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On measures of weak noncompactness

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Abstract. In this paper we give an axiomatic definition of a measure of weak noncompactness and at the same time we present a general scheme of construction of such measures in a useful way. This allows us to prove their important properties. Furthermore, we prove a theorem dealing with the existence of bounded weak solutions of a nonlinear differential equation on $\mathbb{R}_+$, as an example of application of our theory.

1. Introduction

The concept of measure of noncompactness is one of the most useful concepts of general topology. This notion defined in many ways ([13], [15], [18]). Some of the authors have tried to introduce a definition of measures of noncompactness in an axiomatic way ([3], [23], [25]). These measures are defined in a Banach space ([3], [13], [15], [18]), in a metric space ([18], [25]) or in a locally convex space ([14], [23]).

In 1977 F. De Blasi introduced a notion of a measure of weak noncompactness in a Banach space [11]. The usefulness of this concept was made clear by a number of authors such as A. R. Mitchell and Ch. Smith [20], E. Cramer, V. Lakshmikantham and A. R. Mitchell [8], I. Kubiaczyk and S. Szufla [17] and many others.

In this paper we give a definition of measures of weak noncompactness in an axiomatic manner, similarly as in [4], and we present a general scheme of construction of these measures in a useful way. It is in our interest to look for a unified treatment of noncompactness measures via requirements naturally imposed on such a class of mappings.

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The convenient criteria of weak noncompactness are rather unkown except for some cases (see examples in chapter 2), also the theory of measures of weak noncompactness seems to be more interesting and at the same time more useful than that concerning measures of strong noncompactness.

Remark that our definition can be used for both the strong and the weak topology on E (cf. [7]). We refer the reader to [3] and [16].

This approach to measures of weak noncompactness is motivated by its usefulness in the theory of nonlinear differential equations.

A theorem which ensures the existence of bounded weak solutions of a nonlinear differential equation on $\mathbb{R}_+$ is given as an example of the applications of our concept of measures.

2. Measures of weak noncompactness

Let $(E, \|\cdot\|)$ denote a Banach space and let us denote by B_1 the closed unit ball in E.

Recall the definition of De Blasi's measure of weak noncompactness:

$$\omega(W) = \inf \{\varepsilon > 0 : \text{there exists a weakly compact subset } K \text{ of } E, \text{ such that } W \subset K + \varepsilon B_1\},$$

where W is a nonempty bounded subset of E.

Fix some further notations for the families of sets that will be used in the sequel:

$\mathcal{M}$ – the family of all nonempty bounded subsets of E,

$\mathcal{N}$ – the family of all nonempty and relatively weakly compact subsets of E.

Throughout this paper, $\bar{X}$ denotes the weak closure of the set X.

Definition 2.1. A nonempty family $\mathcal{P} \subset \mathcal{N}$ is said to be a kernel if it satisfies the following conditions:

(i) $X \in \mathcal{P} \implies \operatorname{conv} X \in \mathcal{P}$,

(ii) $Y \neq \emptyset,\ Y \subset X,\ X \in \mathcal{P} \implies Y \in \mathcal{P}$,

(iii) A subfamily of all weakly compact sets in $\mathcal{P}$ is closed in the family of all bounded and closed subsets of E with the topology generated by the Hausdorff distance.

Notice that the condition $\mathcal{P} \subset \mathcal{N}$ is really essential. In the space $L^1(I, E)$ the family $\mathcal{R}$ of all uniformly integrable subsets satisfies the above conditions (cf. [5], Prop. 4) — see Example 2.

Now we give our axiomatic definition of a measure of weak noncompactness:

Definition 2.2. A function $\mu : \mathcal{M} \to [0,\infty)$ is said to be a measure of weak noncompactness with the kernel $\mathcal{P}$ if it is subject to the following conditions:

(i) $\mu(X) = 0 \iff X \in \mathcal{P}$,
(ii) $\mu(X) = \mu(\bar{X})$,
(iii) $\mu(\text{conv } X) = \mu(X)$,
(iv) $X, Y \in \mathcal{M},\ X \subset Y \implies \mu(X) \leq \mu(Y)$.

Example 1 (cf. [1]). Let ϕ be a continuous, convex and nondecreasing function from $\mathbb{R}$ into $\mathbb{R}_+$ satisfying the following conditions (cf. [22]):

(E1) $\phi(x) = 0 \iff x = 0$,
(E2) $\phi(-x) = \phi(x)$,
(E3) $\phi(x) > 0$ whenever $x \neq 0$,
(E4) $\lim_{x\to\infty} \phi(x)/x = +\infty,\ \lim_{x\to 0} \phi(x)/x = 0$,
(E5) there exists a constant $k > 0$, such that for $x \geq 0$ we have

$$\phi(2x) \leq k \cdot \phi(x) \quad \text{whenever} \quad x \geq 0\,,$$

(Δ_2-condition).

Let $I = [0,1]$. In an Orlitcz-space $L^\phi(I)$, for each bounded subset $S \subset L^\phi(I)$ we define:

$$(*) \qquad \mu(S) = \lim_{\varepsilon\to 0} \sup_{f\in S} \frac{1}{\varepsilon} \cdot \int_I \phi(\varepsilon f(x))dx\,.$$

Under the above assumptions $\mu : \mathcal{M} \to [0,\infty)$ is a measure of weak noncompactness on $L^\phi(I)$. Indeed, (iv) is obvious.

Put $\mu(A,\varepsilon) = \sup_{f\in A} \frac{1}{\varepsilon} \cdot \int_I \phi(\varepsilon f(x))dx$ for each $\varepsilon > 0$.

Fix an arbitrary $\varepsilon > 0$ and let $A, B \in \mathcal{M}$. Consider an arbitrary $g \in \text{conv}A$. First, let us remark that there exist $\lambda \in [0,1]$ and $f_1, f_2 \in A$ such that

$$g = \lambda \cdot f_1 + (1-\lambda) \cdot f_2\,.$$

By the properties of ϕ we have:

$$\phi(\varepsilon g) = \phi(\varepsilon \cdot \lambda \cdot f_1 + \varepsilon \cdot (1-\lambda) \cdot f_2) \leq \lambda \cdot \phi(\varepsilon f_1) + (1-\lambda) \cdot \phi(\varepsilon f_2), \quad \text{so}$$

$$\frac{1}{\varepsilon} \cdot \int_I \phi(\varepsilon g(x))dx \leq \lambda \cdot \mu(A,\varepsilon) + (1-\lambda) \cdot \mu(A,\varepsilon) = \mu(A,\varepsilon)\,.$$

Thus $\mu(\text{conv } A, \varepsilon) \leq \mu(A,\varepsilon)$, and letting $\varepsilon \to 0$, $\mu(\text{conv } A) \leq \mu(A)$.

The converse inequality follows from (iv).

Now, let A be a convex set in $L^{\phi}(I)$ and let $h \in \bar{A}$. Then there exists a sequence $(f_n) \subset A$ such that f_n tends to h in $L^{\phi}(I)$. Since ϕ satisfies the Δ_2-condition, we have

$$\int_I \phi(\varepsilon f_n(x))dx \to \int_I \phi(\varepsilon h(x))dx \quad \text{when} \quad n \to \infty .$$

Next, we observe that $\frac{1}{\varepsilon} \cdot \int_I \phi(\varepsilon f_n(x))dx \le \mu(A, \varepsilon)$, since $f_n \in A$ ($n = 1, 2, \dots$), and finally:
$\frac{1}{\varepsilon} \cdot \int_I \phi(\varepsilon h(x))dx \le \mu(A, \varepsilon)$, hence $\mu(\bar{A}, \varepsilon) \le \mu(A, \varepsilon)$. By (iv) again and letting $\varepsilon \to 0$ we have $\mu(\bar{A}) = \mu(A)$.

For a moment we distinguish a weak and a strong closure of a subset C. However, for arbitrary $C \subset L^{\phi}(I)$ we have $C \subset \bar{C}^w \subset \overline{\mathrm{conv}}^w C = \overline{\mathrm{conv}} C$, and by the above, $\mu(C) \le \mu(\bar{C}^w) \le \mu(C)$, so μ has the property (ii).

Our condition (i) follows from Theorem 3 in [22] (page 144) together with the Eberlein–Šmulian theorem.

Example 2. Let E be a reflexive Banach space, and let W be a bounded subset of $L^1(I, E)$.

Put

$$\gamma(W) = \lim_{\varepsilon \to 0} \sup_{x \in W} \left(\sup \left(\int_D \|x(t)\| dt : \ D \subset I, \ \mathrm{mes}\, D \le \varepsilon \right) \right) .$$

It is not difficult to prove that γ is a measure of weak noncompactness in $L^1(I, E)$ (cf. [4]) such that $\gamma(W) \le \omega(W)$. It suffices to use Prop. 3 from [5].

It is very difficult to construct formulas allowing us to express measures of weak noncompactness in a form convenient for applications in a concrete Banach space E. For example, the result of this type for De Blasi's measure is known only for $L^1(I, \mathbb{R})$ (see [2]).

Let us denote by $\mathcal{B}$ a basis of neighbourhoods of the zero in a locally convex space composed of closed convex sets. Let $\mathcal{B}' = \{rB : r > 0, \ B \in \mathcal{B}\}$. We will need the definition of a so-called p–function:

Definition 2.3. (cf. [7]). A function $p : \mathcal{B}' \to [0, \infty)$ is said to be a p–function if it satisfies the following conditions:

(i) $V, U \in \mathcal{B}', \ V \subset U \implies p(V) \le p(U)$,

(ii) $\forall \varepsilon > 0 \ \exists V \in \mathcal{B}' \ p(V) \le \varepsilon$,

(iii) $p(U) > 0$ whenever $U \notin \mathcal{P}$.

The following definition is a key point of this paper:

Definition 2.4. A function $\mu : \mathcal{M} \to [0, \infty)$ is said to be a $(\mathcal{P}, \mathcal{B}, p)$–measure of weak noncompactness [$(\mathcal{P}, \mathcal{B}, p)$-mwnc] iff

$$\mu(W) = \inf\{\varepsilon > 0 : \exists H \in \mathcal{P},\ V \in \mathcal{B}',\ W \subset H + V,\ p(V) \leq \varepsilon\},$$

where $W \in \mathcal{M}$.

Theorem 2.1. *Each $(\mathcal{P}, \mathcal{B}, p)$-mwnc μ is a measure of weak noncompactness (in the sense of Definition 2.2.) and $\mu(X) = 0 \iff X \in \mathcal{P}$.*

Proof. (iv) Fix an arbitrary $\varepsilon > 0$. Let $X, Y \in \mathcal{M},\ X \subset Y,\ H \in \mathcal{P}$ and $V \in \mathcal{B}'$ be such that $H \subset H + V,\ p(V) \leq \mu(Y) + \varepsilon$. So $X \subset H + V$ and $\mu(X) \leq (Y) + \varepsilon$. As ε is arbitrary we have $\mu(X) \leq \mu(Y)$.

(iii) $X \subset \operatorname{conv} X$, so by (iv) $\mu(X) \leq \mu(\operatorname{conv} X)$, and if $H \in \mathcal{P}$ and $V \in \mathcal{B}'$ are such that $X \subset H + V,\ p(V \leq \mu(X) + \varepsilon$ then $\operatorname{conv} X \subset \operatorname{conv} H + V$. But $\operatorname{conv} H \in \mathcal{P}$, so $\mu(\operatorname{conv} X) \leq p(V) \leq \mu(V) + \varepsilon$.

Finally $\mu(X) = \mu(\operatorname{conv} X)$.

(ii) As in (iv), let $X \subset H + V$ and $p(V) \leq \mu(X) + \varepsilon$, so $\bar{X} \subset \bar{H} + V$ and $\mu(\bar{X}) \leq p(V) \leq \mu(X) + \varepsilon$. Consequently $\mu(\bar{X}) = \mu(X)$ (by (iv)).

(i) $\mu(X) = 0 \implies \mu(\bar{X}) = 0 \implies \forall V \in \mathcal{B}'\ \exists H \in \mathcal{P}\ \bar{X} \subset H + V$ and by the closure of $\mathcal{P}$ in the Hausdorff topology $\bar{X} \in \mathcal{P}$. Thus $X \subset \bar{X}$ and by Definition 2.1. $X \in \mathcal{P}$.

$X \in \mathcal{P} \implies X \subset \bar{X} + V$ for each $V \in \mathcal{B}'$. Obviously $\bar{X} \in \mathcal{P}$. But for every $\varepsilon > 0$ there exists $V \in \mathcal{B}'$ such that $p(V) \leq \varepsilon$, so finally $\mu(X) = 0$.

This is a very interesting result, since a theorem of Darbo type ([4], th. 11) guarantees that the set of all fixed points of a weakly continuous operator, which is condensing with respect to the measure of weak noncompactness, belongs to $\mathcal{P}$.

We see that this paper establishes a relation between the strong and weak measures of noncompactness. Our approach to the notion of a measure of weak noncompactness is very similar to the approach associated with the notion of a measure of noncompactness in a strong sense. This is the reason why many properties of these measures are the same as in the strong case (cf. [3], [7]).

Our definition 2.4. parallels that given in [4]. The properties of $(\mathcal{P}, \mathcal{B}, p)$-measures are similar to the properties of De Blasi's measure [11] and are dependent on the properties of $\mathcal{P}, \mathcal{B}$ and p. Our definition of $(\mathcal{P}, \mathcal{B}, p)$-measures of weak noncompactness allows us to look on measures of strong and of weak noncompactness as on similar objects. Thus all proofs of their properties run as in the strong case (see [7]).

Moreover, if by $\bar{\mathcal{N}}$ we denote a subfamily of all weakly closed subsets of $\mathcal{N}$ and by $\mathcal{B}^0$ a family $\{rB_1 : r > 0\}$, then the $(\bar{\mathcal{N}}, \mathcal{B}^0, \|\cdot\|)$-mwnc coincides with De Blasi's measure.

Denote by $L(E)$ the space of all continuous linear operators from E to E with the norm $|\cdot|$. Now we can prove some properties of $(\mathcal{P},\mathcal{B},p)$–measures of weak noncompactness, which are very important. In the sequel we will use the following lemmas.

Lemma 2.1. *Denote by μ an $(\mathcal{N},\mathcal{B},p)$-*mwnc *where:*
(i) *$\mathcal{B}$ is composed of balanced sets,*
(ii) *$p(kV)=k\cdot p(V)$, for each $V\in\mathcal{B}'$ and $k\in\mathbb{R}$.*
Thus for each bounded subset W of E and for each $L\in L(E)$ we have:

$$\text{(L)}\qquad \mu(LW)\le |L|\cdot\mu(W)\,.$$

Proof. Fix an arbitrary $\varepsilon>0$. Let $H\in\mathcal{N}$ and $V\in\mathcal{B}'$ be such that $W\subset H+V$ with $p(V)\le\mu(W)+\varepsilon$. Therefore $LW\subset LH+LV\subset H'+|L|\cdot V$, where $H'=LH\in\mathcal{N}$. Moreover

$$p\left(|L|\cdot V\right)=|L|\cdot p(V)\le |L|\cdot(\mu(W)+\varepsilon)=|L|\cdot\mu(W)+|L|\cdot\varepsilon\,.$$

As ε is arbitrary we obtain our assertion.

Lemma 2.2. *Let μ be as above. If K is a continuous mapping from a compact interval I of $\mathbb{R}$ into $L(E)$ and W is a bounded subset of E then*

$$\mu\left(\bigcup_{t\in I}K(t)W\right)\le\sup_{t\in I}|K(t)|\cdot\mu(W)\,.$$

Proof. As W is bounded there exists $b>0$ such that $\|W\|\le b$. Fix an arbitrary $\varepsilon>0$. Let $W\subset P+V$ for some $P\in\mathcal{N}$ and $V\in\mathcal{B}'$, $p(V)\le\mu(W)+\varepsilon$. Put $U=\varepsilon\cdot V$, so $U\in\mathcal{B}'$. Let $\delta>0$ be such that $B(0,\delta)=\{x\in E:\|x\|\le\delta\}\subset U$. Divide the interval I in such that a way that $t_1<t_2<\dots<t_n$ with $|K(t_1)-K(t_{i-1})|<\delta/b$ (by continuity of K). For $t\in[t_{i-1},t_i]=:I_i$, letting $K(t)W\div K(t_i)W:=(K(t)-K(t_i))W=\{K(t)w-K(t_i)w:w\in W\}$ we obtain

$$K(t)W\subset(K(t)W\div K(t_i)W)+K(t_i)W\,.$$

But $\|K(t)W\div K(t_i)W\|\le(\delta/b)\cdot b=\delta$, so $K(t)W\div K(t_i)W\subset B(0,\delta)\subset U$. Since $W\subset P+V$, thus by Lemma 2.1 we get

$$K(t_i)W\subset K(t_i)P+K(t_i)V\subset K(t_i)P+\sup_{t\in I}|K(t)|\cdot V\,.$$

Now we get

$$\bigcup_{t\in I}K(t)W=\bigcup_{i=1}^{n}\bigcup_{t\in I_i}K(t)W\subset$$

$$\subset \bigcup_{i=1}^{n} \bigcup_{t\in I_i} [(K(t)W \div K(t_i)W) + K(t_i)W] \subset$$

$$\subset \bigcup_{i=1}^{n} \left[U + K(t_i)P + \sup_{t\in I} |K(t)| \cdot V \right] \subset$$

$$\subset U + \sup_{t\in I} |K(t)| \cdot V + \bigcup_{i=1}^{n} K(t_i)P =$$

$$= P' + \left(\varepsilon + \sup_{t\in I} |K(t)| \right) \cdot V \quad \text{(by convexity of } V) ,$$

where $P' = \bigcup_{i=1}^{n} K(t_i)P$.

Consequently

$$p\left(\left(\varepsilon + \sup_{t\in I} |K(t)| \right) \cdot V \right) = \left(\varepsilon + \sup_{t\in I} |K(t)| \right) \cdot p(V) =$$

$$= \left(\varepsilon + \sup_{t\in I} |K(t)| \right) \cdot (\mu(W) + \varepsilon) \ ,$$

and since ε is arbitrary we obtain

$$\mu \left(\bigcup_{t\in I} K(t)W \right) \le \sup_{t\in I} |K(t)| \cdot \mu(W) \, .$$

The above properties of measures of weak noncompactness are very useful in the theory of linear differential equations. Property (L) from Lemma 2.1. does not follow from other properties of such measures and in the case of a purely axiomatic approach to this problem it is necessary to assume our (L)-property (cf. [21]).

Now we can state a theorem on the existence of bounded weak solutions of nonlinear differential equations of the form:

$$x'(t) = A(t)x(t) + f(t, x(t)), \qquad t \in \mathbb{R}_+ \tag{1}$$

where $A(t) \in L(E)$ for $t \in \mathbb{R}_+$, x' denotes the weak derivative of x, $(t,x) \to f(t,x)$ is a function from $\mathbb{R}_+ \times B_r$ into E which is weakly – weakly continuous (i.e. continuous with respect to the weak topologies on B_r and E) and $B_r = \{x \in E : \|x\| \le r\}$.

In this chapter we use some notations and definitions from the book of Massera–Schäffer [19].

Assume that $E_w = (E, \sigma(E, E^*))$ is sequentially complete. Moreover, we introduce the following notations: $L = L(\mathbb{R}_+, E)$ is the space of all measurable functions $u : \mathbb{R}_+ \to E$ integrable in the Bochner sense on every finite subinterval I of $\mathbb{R}_+$, with the topology of convergence in the mean on every such I, i.e., the convergence in $L^1(I,E)$ of the restrictions to I.

Let $B = B(\mathbb{R}_+, E)$ be a Banach space of measurable functions $x : \mathbb{R}_+ \to E$ such that $\|x\| \in B(\mathbb{R}_+, \mathbb{R})$ with the norm $\|x\|_B = \| \, \|x\| \, \|_{B(\mathbb{R})}$, where $B(\mathbb{R}) = B(\mathbb{R}_+, \mathbb{R})$ is a function space such that:

(i) $B(\mathbb{R}_+, \mathbb{R}) \subset L(\mathbb{R}_+, \mathbb{R})$ and $B(\mathbb{R}_+, \mathbb{R})$ is stronger than $L(\mathbb{R}_+, \mathbb{R})$,

(ii) $B(\mathbb{R}_+, \mathbb{R})$ contains all essentially bounded functions with compact support,

(iii) if $u \in B(\mathbb{R}_+, \mathbb{R})$, $v : \mathbb{R}_+ \to \mathbb{R}$ is measurable and $|v| \le |u|$, then $v \in B(\mathbb{R}_+, \mathbb{R})$ and $\|v\|_{B(\mathbb{R})} \le \|u\|_{B(\mathbb{R})}$,

(iv) if $u \in B(\mathbb{R}_+, \mathbb{R})$, $v_n \in L(\mathbb{R}_+, \mathbb{R})$ $(n \in \mathbb{N})$, $|v_n| \le |u|$ and $\lim\limits_{n\to\infty} v_n(t) = 0$, a.e. on $\mathbb{R}_+$, then $\lim\limits_{n\to\infty} \|v_n\|_{B(\mathbb{R})} = 0$.

Let B' denote the space associate to B (cf. [19], p. 50).

Remark that as examples of such spaces we can consider some Orlicz–spaces (see [19], [22]).

Let E_0 be the subspace consisting of all points of E which are values for $t = 0$ of bounded weak solutions of the linear differential equation

$$x' = A(t)x \, . \tag{2}$$

We assume that E_0 is closed and has a closed complement E_1, i.e., E is the direct sum of E_0 and E_1. Take the Green function for (2):

$$G(t,s) = \begin{cases} U(t)PU^{-1}(s) & \text{for } 0 \le s \le t \\ -U(t)(Id - P)U^{-1}(s) & \text{for } 0 \le t \le s \, , \end{cases}$$

where $U : \mathbb{R}_+ \to L(E)$ is the solution of the differential equation

$$U' = A(t)U, \qquad U(0) = Id \, ,$$

and P is the projection of E onto E_0 with $\ker P = E_1$ (cf. [18]). Thus there exists a positive number N such that $\|U(t)P\| \le N$ for every $t \in \mathbb{R}_+$, i.e., $\|G(t,0)\| \le N$.

Furthermore, we assume that:

(A0) $f : \mathbb{R}_+ \times B_r \to E$ is weakly–weakly continuous,

(A1) $A : \mathbb{R}_+ \to L(E)$ is strongly measurable and Bochner integrable on every finite subinterval of $\mathbb{R}_+$,

(A2) $G(t, \cdot) \in B'$ and $\|G(t, \cdot)\|_{B'} \le k$ for every $t \in \mathbb{R}_+$, (A3) there exists $m \in B'$ such that $k \cdot \|m\|_B < r$ and $\|f(t,x)\| \le m(t)$ for every $(t,x) \in \mathbb{R}_+ \times B_r$,

(A4) $q : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function such that $q(0) = 0,\ q(u) < u$ for all $u > 0$,

(A5) there exists $h \in B$ such that $k \cdot \|h\|_B < 1$ and $\forall a > 0,\ \forall \varepsilon > 0,$ $\forall X \subset B_r\ \exists$ open $Z \subset [0,a]$, $\mathrm{mes}(Z) < \varepsilon$ and

$$\gamma(f(T \times X)) \le \sup\{h(t) : t \in T\} \cdot q(\gamma(X))$$

for every compact subset T of $[0,a] \setminus Z$, where γ is a $(\mathcal{P}, \mathcal{B}, p)$–measure of weak noncompactness satisfying the following assumptions:

(M1) $$\mathcal{P} = \mathcal{N}\,,$$

(M2) $$\forall U, V \in \mathcal{B}' \ \ \forall k \in \mathbb{R} \ \ U + V \in \mathcal{B}', \ \ k \cdot V \in \mathcal{B}' \text{ and}$$
$$p(U+V) \le p(U) + p(V), \ \ p(k \cdot V) = |k| \cdot p(V)\,,$$

(M3) $$\forall U, V \in \mathcal{B}' \ \ \exists W \in \mathcal{B}' \ \ U, V \subset W,$$
$$p(W) = P(U) \text{ or } p(W) = p(V)$$

(cf. [6]).

Theorem 2.2. *If* (A0)–(A5) *hold then for every* $x_0 \in E_0$ *such that* $\|x_0\| \le (r - k\|m\|_B)/N$ *there exists at least one bounded weak solution of* (1) *with* $Px(0) = x_0$.

We omit the proof, because it runs as in [6]. Looking at previous of this type ([6], [10], [21], [23] for instance), our example of an application of $(\mathcal{P}, \mathcal{B}, p)$-measures of weak noncompactness sheds some light on the main idea of the construction of such measures. In fact, we can enlarge a class of measures of weak noncompactness for which (1) has a bounded weak solution on $\mathbb{R}_+$ (cf. [6], [21], [24]).

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