

Nonlinear contractions in metrically convex space

By JANUSZ MATKOWSKI (Bielsko-Biała)

Abstract. In this paper we prove among other things the following fixed point theorem. Let T be a selfmapping of a complete Menger convex metric space (X, d) and $\psi : [0, \infty) \rightarrow [0, \infty)$ a function such that

$$d(T(x), T(y)) \leq \psi(d(x, y)), \quad (x, y \in X).$$

Suppose that ψ is continuous at 0 and that there exists a positive sequence t_n , ($n \in \mathbb{N}$), such that $\lim_{n \rightarrow \infty} t_n = 0$ and $\psi(t_n) < t_n$, ($n \in \mathbb{N}$). Then T has a unique fixed point. Moreover T is γ -contractive for an increasing concave function γ and such that $\gamma(t) < t$ for all $t > 0$.

An application to a functional equation is also given.

Introduction

Let (X, d) be a metric space and $T : X \rightarrow X$ a selfmapping of X . If there exists a function $\psi : [0, \infty) \rightarrow [0, \infty)$ such that

- 1°. $d(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$;
- 2°. $\psi(t) < t$ for every $t > 0$,

then we say that T is ψ -contractive.

A metric space (X, d) is said to be *Menger convex* or *metrically convex* iff for every $x, y \in X$, $x \neq y$, there is $z \in X$ such that $x \neq z \neq y$ and

$$d(x, y) = d(x, z) + d(z, y).$$

Let T be a ψ -contractive selfmap of a Menger convex metric space. D. W. BOYD and J. S. W. WONG [4] proved that T has a unique fixed

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point. Moreover, there exists a subadditive and right continuous function $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that T is γ -contractive. The last statement of the Boyd and Wong result has been improved by C. S. Wong [12] who showed that T is γ -contractive with γ such that the function $t \rightarrow \gamma(t)/t$ is nonincreasing in $(0, \infty)$. Afterwards J. MATKOWSKI and R. WĘGRZYK [8] proved that T is γ -contractive with an increasing and concave function γ .

In this paper we prove the following generalization of the result of D. W. BOYD and J. S. W. WONG.

Let T be a selfmapping of a complete Menger convex metric space (X, d) and $\psi : [0, \infty) \rightarrow [0, \infty)$ a function such that

$$d(T(x), T(y)) \leq \psi(d(x, y)), \quad (x, y \in X).$$

If ψ is continuous at 0 and there exists a positive sequence t_n , ($n \in \mathbb{N}$), such that

$$\lim_{n \rightarrow \infty} t_n = 0, \quad \psi(t_n) < t_n, \quad (n \in \mathbb{N}),$$

then T has a unique fixed point $a \in X$ and $\lim_{n \rightarrow \infty} T^n(x) = a$ for every $x \in X$. Moreover there exists an increasing and concave function $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that T is γ -contractive.

The arguments of the present paper strongly depend on some properties of subadditive functions discussed in section 1. We wish to emphasise that, due to them, the proof of the above result is short and elementary.

In section 4, as a consequence of our main result, we obtain the following theorem. Let T be a uniformly continuous selfmapping of a nonempty closed convex subset X of a Banach space. If for a positive sequence t_n , ($n \in \mathbb{N}$), with $\lim_{n \rightarrow \infty} t_n = 0$ we have

$$\sup \{ \|T(x) - T(y)\| : \|x - y\| = t_n; x, y \in X \} < t_n, \quad (n \in \mathbb{N}),$$

then T has a unique fixed point. Moreover T is γ -contractive for an increasing and concave function γ .

In the same section we improve some results obtained by NADIM A. ASSAD and W. A. KIRK in [1] and [2].

At the end of this paper we apply the main result to the theory of integrable solutions of a nonlinear iterative functional equation. The concavity of γ plays there an important role.

Let us also mention that using the methods applied in this paper one can generalize the BROWDER–GOEHDE–KIRK fixed point theorem for nonexpansive mappings (cf. [11]).

1. Some remarks on subadditive functions

A function $\lambda : [0, \infty) \rightarrow [0, \infty)$ is said to be *subadditive* if

$$\lambda(s + t) \leq \lambda(s) + \lambda(t), \quad (s, t \geq 0).$$

The following result plays an important role in this paper.

Proposition 1. *Let $\lambda : (0, \infty) \rightarrow [0, \infty)$ be subadditive and let $g : (0, \infty) \rightarrow [0, \infty)$ be defined by*

$$g(t) := \frac{\lambda(t)}{t}, \quad t > 0.$$

If $\lim_{t \rightarrow 0} \lambda(t) = 0$ then

- a) there exists $g(0) := \lim_{t \rightarrow 0} g(t)$ and $g(0) = \sup_{t > 0} g(t)$;
- b) there exists $g(\infty) := \lim_{t \rightarrow \infty} g(t)$ and $g(\infty) = \inf_{t > 0} g(t)$;
- c) for every positive r there exist the one-sided limits $g(r-)$, $g(r+)$ and $g(r+) \leq g(r) \leq g(r-)$. If moreover $g(0) < \infty$ and there is an $s > 0$ such that $g(s) < g(0)$ then for every $r \in (s, \infty)$ we have

$$g(r-) < g(0).$$

PROOF. Part a) is a reformulation of Theorem 7.11.1 in [6]. Part b) follows from Theorem 7.6.1 in [6] and c) is an immediate consequence of Theorem 7.8.3 in [6]. To prove the last statement of the proposition note that $g(t) \leq g(0)$ for all $t > 0$. Now the definition of g and subadditivity of λ imply that

$$\lambda(t) \leq \lambda(s) + \lambda(t - s) \leq \lambda(s) + g(0)(t - s), \quad t \in (s, r).$$

Letting here t tend to r , we obtain $\lambda(r-) \leq \lambda(s) + g(0)(r - s)$. Making use of the inequality $g(s) < g(0)$, (i.e. $\lambda(s) < g(0)s$), we hence get $\lambda(r-) < g(0)r$, which means that $g(r-) < g(0)$. This completes the proof.

Corollary 1. *Let $\lambda : [0, \infty) \rightarrow [0, \infty)$ be a subadditive and continuous at 0. Suppose that there exist $c > 0$ and $t_n > 0$, ($n \in \mathbb{N}$), such that*

$$\lim_{n \rightarrow \infty} t_n = 0, \quad \lambda(t_n) < ct_n, \quad (n \in \mathbb{N}).$$

Then the function $\mu : (0, \infty) \rightarrow [0, \infty)$ defined by the formula

$$\mu(t) := \sup \left\{ \frac{\lambda(u)}{u} : u > t \right\}$$

is decreasing and

$$\mu(t) < c, \quad (t > 0).$$

In particular we have $\lambda(t) < ct$, $(t > 0)$.

PROOF. It is enough to note that $g(0) \leq c$ and apply Proposition 1 with $s := t_n$, $(n \in \mathbb{N})$.

Corollary 2. *If $\lambda : (0, \infty) \rightarrow [0, \infty)$ is subadditive, moreover there exists a $c > 0$ such that $\lambda(t) \leq ct$, $t > 0$, and $\limsup_{t \rightarrow r} \lambda(t) = cr$ for some positive r , then $\lambda(t) = ct$ for all $t \in (0, r)$.*

Remark 1. The above Proposition 1 shows that every subadditive function $\lambda : (0, \infty) \rightarrow [0, \infty)$ such that $\lambda(t) < ct$, $(t > 0)$, for some $c > 0$ satisfies the following condition: for any $s > 0$ we have

$$\sup \left\{ \frac{\lambda(t)}{t} : t > s \right\} < c.$$

Taking $c = 1$ we hence obtain a negative answer to the problem posed by D. W. BOYD and J. S. W. WONG at the end of the paper [4]. A longer argument is given in [12].

In the sequel we need the following

Lemma 1. *Suppose that $\lambda : (0, \infty) \rightarrow [0, \infty)$ and $c > 0$. If*

$$\sup \left\{ \frac{\lambda(t)}{t} : t > s \right\} < c$$

for every $s > 0$ then there exists an increasing and concave function $\gamma : (0, \infty) \rightarrow [0, \infty)$ such that $\lambda(t) \leq \gamma(t) < ct$, $(t > 0)$.

PROOF. Denote by \mathcal{L} the family of all the functions $\ell : (0, \infty) \rightarrow [0, \infty)$ of the form $\ell(t) = at + b$, $(a, b \geq 0)$, such that $\lambda(t) \leq \ell(t)$ for every $t > 0$ and put $\gamma(t) := \inf_{\ell \in \mathcal{L}} \ell(t)$.

For every $\alpha, \beta > 0$; $\alpha + \beta = 1$, and $u, v > 0$ we have

$$\begin{aligned} \gamma(\alpha u + \beta v) &= \inf_{\ell \in \mathcal{L}} \ell(\alpha u + \beta v) = \inf_{\ell \in \mathcal{L}} (\alpha \ell(u) + \beta \ell(v)) \geq \\ &\geq \alpha \inf_{\ell \in \mathcal{L}} \ell(u) + \beta \inf_{\ell \in \mathcal{L}} \ell(v) = \alpha \gamma(u) + \beta \gamma(v), \end{aligned}$$

which shows that γ is concave in $(0, \infty)$. Since the function $\ell(t) := ct$, $(t > 0)$, belongs to \mathcal{L} , we have

$$\gamma(t) \leq ct, \quad (t > 0).$$

Now we have to show that the set $A := \{t > 0 : \gamma(t) = ct\}$ is empty. By assumption there are $s > 0$ and k , $0 < k < c$, such that $\lambda(t) \leq kt$ for $t > s$, and therefore,

$$\lambda(t) \leq k(t - s) + cs = kt + (c - k)s, \quad (t > 0).$$

In view of the definition of γ we have $\gamma(t) < c(t - s) + cs = ct$ for $t > s$. This proves that $A \subset (0, s)$. Suppose for an indirect proof that $A \neq \emptyset$ and put $t_0 := \sup A$. The concavity of γ and the inequality $\gamma(t) \leq ct$, ($t > 0$), imply that

$$\gamma(t) = ct, \quad (0 < t \leq t_0); \quad \gamma(t) < ct, \quad (t > t_0).$$

According to the assumption there are t_1, t_2 ; $t_1 < t_0 < t_2$, such that

$$\lambda(t) < ct_1, \quad (t_1 \leq t \leq t_2).$$

Let $m := \max\{ct_1, \gamma(t_2)\}$. By the concavity of γ the function

$$\ell(t) := \frac{m - ct_1}{t_2 - t_1}(t - t_1) + ct_1, \quad (t > 0),$$

satisfies the inequality $\lambda(t) \leq \ell(t)$ for all $t > 0$ and, consequently,

$$\gamma(t_0) \leq \ell(t_0) = \frac{m - ct_1}{t_2 - t_1}(t_0 - t_1) + ct_1 < c(t_0 - t_1) + ct_1 = ct_0.$$

This contradiction shows that $A = \emptyset$. Since every $\ell \in \mathcal{L}$ is increasing it follows that so is γ . This completes the proof.

2. A fixed point theorem for an arbitrary complete metric space

In this section we present the following

Proposition 2. *Let (X, d) be an arbitrary complete metric space and $T : X \rightarrow X$ a selfmapping of X . Suppose that $\lambda : [0, \infty) \rightarrow [0, \infty)$ is continuous at 0, subadditive and there exists a sequence $t_n > 0$, ($n \in \mathbb{N}$), such that*

$$\lim_{n \rightarrow 0} t_n = 0, \quad \lambda(t_n) < t_n, \quad (n \in \mathbb{N}).$$

If

$$d(T(x), T(y)) \leq \lambda(d(x, y)), \quad (x, y \in X),$$

then T has a unique fixed point $a \in X$ and $\lim_{n \rightarrow \infty} T^n(x) = a$ for every $x \in X$. Moreover T is a γ -contractive with an increasing and concave γ .

PROOF. By Corollary 1 and Lemma 1 the mapping T is a γ -contractive with an increasing and concave γ . Denote by γ^n the n th iterate of

γ . Since $\gamma(t) < t$ for $t > 0$, we have

$$\lim_{n \rightarrow \infty} \gamma^n(t) = 0, \quad (t > 0).$$

Now the existence of a unique fixed point of T and the convergence of every sequence of successive approximations follows from [9] p. 8, Theorem 1.2, (cf. also J. DUGUNDJI, A. GRANAS [5] p. 12, Theorem 3.2, and [10], Theorem 2).

3. A family of nonlinear contractions in Menger convex space

We begin this section with the following well known result of MENGER (cf. [3], p. 41).

Lemma 2. *If (X, d) is a complete and Menger convex metric space then any two points are the endpoints of at least one metric segment. More precisely, for every $x, y \in X$, $x \neq y$, there exists a function $F : [0, d(x, y)] \rightarrow X$ such that*

$$F(0) = x, \quad F(d(x, y)) = y$$

and for every $s, t \in [0, d(x, y)]$ we have

$$d(F(s), F(t)) = |s - t|.$$

In particular, for every $x, y \in X$ and $\alpha \in (0, 1)$ there is $z \in X$ such that

$$d(x, z) = \alpha d(x, y), \quad d(z, y) = (1 - \alpha)d(x, y).$$

By this Lemma $P := d(X \times X)$, the range of the metric d , is an interval of the form $[0, b)$, $(0 \leq b \leq \infty)$, or $[0, b]$, $(0 \leq b < \infty)$.

Now, applying Lemma 1, Lemma 2 and Corollary 1, we can prove the following basic

Proposition 3. *Let (X, d) be a complete Menger-convex metric space, (Y, ρ) a metric space, $T_\iota : X \rightarrow Y$, $(\iota \in I)$, a family of mappings and $\psi : [0, \infty) \rightarrow [0, \infty)$ continuous at 0. Suppose that there exist $c > 0$ and a positive sequence t_n , $(n \in \mathbb{N})$, such that*

$$\lim_{n \rightarrow \infty} t_n = 0, \quad \psi(t_n) < ct_n, \quad (n \in \mathbb{N}).$$

If

$$\rho(T_\iota(x), T_\iota(y)) \leq \psi(d(x, y)), \quad (x, y \in X; \iota \in I),$$

then there exists an increasing concave function $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that $\gamma(t) < ct$, $(t > 0)$, and

$$\rho(T_\iota(x), T_\iota(y)) \leq \gamma(d(x, y)), \quad (x, y \in X; \iota \in I).$$

PROOF. Let us define a function $\lambda : [0, \infty) \rightarrow [0, \infty)$ by the formula

$$\lambda(t) := \sup \{ \rho(T_\iota(x), T_\iota(y)) : x, y \in X, \iota \in I, d(x, y) = t \},$$

$$t \in P = d(X \times X),$$

and, if $\bar{P} = [0, b]$ with $b < \infty$, we put

$$\lambda(t) := 0, \quad t \in [0, \infty) \setminus P.$$

Applying an idea of BOYD and WONG, (cf. [4], Lemma 2), we first prove that λ is subadditive, i.e. that $\lambda(s + t) \leq \lambda(s) + \lambda(t)$, ($s, t \geq 0$). Take arbitrary $s, t \geq 0$. This inequality is obviously true if $b < \infty$ and $s + t \in [0, \infty) \setminus P$. Suppose that $s + t \in P$. Thus $s + t = d(x, y)$ for some $x, y \in X$ and, in view of Lemma 2, there exists a point $z \in X$ such that $d(x, z) = s$ and $d(z, y) = t$. Then we clearly have

$$\rho(T_\iota(x), T_\iota(y)) \leq \rho(T_\iota(x), T_\iota(z)) + \rho(T_\iota(z), T_\iota(y)) \leq \lambda(s) + \lambda(t)$$

for all $\iota \in I$. Now, taking the supremum over all $x, y \in X$ with $d(x, y) = s + t$ and $\iota \in I$, we get $\lambda(s + t) \leq \lambda(s) + \lambda(t)$.

By the definition of λ we clearly have

$$\lambda(t) \leq \psi(t) \leq ct, \quad (t \geq 0).$$

Moreover, according to the assumption, we hence get

$$\lambda(t_n) \leq \psi(t_n) < ct_n, \quad (n \in \mathbb{N}).$$

Now the proposition results from Corollary 1 and Lemma 1.

4. Fixed point theorems in Menger convex space

We begin this section with the following

Theorem 1. *Let T be a selfmapping of a complete Menger convex metric space (X, d) and $\psi : [0, \infty) \rightarrow [0, \infty)$ a function such that*

$$d(T(x), T(y)) \leq \psi(d(x, y)), \quad (x, y \in X).$$

If ψ is continuous at 0 and there exists a positive sequence t_n , ($n \in \mathbb{N}$), such that

$$\lim_{n \rightarrow \infty} t_n = 0, \quad \psi(t_n) < t_n, \quad (n \in \mathbb{N}),$$

then T has a unique fixed point $a \in X$ and $\lim_{n \rightarrow \infty} T^n(x) = a$ for every $x \in X$. Moreover T is γ -contractive for an increasing and concave γ .

PROOF. Taking in Proposition 3 : $(Y, \rho) := (X, d)$, $T_\iota := T$, ($\iota \in I$), and $c := 1$ we infer that there exists an increasing concave $\gamma : [0, \infty) \rightarrow [0, \infty)$,

$\gamma(t) < t$, ($t > 0$), such that

$$d(T(x), T(y)) \leq \gamma(d(x, y)), \quad (x, y \in X).$$

Since $\lim_{k \rightarrow \infty} \gamma^k(t) = 0$ for every $t > 0$, the existence of a unique fixed point of T and the convergence of every sequence of successive approximations easily follows (cf. [9] p. 8, Theorem 1.2, [5] p. 12, Theorem 3.2, and [10], Theorem 2).

Theorem 2. *Let T be a uniformly continuous selfmapping of a complete Menger convex metric space (X, d) . If there exists a positive sequence t_n , ($n \in \mathbb{N}$), $\lim_{n \rightarrow \infty} t_n = 0$, such that*

$$\sup \{d(T(x), T(y)) : d(x, y) = t_n; x, y \in X\} < t_n, \quad (n \in \mathbb{N}),$$

then T has a unique fixed point $a \in X$ and $\lim_{n \rightarrow \infty} T^n(x) = a$ for every $x \in X$. Moreover T is γ -contractive for an increasing and concave γ .

PROOF. According to the assumptions, given $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for every $x, y \in X$, $d(x, y) < \delta(\varepsilon)$ implies $d(T(x), T(y)) < \varepsilon$. Take $\varepsilon := 1$, an arbitrary $t \in P := d(X \times X)$ and $x, y \in X$ such that $d(x, y) = t$. In view of Lemma 2 there exist $n=n(t) \in \mathbb{N}$ and $z_0, \dots, z_n \in X$ such that $z_0 = x$, $z_n = y$; $d(z_{i-1}, z_i) = n^{-1}d(x, y) < \delta(1)$. Hence

$$d(T(x), T(y)) \leq \sum_{i=1}^n d(T(z_{i-1}), T(z_i)) < n = n(t).$$

This proves that for every $t \in P$ the number

$$\psi(t) := \sup \{d(T(x), T(y)) : d(x, y) = t, x, y \in X\}$$

is finite. Put $\psi(t) := 0$ for $t \in (0, \infty) \setminus P$. Then $\psi : [0, \infty) \rightarrow [0, \infty)$ and

$$d(T(x), T(y)) \leq \psi(d(x, y)), \quad (x, y \in X).$$

By the uniform continuity of T the function ψ is continuous at 0. Moreover, according to the remaining assumption, we have $\psi(t_n) < t_n$ for all $n \in \mathbb{N}$. Now the result follows from Theorem 1.

Let us note the following obvious

Corollary 3. *Let T be a uniformly continuous selfmapping of a non-empty closed convex subset X of a Banach space. If for a positive sequence t_n , ($n \in \mathbb{N}$), with $\lim_{n \rightarrow \infty} t_n = 0$ we have*

$$\sup \{\|T(x) - T(y)\| : \|x - y\| = t_n; x, y \in X\} < t_n, \quad (n \in \mathbb{N}),$$

then T has a unique fixed point. Moreover T is γ -contractive for an increasing and concave function γ .

For a subset K of a metric space (X, d) denote by ∂K the boundary of K . NADIM A. ASSAD [2] proved the following theorem.

Let (X, d) be a complete Menger convex metric space and K a non-empty closed subset of X . Suppose that $T : K \rightarrow X$ satisfies the following condition: given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(T(x), T(y)) < \varepsilon, \quad (x, y \in K),$$

and $T(x) \in K$ for $x \in \partial K$. Then T has a unique fixed point in K .

Applying this results we prove the following

Theorem 3. Let (X, d) be a complete Menger convex metric space and K a nonempty closed and Menger convex subset of X . Suppose that $T : K \rightarrow X$ satisfies the following conditions: $T(x) \in K$ for $x \in K$ and

(*) there exist a continuous at 0 function $\psi; [0, \infty) \rightarrow [0, \infty)$ and a positive sequence $t_n, (n \in \mathbb{N})$, such that

$$\lim_{n \rightarrow \infty} t_n = 0, \quad \psi(t_n) < t_n, \quad (n \in \mathbb{N}),$$

and

$$d(T(x), T(y)) \leq \psi(d(x, y)), \quad (x, y \in K).$$

Then T has a unique fixed point in K . Moreover there exists an increasing and concave function $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that $\gamma(t) < t$ for $t > 0$ and

$$d(T(x), T(y)) \leq \gamma(d(x, y)), \quad (x, y \in K).$$

PROOF. Taking in Proposition 3 : $X := K, Y := X$ and the one-element family $\{T\}$ we get the existence of the function γ . The existence of a unique fixed point results from the above Assad theorem.

Remark 2. Let us note that the above theorem remains true on replacing (*) by each of the following conditions:

(**) T is uniformly continuous and there exists a positive sequence $t_n, (n \in \mathbb{N}), \lim_{n \rightarrow \infty} t_n = 0$ such that

$$\sup \{d(T(x), T(y)) : d(x, y) = t_n; x, y \in K\} < t_n, \quad (n \in \mathbb{N});$$

(***) given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < d(x, y) < \varepsilon + \delta \implies d(T(x), T(y)) \leq \varepsilon, \quad (x, y \in K).$$

For a metric space (X, d) denote by $(\mathcal{B}(X), D)$ the metric space of all nonempty bounded closed subsets of X with the Hausdorff metric D induced by d . NADIM A. ASSAD and W. A. KIRK [1] proved the following fixed point theorem for set-valued contractive mappings.

Let (X, d) be a complete Menger convex metric space, K a nonempty closed subset of X and $T : K \rightarrow \mathcal{B}(X)$ a mapping such that $T(x) \subseteq K$ for every $x \in \partial K$. If there is a constant $c < 1$ such that

$$D(T(x), T(y)) \leq cd(x, y), \quad (x, y \in K),$$

then there exists $a \in K$ such that $a \in T(a)$.

Using this results, Proposition 3 and Corollary 1 one can prove

Theorem 4. Let (X, d) be a complete Menger convex metric space, K a nonempty closed Menger convex subset of X and $T : K \rightarrow \mathcal{B}(X)$ a mapping such that $T(x) \subseteq K$ for every $x \in \partial K$. If there is a continuous at 0 function $\psi : [0, \infty) \rightarrow [0, \infty)$, a sequence $t_n > 0$, $(n \in \mathbb{N})$, and $c < 1$ such that

$$D(T(x), T(y)) \leq \psi(d(x, y)), \quad (x, y \in K),$$

and

$$\lim_{n \rightarrow \infty} t_n = 0, \quad \psi(t_n) \leq ct_n, \quad (n \in \mathbb{N}),$$

then there exists a point $a \in K$ such that $a \in T(a)$. Moreover

$$D(T(x), T(y)) \leq cd(x, y), \quad (x, y \in K).$$

5. An application to a functional equation

In this section we apply Proposition 3 and Theorem 1 to the theory of integrable solutions of the functional equation

$$(1) \quad \phi(x) = h(x, \phi[f(x)])$$

where ϕ is an unknown function. We assume that the given functions f and h satisfy the following hypotheses:

- (i) $f : [0, 1] \rightarrow [0, 1]$ is increasing and absolutely continuous;
- (ii) $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and
 - (a) for every $y \in \mathbb{R}$ the function $h(\cdot, y) : [0, 1] \rightarrow \mathbb{R}$ is measurable,
 - (b) for almost all $x \in [0, 1]$ the function $h(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous;

(iii) *there exist $\eta : [0, 1] \rightarrow (0, \infty)$ and $\psi : [0, \infty) \rightarrow [0, \infty)$ such that*

$$|h(x, y_1) - h(x, y_2)| \leq \eta(x)\psi(|y_1 - y_2|), \quad (x \in [0, 1]; y_1, y_2 \in \mathbb{R})$$

ψ is continuous at 0, and there exists a positive sequence t_n , ($n \in \mathbb{N}$), such that

$$\lim_{n \rightarrow \infty} t_n = 0, \quad \psi(t_n) < t_n, \quad (n \in \mathbb{N}).$$

Lemma 3. *If $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies (iii) then there exists an increasing concave function $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that*

$$|h(x, y_1) - h(x, y_2)| \leq \eta(x)\gamma(|y_1 - y_2|), \quad (y_1, y_2 \in \mathbb{R}),$$

$$\gamma(t) < t, \quad (t > 0).$$

PROOF. With the following specification: $X = Y := \mathbb{R}$; $I := [0, 1]$, ($\iota \equiv x$); and $T_\iota = T_x : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T_x(y) := [\eta(x)]^{-1}h(x, y)$, the lemma is an immediate consequence of Proposition 3.

In the sequel \mathbf{L}^1 stands for the Banach space of all the Lebesgue integrable functions $\phi : [0, 1] \rightarrow \mathbb{R}$.

Theorem 2. *Suppose that conditions (i)–(iii) are fulfilled. If*

$$h(\cdot, 0) \in \mathbf{L}^1 \quad \text{and} \quad \eta \leq f' \text{ a.e. in } [0, 1]$$

then equation (1) has exactly one solution $\phi \in \mathbf{L}^1$. Moreover, for every $\phi_0 \in \mathbf{L}^1$ the sequence of successive approximations $(\phi_n)_{n=0}^\infty$ given by

$$\phi_{n+1}(x) := h(x, \phi_n[f(x)]), \quad (n = 0, 1, \dots),$$

converges (in the \mathbf{L}^1 -norm) to ϕ .

PROOF. By (i)–(ii) and Caratheodory's theorem, the function

$$T(\phi)(x) := h(x, \phi[f(x)]), \quad (x \in [0, 1]),$$

is measurable for every $\phi \in \mathbf{L}^1$. Moreover for $\phi \in \mathbf{L}^1$ we have

$$|h(x, \phi[f(x)])| \leq \eta(x)|\phi[f(x)]| + |h(x, 0)|$$

and, consequently, making use of the inequality $\eta \leq f'$, we get

$$\begin{aligned} \int_I |T(\phi)(x)|dx &\leq \int_I f'(x)|\phi[f(x)]|dx + \int_I |h(x, 0)|dx = \\ &= \int_{f(I)} |\phi(x)|dx + \int_I |h(x, 0)|dx < \infty, \end{aligned}$$

which shows that $T : \mathbf{L}^1 \rightarrow \mathbf{L}^1$. Take arbitrary $\phi_1, \phi_2 \in \mathbf{L}^1$. Applying in turn Lemma 3, inequality $\eta \leq f'$ and the Jensen integral inequality for concave functions, (cf. M. KUCZMA [7], p. 181), we obtain

$$\begin{aligned} \|T(\phi_1) - T(\phi_2)\| &= \int_I |h(x, \phi_1[f(x)]) - h(x, \phi_2[f(x)])| dx \leq \\ &\leq \int_I \gamma(|\phi_1[f(x)] - \phi_2[f(x)]|) f'(x) dx = \int_{f(I)} \gamma(|\phi_1(x) - \phi_2(x)|) dx \leq \\ &\leq \int_I \gamma(|\phi_1(x) - \phi_2(x)|) dx \leq \gamma \left(\int_I |\phi_1(x) - \phi_2(x)| dx \right) = \gamma(\|\phi_1 - \phi_2\|). \end{aligned}$$

Thus the theorem follows from Theorem 1.

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JANUSZ MATKOWSKI
DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY
WILLOWA 2
43-300 BIELSKO-BIAŁA
POLAND

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