$ account for the atoms of $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. We now define measures $f(V)$ and $g(W)$ on $f(X)$ and $g(Y)$ by putting $f(V)(E) = V(f^{-1}(E))$ and $g(W)(E) = W(g^{-1}(E))$. Then $f(V)$ and $g(W)$ are measures on the separable metric spaces $f(X)$ and $g(Y)$ satisfying the hypotheses of the theorem. Case 1 may then be applied for construct $f(V) \otimes f(W)$ on $f(X) \times f(Y)$. Define $F : X \times Y \to f(X) \times f(Y)$ by $F(x, y) = (f(x), g(y))$. Then $V \otimes W$ may be defined by $(V \otimes W)(E) = (f(V) \otimes f(W))(F(E))$. Verification is routine. Uniqueness follows from [3; p. 27].

Case 3: (For general measurable spaces) Let $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ and $\{\mathcal{B}_\mu : \mu \in M\}$ be listings of all countably generated sub-σ-fields of $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. For each λ and μ, let V_λ and W_μ be the restrictions of V and W to $\mathcal{A}_\lambda$ and $\mathcal{B}_\mu$, respectively. Case 2 applies to produce a unique product measure $\rho_{\lambda\mu} = V_\lambda \otimes W_\mu$ on $\mathcal{A}_\lambda \otimes \mathcal{B}_\mu$. By uniqueness, we have that if $\mathcal{A}_\lambda \subseteq \mathcal{A}_{\lambda'}$ and $\mathcal{B}_\mu \subseteq \mathcal{B}_{\mu'}$, then $\rho_{\lambda\mu}$ is the restriction of $\rho_{\lambda'\mu'}$ to $\mathcal{A}_\lambda \otimes \mathcal{B}_\mu$. Thus, we may define $V \otimes W$ to be the coherent union of the functions $\rho_{\lambda\mu}$. Countable additivity follows from the fact that if $\mathcal{A}_{\lambda_1}\ \mathcal{A}_{\lambda_2} \ldots$ and $\mathcal{B}_{\mu_1}\ \mathcal{B}_{\mu_2} \ldots$ are countably generated, so are the σ-fields they generate. Uniqueness again comes from [3; p. 27]. $\square$

Now that existence of product measure has been established in full generality, it remains only to develop a Fubini-type theorem to complete the integration theory.

2.9 Lemma. *With measures V and W as in Theorem 2.8 with $|V| < \infty$. Then $|V \otimes W| \le |V| \otimes |W|$.*

Proof. Let $E \subseteq X \times Y$ be a Borel set and let $E = E_1 \cup \ldots \cup E_n$ be a finite partition of E into Borel sets. If $|V| \otimes |W|(E) = \infty$, there is nothing to prove; if $|V| \otimes |W|(E) < \infty$,

$$|V| \otimes |W|(E) = \sum |V| \otimes |W|(E_i) = \sum \int |W|((E_i)_x)\, d|V|(x) \text{ (Fubini)}$$

$$\ge \sum \left\| \int W((E_i)_x)\, dV(x) \right\| = \sum \|(V \otimes W)(E_i)\| . \text{ (Lemma 2.7)}$$

Taking the supremum over all partitions of E yields $|V| \otimes |W|(E) \ge |V \otimes W|(E)$. □

We say that $f : X \times Y \to B$ is *product integrable* if $\int \|f\| d|V| \otimes |W| < \infty$. If $|V| < \infty$, then the previous lemma implies that f is absolutely $V \otimes W$ -integrable and that $\int f dV \otimes W$ exists.

2.10 Theorem. *Let $(X, B(X))$ and $(Y, \mathcal{B}(Y))$ be measurable spaces and let B be a Banach algebra with full spectrum. Suppose that $V \in$ ca$(\mathcal{B}(X), B)$ and $W \in$ ca$(\mathcal{B}(Y), B)$ are measures satisfying the hypotheses of Theorem 2.8. Let $f : X \times Y \to B$ be product integrable. Then*

- i) *for $|V|$ - a.e. $x \in X$, $f_x(y) = f(x, y)$ is absolutely W-integrable on Y;*
- i') *for $|W|$ - a.e. $y \in Y$, $f^y(x) = f(x, y)$ is absolutely V-integrable on X;*
- ii) *the function $x \to \int f_x dW$ is absolutely V-integrable on X;*
- ii') *the function $y \to \int f^y dV$ is absolutely W-integrable on Y;*
- iii) *$\int f dV \otimes W = \iint f dW dV$.*

Proof. The ordinary Fubini Theorem applies to $\|f(x, y)\|$ and the real measures $|V|$ and $|W|$ to establish statements *i*), *i'*), *ii*), and *ii'*). Lemma 2.7 may be applied to show that *iii.*) holds when f is a simple function. In general, if f is $|V| \otimes |W|$-integrable, there are simple functions, $f_n : X \in Y \to B$ such that $\int \|f - f_n\| d|V| \otimes |W| \to 0$. The functions f_n

can be chosen with $\|f_n\| \le \|f\| + 1$ and $f_n \to f$ pointwise. Then

$$\left\| \int f d(V \otimes W) - \iint f dW dV \right\| \le \left\| \int f d(V \otimes W) - \int f_n d(V \otimes W) \right\|$$
$$+ \left\| \int f_n d(V \otimes W) - \iint f_n dW dV \right\| + \left\| \iint f_n dW dV - \iint f dW dV \right\|$$
$$\le \int \|f - f_n\| \, d|V| \otimes |W| + \iint \|f_n - f\| \, d|W| d|V|.$$

The first term tends to zero as $n \to \infty$. To the second we apply the usual Lebesgue Dominated Convergence Theorem: it, too, tends to zero. $\square$

Note: If we also have $|W| < \infty$, then *iii*) can read $\int f d(V \otimes W) = \iint f dW dV = \iint f dV dW$.

3. Banach lattices; Strassen's Theorem

In this section, we developed the theory of weak convergence for measures with values in the positive cone of a Banach lattice. Analogies with the classical theory are quite close and facilitate applications. We generalise a well-known result of Strassen to the context of vector measures.

We recall that a vector space V with a partial ordering $\le$ is a *vector lattice* if

1) $x \le y$ implies $x + z \le y + z$ for all $x, y, z \in V$;
2) $x \le y$ implies $\alpha x \le \alpha y$ for all $x, y \in V$ and $\alpha \in \mathbb{R}^+$;
3) every pair of elements x, y of V has a supremum $x \vee y$ and an infimum $x \wedge y$.

If (V, S) is a vector lattice and $x \in V$, we define $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$ and $|x| = x \vee (-x)$. Then $x = x^+ - x^-$ and $|x| = x^+ + x^-$. A vector lattice $(B, \le)$ is a *Banach lattice* if the norm $\|\cdot\|$ on B satisfies the implication

$$(*) \qquad |x| \le |y| \text{ implies } \|x\| \le \|y\|.$$

For further information on Banach lattices, we refer the reader to the monograph of Schaefer [11] or the shorter exposition by Kelly, Namioka *et al.* [7]. The theory of vector measures with values in the positive cone $B^+ = \{x \in B : x \ge 0\}$ of a Banach lattice more nearly mirrors the classical theory of measures that does general vector measure theory. The following observation greatly facilitates this analysis.

3.1 Lemma. *Let $(X, \mathcal{F})$ be a measurable space and let $V : \mathcal{F} \to B^+$ be a countably additive measure taking values in the positive cone of a Banach lattice B. Then, for each $E \in \mathcal{F}$, we have $\|V\|(E) = \|V(E)\|$.*

Proof. Let $\varphi \in B^*$ be a functional with $\|\varphi\| \leq 1$. An argument in [7; p. 239] shows that if $\varphi = \varphi^+ - \varphi^-$ and $|\varphi| = \varphi^+ + \varphi^-$, then $\||\varphi|\| = \|\varphi\| \leq 1$. So then $|\varphi(V)|(E) = |\varphi^+(V) - \varphi^-(V)|(E) \leq |\varphi^+(V)|(E) + |\varphi^-(V)|(E)$. Since φ^+ and φ^- are positive functionals, the latter is $\varphi^+(V(E)) + \varphi^-(V(E)) = |\varphi|(V(E)) \leq \||\varphi|\| \, \|V(E)\| \leq \|V(E)\|$, so that $|\varphi(V)|(E) \leq \|V(E)\|$. The reverse inequality always holds. $\square$

The following generalises a result of Ulam. See e.g. [4; 7.1.4].

3.2 Theorem. *Let (X, d) be a complete, separable metric space with Borel σ-field $\mathcal{B}(X)$ and let $(B, \leq)$ be a Banach lattice. Every countably additive vector measures $V : \mathcal{B}(X) \to \mathcal{B}^+$ is tight.*

Proof. Let $x_1, x_2, \ldots$ be a sequence dense in X. For each $\delta > 0$ and $x \in X$, define $\bar{B}(x; \delta) = \{y \in X : d(x, y) \leq \delta\}$. Given $\varepsilon > 0$ and a positive integer m, we take $n(m) < \infty$ such that

$$\left\| V\left(X - \bigcup_{n=1}^{n(m)} \bar{B}\left(x_n; \frac{1}{m}\right)\right) \right\| < \frac{\varepsilon}{2^m}.$$

(Countable additivity makes this choice possible.) Define

$$K = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{n(m)} \bar{B}\left(x_n; \frac{1}{m}\right).$$

This set K is closed and totally bounded in X and so is compact. Then, using Lemma 3.1, we have

$$\|V\|(X - K) = \|V(X - K)\| \leq \sum \frac{\varepsilon}{2^m} = \varepsilon,$$

as desired. $\square$

Weak convergence for general vector or signed measures exhibits certain pathologies. We note the following

3.3 Example. Define $V_n = \delta_{\frac{1}{n}} - \delta_{-\frac{1}{n}}$, the difference of point masses at $\frac{1}{n}$ and $-\frac{1}{n}$. The V_n are real-valued measures converging weakly to zero. Yet for $F = [0, \infty)$, we have $V_n(F) = 1$ for all n.

As we shall see, restriction to non-negative measures avoids such unusual behaviour.

Let $(B, \leq)$ be a Banach lattice with positive cone $B^+ = \{x \in B : x \geq 0\}$. For measures taking values in the positive cone B^+, the theory of weak convergence assumes a more familiar form. The following theorem bears comparison with recent work of Dekiert [2] and generalises the classical Portmanteau Theorem [4; 11.1.1].

3.4 Theorem. *Let (X, d) be a metric space with Borel σ-field $\mathcal{B}(X)$. Let V and (V_n) be measures in $\mathrm{ca}(\mathcal{B}(X), B)$ taking values in B^+. The following are equivalent:*

- i) *the sequence (V_n) converges weakly to V;*
- ii) *$\varphi(\int f dV_n) \to \varphi(\int f dV)$ as $n \to \infty$ for all bounded, uniformly continuous $f : X \to \mathbb{R}$ and all positive $\varphi \in B^*$;*
- iii) *$\limsup \varphi(V_n(F)) \leq \varphi(V(F))$ for all closed $F \subseteq X$ and positive $\varphi \in B^*$;*
- iv) *$\liminf \varphi(V_n(G)) \geq \varphi(V(G))$ for all open $G \subseteq X$ and positive $\varphi \in B^*$.*

Proof. $i \Rightarrow ii$: Trivial.

$ii \Rightarrow iii$: Given a positive $\varphi \in B^*$, F closed, and $\delta > 0$, we use the regularity of the measure $\varphi \circ V$ to select an $\varepsilon > 0$ such that $G = \{x : d(x, F) < \varepsilon\}$ satisfies $\varphi(V)(G - F) < \delta$. Define $f : X \to \mathbb{R}$ by

$$f(x) = 1 - (d(x, F)/\varepsilon) \wedge 1.$$

Then $f(x) = 1$ on F and $f(x) = 0$ on $X - G$. Also $0 \leq f \leq 1$, so that

$$\varphi(V_n(F)) = \int_F f d\varphi(V_n) \leq \int f d\varphi(V_n)$$

and

$$\int f d\varphi(V) = \int_G f d\varphi(V) \leq \varphi(V(G)) < \varphi(V(F)) + \delta,$$

whence follows (using uniform continuity of f),

$$\limsup \varphi(V_n(F)) \leq \lim \int f d\varphi(V_n) = \int f d\varphi(V) < \varphi(V(F)) + \delta.$$

Letting δ evaporate gives the desired inequality.

$iii \Rightarrow i$: We may assume that $0 < f(x) < 1$. For each k, let $F_i = \{x : i/k \leq f(x)\}$, $i = 0, 1, \ldots, k$. Each F_i is closed. We have, for $\varphi \in B^*$

positive,

$$\sum_{i=1}^{k} \frac{i-1}{k} \varphi(V) \left\{ x : \frac{i-1}{k} \leq f(x) < \frac{i}{k} \right\} \leq \int f d\varphi(V)$$
$$< \sum_{i=1}^{k} \frac{i}{k} \varphi(V) \left\{ x : \frac{i-1}{k} \leq f(x) < \frac{i}{k} \right\}.$$

The rightmost sum is

$$\sum_{i=1}^{k} \frac{i}{k} \big[\varphi(V)(F_{i-1}) - \varphi(V)(F_i) \big] = \frac{1}{k} + \frac{1}{k} \sum_{i=1}^{k} \varphi(V)(F_i).$$

A similar formula holds for the leftmost sum, and we obtain

$$\frac{1}{k} \sum_{i=1}^{k} \varphi(V(F_i)) \leq \int f d\varphi(V) < \frac{1}{k} + \frac{1}{k} \sum_{i=1}^{k} \varphi(V(F_i)).$$

If now $\limsup \varphi(V_n(F_i)) \leq \varphi(V(F_i))$ for each i, then

$$\limsup \int f d\varphi(V_n) \leq \frac{1}{k} + \int f d\varphi(V).$$

Letting $k \to \infty$ yields $\limsup \int f d\varphi(V_n) \leq \int f d\varphi(V)$. The same argument applied to $-f$ gives $\liminf \int f d\varphi(V_n) \geq \int f d\varphi(V)$, so that $\int f d\varphi(V_n) \to \int f d\varphi(V)$ as $n \to \infty$.

To conclude de demonstration, it remains only to note that every $\varphi \in B^*$ may be written as a difference of positive functionals in B^*.

$iii \Leftrightarrow iv$: This follows by complementation. $\square$

The following result is patterned after theorems of Le Cam [4; 11.5.3] and Shortt [12: Lemma 6].

3.5 Lemma. *Let (X, d) be a separable metric space with Borel σ-field $\mathcal{B}(X)$. There are, for each positive integer k, partitions π_k of X into finitely many Borel sets such that if V and V_k are countably additive tight vector measures on $\mathcal{B}(X)$ taking values in the positive cone B^+ of a Banach lattice $(B, \leq)$ and satisfying $V_k(A) = V(A)$ for all $A \in \pi_k$, then*

i) $V_n \to V$ *weakly;*
ii) $\limsup \|V_n(F)\| \leq \|V(F)\|$ *for each closed set $F \subseteq X$;*
iii) *the sequence (V_n) is uniformly tight.*

PROOF. Since X is separable, there is a totally bounded metric $\bar{d}$ on X topologically equivalent to d. For each k, choose points $x_1, \dots, x_n$ such that the open balls $B(x_i, \frac{1}{k})$, $(i = 1, \dots, n)$, cover all of X. Put $A_1 = B(x_1; \frac{1}{k})$ and in general

$$A_j = B\left(x_j; \frac{1}{k}\right) - (A_1 \cup \ldots \cup A_{j-1}) \text{ for } j \leq n.$$

Put $\pi_k = \{A_1, \dots, A_n\}$. Now suppose that $g : X \to \mathbb{R}$ is $\bar{d}$-uniformly continuous. Define $\alpha_A = \inf\{g(x) : x \in A\}$ and $\beta_A = \sup\{g(x) : x \in A\}$ for $A \in \pi_k$. Then

$$\int g\, dV_k - \int g\, dV = \sum_{A \in \pi_k} \left(\int_A g\, dV_k - \int_A g\, dV \right).$$

We have also

$$\begin{aligned} &\alpha_A V(A) \leq \int_A g\, dV \leq \beta_A V(A) \text{ and} \\ &\alpha_A V(A) = \alpha_A V_k(A) \leq \int g\, dV_n \leq \beta_A V_k(A) = \beta_A V(A), \end{aligned}$$

so that

$$\alpha_A V(A) - \beta_A V(A) \leq \int_A g\, dV_k - \int_A g\, dV \leq \beta_A V(A) - \alpha_A V(A).$$

Putting $S = \sup_A(\beta_A - \alpha_A)$, we obtain

$$\begin{aligned} -S \cdot V(X) \leq -\sum_A (\beta_A - \alpha_A) V(A) &\leq \int g\, dV_k - \int g\, dV \\ &\leq \sum_A (\beta_A - \alpha_A) V(A) \leq S \cdot V(X), \end{aligned}$$

so that $\| \int g dV_k - \int g dV \| \leq S \| V(X) \|$.

Since g is $\bar{d}$-uniformly continuous and since the diameter of each A is less that $2/k$, we have that $S \to 0$ as $k \to \infty$. We have proved that $\int g dV_k \to \int g dV$ as $k \to \infty$ for each such $g : X \to \mathbb{R}$. (This establishes i.)

Now suppose that $F \subseteq X$ is closed. As in the proof of Theorem 3.4, we define, for each $\delta > 0$, on $\varepsilon > 0$ such that $G = \{\bar{x} : \bar{d}(x, F) < \varepsilon\}$ satisfies $\|V\|(G - F) < \delta$. (We use regularity — see Lemma 1.2.) Define $f : X \to \mathbb{R}$ by

$$f(x) = 1 - (\bar{d}(x, F)/\varepsilon) \wedge 1.$$

Then for $f = 1$ on F and $f = 0$ on $X - G$. Also, f is $\bar{d}$-uniformly continuous, and $0 \leq f \leq 1$. We have

$$0 \leq V_n(F) = \int_F f dV_n \leq \int f dV_n$$

and

$$0 \leq \int f dV = \int_G f dV \leq V(G) = V(F) + V(G - F),$$

whence follows (we have the first part of the proof)

$$\begin{aligned} \limsup \|V_n(F)\| &\leq \lim \left\| \int f dV_n \right\| = \left\| \int f dV \right\| \\ &\leq \|V(F)\| + \|V(G - F)\| \leq \|V(F)\| + \delta\,. \end{aligned}$$

Letting $\delta \to 0$ yields the assertion *ii*) of the theorem.

Given $\varepsilon > 0$, we use tightness of V to choose a compact K with $\|V(X - K)\| < \varepsilon$. For each $\delta > 0$ and $n \geq 1$, we define $K^\delta = \{x : d(x, K) < \delta\}$ and

$$a(n) = \inf\{\delta > 0 : \|V_n(X - K^\delta)\| < \varepsilon\}.$$

From part *ii*) of the theorem, we have that $a(n) \to 0$ as $n \to \infty$ (in fact $a(n) = 0$ for all large n). Using regularity together with tightness of V_n, we choose a compact $K_n \subseteq K^{2a(n)}$ with $\|V_n(K^{2a(n)} - K_n)\| < \varepsilon$. Define $L = K \cup K_1 \cup K_2 \cup \dots$. Then L is compact, since every open cover of K covers all but finitely many of the K_n. Then $\|V_n(X - L)\| \leq \|V_n(X - K_n)\| \leq \|V_n(X - K^{2a(n)})\| + \|V_n(K^{2a(n)} - K_n)\| < 2\varepsilon$. $\square$

We are now ready for our major application, a generalisation of the celebrated theorem of Strassen (see [4], [6], [9], [12], [14]). First we solve a finitary version of the problem and use weak convergence to pass to the continuous case.

3.6 Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite fields on a base set X and let $(B, \leq)$ be a vector lattice. Suppose that $V_0 : \mathcal{A} \cup B \to B^+$ be a set function such that the restriction of V_0 to $\mathcal{A}$ or to $\mathcal{B}$ is a vector measure. The following conditions are equivalent:*

i) *V_0 extends to a vector measure $V : \mathcal{C} \to B^+$ on the field $\mathcal{C}$ generated by $\mathcal{A} \cup \mathcal{B}$;*
ii) *if $E \subseteq F$ with $E \in \mathcal{A}$ and $F \in \mathcal{B}$, then $V_0(E) \leq V_0(F)$.*

Indication. *ii* $\Rightarrow$ *i* : This follows easily from a theorem of SHORTT and WEHRUNG [13; Corollary 3.7]. It holds for semigroups much more generally than for positive cones B^+ of vector lattices.

i $\Rightarrow$ *ii*; Obvious.

3.7 Theorem. *Let X and Y be absolute Borel spaces with Borel σ-fields $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ and suppose that $S \subseteq X \times Y$ is the complement of a countable union of Borel rectangles (e.g. S may be closed). Let $(B, \leq)$ be a reflexive Banach lattice and let $V : \mathcal{B}(X) \to B^+$ and $W : \mathcal{B}(Y) \to B^+$ be countably additive vector measures with values in the cone B^+. The following conditions are equivalent:*

i) *there is a countably additive vector measure $\rho : \mathcal{B}(X) \otimes \mathcal{B}(Y) \to B^+$ such that $\rho(E \times Y) = V(E)$ and $\rho(X \times F) = W(F)$ for all $E \in \mathcal{B}(X)$ and $F \in \mathcal{B}(Y)$ and such that $\|\rho\|(X \times Y - S) = 0$;*

ii) *if $(E \times Y) \cap S \subseteq (X \times F) \cap S$ for $E \in \mathcal{B}(S)$ and $F \in \mathcal{B}(Y)$, then $V(E) \leq W(F)$.*

PROOF. *i* $\Rightarrow$ *ii* : Applying ρ to the given inclusion yields $V(E) = \rho(E \times Y) = \rho((E \times Y) \cap S) \leq \rho((X \times F) \cap S) = \rho(X \times F) = W(F)$.

ii $\Rightarrow$ *i*: We know that $X - S = \bigcup (E_n \times F_n)$ for various $E_n \in \mathcal{B}(X)$ and $F_n \in \mathcal{B}(Y)$. We let $\mathcal{B}_0(X)$ and $\mathcal{B}_0(Y)$ be countable collections of sets that generate the σ-fields $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ and are such that $E_n \in \mathcal{B}_0(X)$ and $F_n \in \mathcal{B}_0(Y)$ for each n. We use a technique of MARCZEWSKI ([8], [12]; Lemma 1]) to choose separable metrics d_1 and d_2 for X and Y, respectively, so that the Borel σ-fields for d_1 and d_2 are exactly $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ and for which all to the sets in $\mathcal{B}_0(X)$ and $\mathcal{B}_0(Y)$ are clopen. Then (X, d_1) and (Y, d_2) are again absolute Borel spaces [1; 8.3.7], and S is a closed subset of the product of (X, d_1) and (Y, d_2).

We now apply Lemma 3.5 to find partitions $\pi_k(X)$ and $\pi_k(Y)$ of (X, d_1) and (Y, d_2) satisfying the assertions of that Lemma. We consider finite fields $\mathcal{A}_k$ and $\mathcal{B}_k$ on $S (k \geq 1)$ as follows: $\mathcal{A}_k$ [resp. $\mathcal{B}_k$] is generated by sets of the form $(E \times Y) \cap S$ [resp. $(X \times F) \cap S$] for $E \in \pi_k(X)$ [resp. $F \in \pi_k(Y)$]. Then Theorem 3.6 can be applied to the fields $\mathcal{A}_k$ and $\mathcal{B}_k$ to produce a measure $\rho_k : \mathcal{C}_k \to B^+$ on $\mathcal{C}_k$, the field generated by $\mathcal{A}_k \cup \mathcal{B}_k$, such that $\rho_k((E \times Y) \cap S) = V(E)$ for $E \in \pi_k(X)$ and $\rho_k((X \times F) \cap S) = W(F)$ for $F \in \pi_k(Y)$. We may choose ρ_k so as to be concentrated on a finite subset of S: thus ρ_k can be extended to all of $\mathcal{B}(X) \otimes \mathcal{B}(Y)$. Define $V_k \in \mathrm{ca}(\mathcal{B}(X), B)$ and $W_k \in \mathrm{ca}(\mathcal{B}(Y), B)$ by putting $V_k(E) = \rho_k(E \times Y)$ and $W_k(F) = \rho_k(X \times F)$. Then Lemma 3.5 implies that the sequences (V_k) and (W_k) are uniformly tight. (We know that V and W are tight, since X and Y are absolutely Borel: see Lemmas 3.2 and 1.2.) Given $\varepsilon > 0$, choose compact sets $K_1 \subseteq X$ and $K_2 \subseteq Y$

such that for each k, $\|V_k(X - K_1)\| < \frac{\varepsilon}{2}$ and $\|W_k(Y - K_k)\| < \frac{\varepsilon}{2}$. Then $\|\rho_k\|((X \times Y) - (K_1 \times K_2)) \leq \|\rho_k\|(X \times (Y - K_2)) + \|\rho_k\|((X - K_1) \times Y) = \|\rho_k(X \times (Y - K_2))\| + \|\rho_k((X - K_1) \times Y)\| = \|W_k(Y - K_2)\| + \|V_k(X - K_1)\| < \varepsilon$, so that the sequence (ρ_k) is uniformly tight.

We now wish to apply Corollary 1.6 to the sequence (ρ_k). This is a bounded sequence, since $\|\rho_k\| = \|\rho_k(X \times Y)\| = \|V(X)\| = \|W(Y)\|$. So there is a subsequence $\rho_{n(k)}$ converging weakly to some measure ρ. We assert first that $\|\rho\|((X \times Y) - S) = 0$: we see that since $(X \times Y) - S$ is open, for each positive $\varphi \in B^*$,

$$0 = \liminf \varphi(\rho_{n(k)}((X \times Y) - S)) \geq \varphi(\rho((X \times Y) - S)),$$

so that $\rho((X \times Y) - S) = 0$. Next, we check that ρ has the desired "marginals": the marginals of $\rho_{n(k)}$ are $V_{n(k)}$ and $W_{n(k)}$, which converge weakly to V and W. These are the marginals of ρ: for each $\varphi \in B^*$ continuous, bounded $f : X \to \mathbb{R}$, we have

$$\varphi\left(\int f(x) dV(x)\right) = \lim \varphi\left(\int f(x) dV_{n(k)}(x)\right)$$
$$= \lim \varphi\left(\int f(x) d\rho_{n(k)}(x, y)\right) = \varphi\left(\int f(x) d\rho(x, y)\right),$$

and likewise with W. $\square$

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