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Circle-preserving transformations in Finsler spaces

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Abstract. Here, by extending the definition of circle to Finsler geometry, we show that, every circle-preserving local diffeomorphism is conformal. This result implies that in Finsler geometry, the definition of concircular change of metrics, a priori, does not require the conformal assumption.

1. Introduction

In Riemannian geometry Vogel proved that every circle-preserving diffeomorphism is conformal, c.f. [14] and [11]. This theorem has been extended to pseudo-Riemannian manifolds in [7]. Here, we shall extend this theorem to Finsler manifolds. Using the Cartan covariant derivative along a curve, the definition of circles in a Finsler manifold is given. This definition is a natural extension of Riemannian one, see for instance, [12]. Some typical examples of circles are helices on a cylinder or a torus. It should be remarked that these circles need not to be closed in general, although it may be closed in some cases as on a torus.

A geodesic circle in a Riemannian geometry, as well as in Finsler geometry, is a curve for which the first Frenet curvature k_1 is constant and the second curvature k_2 vanishes. In other words a geodesic circle is a torsion free constant curvature curve. A *concircular* transformation is defined by [15] and [8] in Riemannian geometry to be a conformal transformation which preserves geodesic circles.

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This notion has been similarly developed in Finsler geometry by Agrawal and Izumi, cf. [1], [9], [10] and studied in [3], [5], [6] by one of the present authors.

The results obtained in this paper shows that in the definition of concircular transformations, a priori, the conformal assumption is not necessary. That is to say, if a transformation preserves geodesic circles then it is conformal.

2. Preliminaries

Let M be a real n-dimensional manifold of class C^∞. We denote by $TM \to M$ the bundle of tangent vectors and by $\pi : TM_0 \to M$ the fiber bundle of non-zero tangent vectors. A *Finsler structure* on M is a function $F : TM \to [0, \infty)$, with the following properties: (I) F is differentiable (C^∞) on TM_0; (II) F is positively homogeneous of degree one in y, i.e. $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$, where we denote an element of TM by (x, y). (III) The Hessian matrix of $F^2/2$ is positive definite on TM_0; $(g_{ij}) := \big(\frac{1}{2}\big[\frac{\partial^2}{\partial y^i \partial y^j} F^2\big]\big)$. A *Finsler manifold* (M, g) is a pair of a differentiable manifold M and a tensor field $g = (g_{ij})$ on TM which defined by a Finsler structure F. The spray of a Finsler structure F is a vector field on TM

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where

$$G^i = \frac{g^{il}}{4} \Big\{ \frac{\partial^2 F^2}{\partial x^m \partial y^l} y^m - \frac{\partial F^2}{\partial x^l} \Big\},$$

and $(g^{ij}) := (g_{ij})^{-1}$.

Let (M, g) be a C^∞ Finsler manifold and let c be an oriented C^∞ parametric curve on M with equation $x^i(t)$. We choose the pair $(x, \dot{x})$, to be the line element along the curve c.

Let (x^i, y^i) be the local coordinates on the slit tangent bundle $TM/0$. Using a Finsler connection we can choose the natural basis $(\delta/\delta x^i, \partial/\partial y^i)$, where $\frac{\delta}{\delta x^j} := \frac{\partial}{\partial x^j} - N^i_j \frac{\partial}{\partial y^i}$, and $N^i_j := \frac{1}{2} \frac{\partial G^i}{\partial y^j}$. The dual basis is given by $(dx^i, \delta y^i)$, where $\delta y^k := dy^k + N^k_l dx^l$.

Let X be a C^∞ vector field $X = X^i(t) \frac{\partial}{\partial x^i}|_{c(t)}$ along $c(t)$. We denote the Cartan covariant derivative of X in direction of $\dot{c} = \frac{dx^j}{dt} \frac{\partial}{\partial x^j}$ by $\nabla_{\dot{c}} X = \frac{\delta X^i}{dt} \frac{\partial}{\partial x^i}|_{c(t)}$. The components $\frac{\delta X^i}{dt}$ can be determined explicitly as follows.

$$\nabla_{\dot{c}} X = \nabla_{\dot{c}} X^i \frac{\partial}{\partial x^i} = \frac{dX^i}{dt} \frac{\partial}{\partial x^i} + X^i \nabla_{\dot{c}} \frac{\partial}{\partial x^i}. \tag{1}$$

The last term in (1) is given by $\nabla_{\dot c}\frac{\partial}{\partial x^i} := \omega_i^j(\dot c)\frac{\partial}{\partial x^j}$, where $\omega_i^j(\dot c) := (\Gamma^j_{ik}dx^k + C^j_{ik}\delta y^k)(\dot c)$, is the connection 1-form of Cartan connection, cf. [4], p. 39. Here, the coefficients Γ^i_{jk} are Christoffel symbols with respect to the horizontal partial derivative $\frac{\delta}{\delta x^j}$, that is,

$$\Gamma^i_{jk} := \frac{1}{2}g^{ih}\left(\frac{\delta g_{hk}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^h}\right),$$

and $C^i_{hk} := \frac{1}{2}g^{im}\frac{\partial g_{mk}}{\partial y^h}$, is the *Cartan torsion tensor.* Plugging δy^k in $\omega_i^j(\dot c)$

$$\begin{aligned}\omega_i^j(\dot c) &= (\Gamma^j_{ik}dx^k + C^j_{ik}(dy^k + N^k_l dx^l)(\dot c),\\ &= (\Gamma^j_{ik}dx^k + C^j_{is}N^s_k dx^k)\left(\frac{dx^l}{dt}\frac{\partial}{\partial x^l}\right),\\ &= (\Gamma^j_{il} + C^j_{is}N^s_l)\left(\frac{dx^l}{dt}\right),\end{aligned}$$

and replacing the resulting term in equation (1), we obtain the components of Cartan covariant derivative of X in direction of $\dot c$.

$$\frac{\delta}{dt}X^i = \frac{dX^i}{dt} + (\Gamma^i_{kh} + C^i_{ks}N^s_h)X^k\frac{dx^h}{dt}. \tag{2}$$

The Cartan covariant derivative $\nabla_{\dot c}$ is metric-compatible along c, that is, for any vector fields X and Y along c,

$$\frac{d}{dt}g(X,Y) = g(\nabla_{\dot c}X, Y) + g(X, \nabla_{\dot c}Y).$$

More details about this preliminaries may be found in [2], [4], [13].

3. Circles in a Finsler manifold

As a natural extension of circles in Riemannian geometry, cf. [12], we recall the definition of a generalized circle in a Finsler manifold, called here simply, circle.

Definition 3.1. Let (M, g) be a Finsler manifold of class C^∞. A smooth curve $c : I \subset \mathbb{R} \to M$ parameterized by arc length s is called a *circle* if there exist a unitary vector field $Y = Y(s)$ along c and a positive constant k such that

$$\nabla_{c'}X = kY, \tag{3}$$

$$\nabla_{c'}Y = -kX, \tag{4}$$

where, $X := c' = dc/ds$ is the unitary tangent vector field at each point $c(s)$. The number $1/k$ is called the *radius* of the circle.

Comparing this definition of circle with definition of a geodesic circle in Finsler geometry, recalled in the introduction, we find out that if in the definition of a geodesic circle we exclude the trivial case, $k_1 = 0$, that is, if we remove geodesics, then we obtain the definition of a circle in a Finsler manifold.

Lemma 3.1. *Let $c = c(s)$ be a unit speed curve on an n-dimensional Finsler manifold (M, g). If c is a circle, then it satisfies the following ODE*

$$\nabla_{c'}\nabla_{c'}X + g(\nabla_{c'}X, \nabla_{c'}X)X = 0, \tag{5}$$

where, $g(\ ,\ )$ denotes scalar product determined by the tangent vector c'. Conversely, c satisfies (5), then it is either a geodesic or a circle.

Proof. Assume that c is a circle parameterized by arc-length. By means of metric compatibility we have

$$g(\nabla_{c'}X, X) = \frac{1}{2}\frac{d}{ds}[g(X, X)] = 0.$$

equations (3) and (4) yield

$$\nabla_{c'}\nabla_{c'}X = k\nabla_{c'}Y = -k^2 X. \tag{6}$$

This implies

$$k^2 = -g(\nabla_{c'}\nabla_{c'}X, X) = \frac{d}{ds}[g(\nabla_{c'}X, X)] + g(\nabla_{c'}X, \nabla_{c'}X).$$

Plugging it into (6), we obtain (5).

Conversely, assume that $c = c(s)$ is a unit speed curve on M which satisfies equation (5). Then by metric compatibility property of $\nabla_{c'}$, we have

$$\frac{d}{ds}g(\nabla_{c'}X, \nabla_{c'}X) = 2g(\nabla_{c'}\nabla_{c'}X, \nabla_{c'}X). \tag{7}$$

Plugging equation (5) into this equation we have

$$g(\nabla_{c'}\nabla_{c'}X, \nabla_{c'}X) = -g(\nabla_{c'}X, \nabla_{c'}X)g(X, \nabla_{c'}X). \tag{8}$$

Taking into account equations (7) and (8) and the fact that $g(X, \nabla_{c'}X) = 0$ for unitary tangent vector field X, we have

$$\frac{d}{ds}g(\nabla_{c'}X, \nabla_{c'}X) = 0.$$

Therefore $k^2 := g(\nabla_{c'}X, \nabla_{c'}X)$ is constant along c. If $k = 0$, then c is a geodesic. If $k \neq 0$, set

$$Y = \frac{1}{k}\nabla_{c'}X, \tag{9}$$

then Y is a unit vector field which satisfies equation (3). The covariant derivative of equation (9) and using equation (5) yields to

$$\nabla_{c'}Y = \frac{1}{k}\nabla_{c'}\nabla_{c'}X = -kX.$$

Thus we have equations (3) and (4), hence c is a circle. This completes the proof. □

For a curve $c = c(s)$ parameterized by arc-length s, $c'(s) := \frac{dc}{ds}(s)$ is the unit tangent vector along c. Let

$$c''(s) := \nabla_{c'}c', \quad c''(s) := \nabla_{c'}\nabla_{c'}c', \quad c'''(s) := \nabla_{c'}\nabla_{c'}\nabla_{c'}c'.$$

We can express (5) as follows

$$c''' + g(c'', c'')c' = 0. \tag{10}$$

Equivalently, differential equation of a circle is given by

$$c''' = -k^2 c', \tag{11}$$

where $k = \sqrt{g(c'', c'')}$ is the constant first Frenet curvature. Hence, $c(s)$ is a circle if and only if c''' is a tangent vector field along c, or equivalently c''' is a scalar multiple of c' or $\dot{c}$.

If $c = c(t)$ is parameterized by an arbitrary parameter t, we denote its successive covariant derivatives by $\dot{c} := \frac{dc}{dt}$, $\ddot{c} =: \nabla_{\dot{c}}\dot{c}$ and $\dddot{c} := \nabla_{\dot{c}}\nabla_{\dot{c}}\dot{c}$. We have the following successive relations between successive covariant derivatives.

$$\dot{c} = |\dot{c}|\ c', \tag{12}$$

$$\ddot{c} = |\dot{c}|^2\ c'' + \frac{g(\dot{c}, \ddot{c})}{|\dot{c}|}\ c', \tag{13}$$

$$\dddot{c} = |\dot{c}|^3\ c''' + 3g(\dot{c}, \ddot{c})c'' + \frac{d}{dt}\left(\frac{g(\dot{c}, \ddot{c})}{|\dot{c}|}\right)\ c'. \tag{14}$$

For an arbitrary parameter t we have the following lemma.

Lemma 3.2. *Let (M, g) be a Finsler manifold and $c(t)$ a curve on M. Then $c(t)$ is a circle with respect to g, if and only if the vector field*

$$V := \dddot{c} - 3\frac{g(\dot{c}, \ddot{c})}{g(\dot{c}, \dot{c})}\ddot{c},$$

is a tangent vector field along c or equivalently a multiple of $\dot{c}$ or c'.

Proof. It follows from (13) and (14) that

$$\dddot{c} - 3\frac{g(\dot{c}, \ddot{c})}{g(\dot{c}, \dot{c})}\ddot{c} = |\dot{c}|^3 c''' + \left\{\frac{d}{dt}\left(\frac{g(\dot{c}, \ddot{c})}{|\dot{c}|}\right) - 3\frac{g(\dot{c}, \ddot{c})^2}{g(\dot{c}, \dot{c})^{3/2}}\right\}c'.$$

Thus c''' is parallel to c' if and only if $\dddot{c} - 3\frac{g(\dot{c},\ddot{c})}{g(\dot{c},\dot{c})}\ddot{c}$ is proportional to c' □

Contrary to the Euclidean circle, the general notion of circle in Riemannian geometry as well as in Finsler geometry, called here, simply circle, is not required that a circle be a closed curve. Although it may happen in some cases as small circles or helicoid curves on the sphere. In general, similar to the Riemannian circles, they are spiral curves on the subordinate spaces, for instance, spiral curves on cylindrical surfaces, conical surfaces and so on. Moreover, their lengths are not required to be bounded as in closed circle in Euclidean spaces.

4. Circle-preserving diffeomorphisms

A local diffeomorphism of Finsler manifolds is said to be *circle-preserving* if it maps circles into circles. More precisely, let M be a differentiable manifold, g a Finsler metric on M, $c(s)$ a C^∞ arc length parameterized curve in a neighborhood $U \subset M$ and δ/ds the Cartan covariant derivative along c, compatible with g.

Let $\phi : M \longrightarrow M$ be a local diffeomorphism on M, then it induces a second Finsler metric $\bar{g}$ and a Cartan covariant derivative $\delta/d\bar{s}$ along $\bar{c}$ on $(M, \bar{g})$ on some neighborhood $\bar{U}$ of M. Here, we denote the induced Finsler manifold by $(M, \bar{g})$, in the sequel. We say that the local diffeomorphism $\phi : (M, g) \longrightarrow (M, \bar{g})$ *preserves circles*, if it maps circles to circles.

Let $c(s)$ be a circle and $\bar{c}(\bar{s})$ its image by ϕ, where $\bar{s} = \bar{s}(s)$. Then using definite positiveness of g and $\bar{g}$ and the related fundamental forms

$$ds^2 = g_{ij}(x, x')dx^i dx^j \quad \text{and} \quad d\bar{s}^2 = \bar{g}_{ij}(x, x')dx^i dx^j, \tag{15}$$

respectively, we can establish a relation between s and $\bar{s}$ and their derivatives.

We have $\frac{\delta}{d\bar{s}} = \frac{\delta}{ds} \cdot \frac{ds}{d\bar{s}}$, where

$$\frac{d\bar{s}}{ds} = \sqrt{\bar{g}_{jk}(x,x')\frac{dx^j}{ds}\frac{dx^k}{ds}} \neq 0, \qquad \frac{ds}{d\bar{s}} = \sqrt{g_{jk}(x,x')\frac{dx^j}{d\bar{s}}\frac{dx^k}{d\bar{s}}} \neq 0. \tag{16}$$

If we have $d\bar{s} = e^{\sigma} ds$, or equivalently by means of equation (15), if $\bar{g} = e^{2\sigma} g$ or $\bar{F} = e^{\sigma} F$, where σ is a scalar function on M, then two Finsler structures $\bar{F}$ and F are said to be *conformal*.

5. Circles in a Minkowski space

Let (V, F) be a Minkowski space where V is a vector space and F is a Minkowski norm on V. In a standard coordinate system in V,

$$G^i = 0, \quad N^i_j = 0. \tag{17}$$

Then for a vector field $X = X^i(t)\frac{\partial}{\partial x^i}|_{c(t)}$ along a curve $c(t)$, the Cartan covariant derivative $\nabla_{\dot{c}} X = \frac{\delta X^i}{dt}\frac{\partial}{\partial x^i}|_{c(t)}$ given by equation (2) reduces to

$$\frac{\delta X^i}{dt} = \frac{dX^i}{dt} + \Gamma^i_{kh} X^k \frac{dx^h}{dt}. \tag{18}$$

In particular, for $X = c'$, we have

$$\frac{\delta X^i}{ds} = \frac{d^2 x^i}{ds^2} + \Gamma^i_{kh}\frac{dx^k}{ds}\frac{dx^h}{ds} = \frac{d^2 x^i}{ds^2} + G^i,$$

where we have used $\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = \gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = G^i$, for which γ^i_{jk} are the formal Christoffel symbols. Thus equation (17) yeilds

$$\frac{\delta X^i}{ds} = \frac{d^2 x^i}{ds^2}.$$

Therefore in a Minkowski space, a curve $c(s)$ with arc-length parameter s is a circle if and only if

$$\frac{d^3 x^i}{ds^3} + k^2 \frac{dx^i}{ds} = 0, \tag{19}$$

where k is a constant. In this case $g_{hk}\frac{d^2 x^h}{ds^2}\frac{dx^k}{ds} = 0$ and $k = \sqrt{g_{hk}\frac{d^2 x^h}{ds^2}\frac{d^2 x^k}{ds^2}}$.

Now let us take a look at circles in a special Minkowski space $(\mathbb{R}^2, F_b)$, where

$$F_b := \sqrt{u^2 + v^2} + bu,$$

where b is a positive constant with $0 < b < 1$. $(\mathbb{R}^2, F_b)$ is called a *Randers plane*. Consider a curve $c(s) = (x(s), y(s))$ in $\mathbb{R}^2$ with unit speed, namely, $c'(s) = (x'(s), y'(s))$ is a unit vector. Thus

$$\sqrt{x'(s)^2 + y'(s)^2} + bx'(s) = 1.$$

We can let

$$x'(s) = \frac{\cos\theta(s) - b}{1 - b^2}, \quad y'(s) = \frac{\sin\theta(s)}{\sqrt{1 - b^2}}.$$

where $\theta(s)$ is a smooth function. Since b is bounded, components of $c'(s)$ are well defined and one can find out explicitly equation of $c(s)$, the unit circle in the Randers plane $(\mathbb{R}^2, F_b)$.

6. Vogel theorem in Finsler geometry

Let $\phi : (M, g) \longrightarrow (\bar{M}, \bar{g})$ be a diffeomorphism. We say that ϕ *preserves circles*, if it maps circles to circles. More precisely, if $c(s)$ is a circle in (M, g), where s is the arc-length of c with respect to g, then $\bar{c}(\bar{s}) := \phi \circ c(s(\bar{s}))$ is a circle in $(\bar{M}, \bar{g})$, where $s = s(\bar{s})$ is the arc-length of $\phi \circ c$ with respect to $\bar{g}$.

We recall the following lemma from linear algebra which will be used in the sequel.

Lemma 6.1. *Let F and G be the two bilinear symmetric forms on $\mathbb{R}^n$, satisfying*

- *F and G are definite positive.*
- *F and G are defined on $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, such that $F(X, Y) = 0, \forall X, Y \in \mathbb{R}^n$, with*

$$G(X, X) \neq 0, G(Y, Y) \neq 0, \quad \text{and} \quad G(X, Y) = 0, \tag{20}$$

then there is a positive real number α such that $F = \alpha G$.

Proof. Let $\{e_i\}$, be an orthonormal basis on $\mathbb{R}^n$ such that $G(e_i, e_j) = \delta_{ij}$ where $i, j = 1, \ldots, n$. equation(20) with definite positiveness of F and G imply that there is a positive real number α_i such that $F(e_i, e_j) = \alpha_i \delta_{ij}$, and hence

$$F(e_i, e_j) = \alpha_i G(e_i, e_j). \tag{21}$$

Let $a, b \in \mathbb{R} - \{0\}$ with $a^2 \neq b^2$, then for $i \neq j$ we have

$$G(ae_i + be_j, ae_i + be_j) = a^2 + b^2 \neq 0,$$

$$G(be_i - ae_j, be_i - ae_j) = b^2 + a^2 \neq 0,$$

$$G(ae_i + be_j, be_i - ae_j) = 0.$$

This equation together with equations(20) and (21) for $i \neq j$ imply

$$0 = F(ae_i + be_j, be_i - ae_j) = ab(\alpha_i G(e_i, e_i) - \alpha_j G(e_j, e_j))$$

and hence $0 = ab(\alpha_i - \alpha_j)$. Therefore we obtain $\alpha_i = \alpha_j = \alpha$, and $F = \alpha G$, which completes the proof. □

Next we prove the following theorem.

Theorem 6.1. *Every circle-preserving local diffeomorphism of a Finsler manifold is conformal.*

Proof. Without loss of generality we can consider two Finsler metrics g and $\bar{g}$ on the same manifold. Fix a point $p \in M$. For arbitrary two unit vectors $X, Y \in T_pM$ such that Y is orthogonal to X with respect to $g = g_X$, let $Ç = \{c_k | k \in \mathbb{R}\}$ be a family of circles with the constant curvature k passing through a fixed point $c_k(0) = p$ on (M, g) such that

$$\frac{dc}{ds}(0) = X, \quad \text{and} \quad \nabla_{c'} X(0) = kY. \tag{22}$$

We are going to show that $\bar{g}(X, Y) = 0$, where $\bar{g} := \bar{g}_X$.

Since c is assumed to be a circle with respect to the Finsler metric g, equation (11) yields, c''' is a multiple of c'. By Lemma 3.2, c is a circle with respect to $\bar{g}$ if and only if $\dddot{c}$ and $\ddot{c}$ are parallel to $\dot{c}$ or c'.

On the other hand, by virtue of equation (13), $\ddot{c}$ is parallel to $\dot{c}$, if and only if c'' so does.

By means of equation (14), we can see that $\dddot{c}$ is parallel to $\dot{c}$ if and only if $\ddot{c}$ so does. Denote the second term in right-hand side of equation (14), by $\overline{W} := 3\bar{g}(\dot{c}, \ddot{c})c''$. Therefore, c is a circle with respect to $\bar{g}$ if and only if $\overline{W}$ is parallel to $c' = X$ at the point $p = c(0)$. At p, by involving equations (12) and (22) we have $\dot{c} = c'\bar{g}(\dot{c}, \dot{c})^{1/2} = X\bar{g}(\dot{c}, \dot{c})^{1/2}$, where $\bar{g}(\dot{c}, \dot{c}) \neq 0$ is constant by means of equation (16). Hence, equation (22) yields $\ddot{c} = \bar{\nabla}_{\dot{c}}\dot{c} = \frac{d}{ds}(\bar{g}(\dot{c}, \dot{c})^{1/2}X)\frac{ds}{dt}$ and $\ddot{c} = kY\bar{g}(\dot{c}, \dot{c})$. Therefore we obtain

$$\overline{W} = 3\bar{g}(\dot{c}, \dot{c})^{3/2}\bar{g}(X, Y)Y\ k^2.$$

Hence, c is a circle with respect to $\bar{g}$ if and only if the vector field $\overline{W}$ is parallel to X or equivalently, $\bar{g}(X, Y)Y$ is parallel to X for every $X \in T_pM$ and every

$Y \in T_pM$ orthonormal to X. This implies $\bar{g}(X,Y) = 0$ whenever $g(X,Y) = 0$ and by Lemma 6.1, there is a positive scalar α^2 where $\bar{g} = \alpha^2 g$. Hence, the Finsler metrics $\bar{g}(x,x')$ and $g(x,x')$ are conformally related. □

A geodesic circle is a curve for which the first Frenet curvature k_1 is constant and the second curvature k_2 vanishes. In the other words a geodesic circle is a torsion free constant curvature curve. In Riemannian geometry as well as in Finsler geometry, a concircular transformation is defined to be a conformal transformation which preserves geodesic circles.

By replacing the positive scalar α in proof of the theorem 6.1 by $\alpha = e^\sigma$, we get $\bar{g} = e^{2\sigma} g$, or equivalently $d\bar{s} = e^\sigma ds$ where σ is a scalar function on M. Therefore, as a corollary of Theorem 6.1 we have

Theorem 6.2. *Every local diffeomorphism of a Finsler manifold which preserve geodesic circles is conformal.*

This result shows that in the definition of concircular transformations the conformal assumption is not necessary.

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References

[1] P. Agrawal, On the concircular geometry in Finsler spaces, *Tensor N.S.* **23** (1972), 333–336.

[2] H. Akbar-Zadeh, Initiation to Global Finslerian Geometry, *North-Holland Mathematical Library*, 2006.

[3] A. Asanjarani and B. Bidabad, A classification of complete Finsler manifolds through a second order differential equation, *Differential Geom. Appl.* **26** (2008), 434-444.

[4] D. Bao, S. S. Chern and Z. Shen, Riemann-Finsler geometry, *Springer-Verlag*, 2000.

[5] B. Bidabad, On compact Finsler spaces of positive constant curvature, *C. R. Acad. Sci. Paris, Ser. I* **349** (2011), 1191-1194.

[6] B. Bidabad, Complete Finsler manifolds and adapted coordinates, *Balkan J. Geom. Appl.* **14**(1) (2009), 21–29.

[7] D. A. Catalano, Concircular Diffeomorphisms of pseudo-Riemannian manifolds, Swiss Federal institute of Technology, Zurich, Thesis, 1999.

[8] A. Fialkow, The conformal theory of curves, *Trans. Amer. Math. Soc.* **51** (1942), 435–497.

[9] H. Izumi, Conformal transformation of Finsler spaces I, *Tensor N.S.* **31** (1977), 33–41.

[10] H. Izumi, Conformal transformation of Finsler spaces II, *Tensor N.S.* **34** (1980), 337–359.

[11] W. Kühnel, Conformal Transformations between Einstein Spaces, Conformal Geometry, Max-Planck-Institut für Mathematik, Bonn, 1988, 105–146.

[12] K. Nomizu and K. Yano, On Circles and Spheres in Riemannian Geometry, *Math. Ann.* **210** (1974), 163–170.

[13] Z. Shen, Lecture on Finsler geometry, *World Scientific Publishing Co.*, 2001.

[14] W. O. K. Vogel, Transformationen in Riemannschen Räumen, *(German) Arch. Math. (Basel)* **21** (1970–71), 641–645.

[15] K. Yano, Concircular geometry I, II, III, IV, *Proc. Imp. Acad., Tokyo* (1940–1944).

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