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On sumset of certain sets

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1. Introduction

In 1971 R. L. GRAHAM raised the following question: Let $\alpha, \beta > 0$ be real numbers and put

$$A_{\alpha\beta} = \{[2^n\alpha], [2^n\beta] \mid n \in \mathbb{N}\}.$$

For which pairs of (α, β) is the sequence $A_{\alpha\beta}$ complete?

We say that A is complete if every sufficiently large integer belongs to $P(A) = \{\sum \varepsilon_i a_i \mid a_i \in A;\ \varepsilon_i = 0 \text{ or } 1\}$. An infinite sequence of integers is said to be subcomplete if it contains an infinite arithmetic progression.

It became clear that the structure of $P(A_{\alpha\beta})$ depends on the dyadic representations of α and β.

Let $\rho > 0$ and let $\rho = \sum\limits_{i=-k}^{\infty} \varepsilon_i(\rho) \cdot 2^{-i}$; $\varepsilon_i(\rho) \in \{0,1\}$ and assume that $\varepsilon_i(\rho) = 0$ infinitely many times. Let us call a real number ρ an infinite diadical fraction (briefly IDF) if $\varepsilon_i(\rho) = 1$ infinitely many times, otherwise let us call ρ a finite diadical fraction (briefly FDF). Actually we can distinguish three cases:

Definition 1. Let us say that the type of $A_{\alpha\beta}$ is

(a) $\mathcal{F}$ if α and β are FDF
(b) $\mathcal{M}$ if α is FDF and β is IDF
(c) $\mathcal{I}$ if α and β are IDF.

In [3] I proved that if the type of $A_{\alpha\beta}$ is $\mathcal{M}$ then $A_{\alpha,\beta}$ is complete and I showed in [4] that if $\alpha > 0$ and $X_\alpha = \{\beta \mid A_{\alpha\beta} \text{ is complete}\}$ then $\mu(X_\alpha) = 0$ or ∞ (μ is the Lebesgue-measure). I also proved in [3]:

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Theorem A. *There are continuum many pairs of (α, β) for which $A_{\alpha\beta}$ is not complete.*

We are going to sharpen this result in section 3, proving the following

Theorem 1. *There are continuum many pairs of (α, β) for which $A_{\alpha\beta}$ is not subcomplete.*

Furthermore we would like to pay a dept: we shall investigate $A_{\alpha\beta}$ if it has type $\mathcal{F}$.

Definition 2. Let α, γ be FDF. Let

(1) $m^*_\gamma = \min\{m \mid \forall k > m \implies \varepsilon_k(\gamma) = 0\}$ and let

(2) $g_\alpha(m) = |\{B \mid m^*_\beta = m \text{ and } A_{\alpha\beta} \text{ is complete}\} \,|/2^m$.

Actually the function $g_\alpha(m)$ counts those β's for which $A_{\alpha\beta}$ is complete and has type $\mathcal{F}$. In section 3 we prove

Theorem 2. *Let $\alpha > 0$ be* FDF. *Then*

$$\lim_{m\to\infty} g_\alpha(m) = 1.$$

2. The structure of $P(A_\alpha)$

The second aim of this note is to investigate the structure of $P(A_\alpha)$. It is easy to see that A_α is subcomplete if and only if α is FDF.

Definition 3. Let

$$f_\alpha(x) = \max\{L \mid \exists y \le x - L \text{ for which } \forall t,\ 1 \le t \le 1,\ y + t \notin P(A_\alpha)\}.$$

In other words $f_\alpha(x)$ is the biggest gap in $P(A_\alpha) \cap [1, x]$. In section 4 we prove the following theorem:

Theorem 3. (1) $\limsup_{x\to\infty} f_\alpha(x)/\log_2 x \le 1$.

(2) *For almost all α we have* $\lim_{x\to\infty} f_\alpha(x) = 1/2$.

(3) *Let*

$$G_A(\eta, x) = G(\eta, x) = \Big\{\alpha \mid A - 1 \le \alpha < A \text{ and } \left(f_\alpha(x) - \frac{\log_2 x}{2}\right) \Big/ \left(\sqrt{\log x}\,/2\right) \le \eta\Big\}.$$

Then

$$\lim_{x\to\infty} \mu(G(\eta, x)) = \Phi(\eta),$$

where $\Phi(\eta) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\eta} e^{-t^2/2} dt.$

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Let $a_n = [2^n\alpha]$ and let $x_n = a_0 + a_1 + \dots + a_n + 1$.

Lemma 1. *Let* $\alpha \geq 2$ *and* $\beta = 2^n\alpha$ $(n \in \mathbb{N})$. *Then* $x_n \notin P(A_{\alpha\beta})$ *for every* $n \in \mathbb{N}$.

See the proof of Theorem 2 in [3].

Thus by Lemma 1 we only have to show that there are continuum many α such that for every $n \in \mathbb{N}$ and $0 \leq r < n$ there is an m for which $x_m \equiv r \pmod{n}$. Let us arrange the set of the arithmetical progressions according to their modulus to a non-decreasing sequence. Let us assume, that we have defined the digits

$$\varepsilon_1(\alpha), \varepsilon_2(\alpha), \dots, \varepsilon_{N-1}(\alpha)$$

$N = N(n)$, so that for every m, $1 \leq m \leq n$ and for every s, $0 \leq s < m$ there are $r \leq N-1$ and $k \in \mathbb{N}$ for which $x_r = k \cdot m + s$.

Now let us choose $\varepsilon_N(\alpha)$ equal to 0 or 1 arbitrarily.

Lemma 2. *Let* $N, m \in \mathbb{N}$. *Then*

$$\begin{aligned} x_{N+m} - x_{N-1} &= a_N + a_{N+1} + \dots + a_{N+m} = \\ &= a_N(2^{m+1}-1) + \varepsilon_{N+1}(\alpha)(2^m - 1) + \\ &+ \varepsilon_{N+2}(\alpha)(2^{m-1}-1) + \dots + \varepsilon_{N+m}(\alpha). \end{aligned}$$

Proof of Lemma 2. It is easy to check that if $\alpha = a_0 + \sum\limits_{i=1}^{\infty} \varepsilon_i(\alpha) \cdot 2^{-i}$ then

$$a_{n+1} = 2 \cdot a_n + \varepsilon_{n+1}(\alpha). \tag{1.1}$$

This implies that for every $k \in \mathbb{N}$

$$a_{N+k} = 2^k \cdot a_N + \varepsilon_{N+1}(\alpha) \cdot 2^{k-1} + \varepsilon_{N+2}(\alpha) \cdot 2^{k-2} + \dots + \varepsilon_{N+k}(\alpha).$$

Thus we get

$$\begin{aligned} x_{N+m} - x_{N-1} &= \sum_{h=0}^{m} a_{N+h} = \sum_{h=0}^{m} \left(2^h \cdot a_N + \sum_{k=1}^{h} \varepsilon_{N+h}(\alpha) \cdot 2^{k-h} \right) = \\ &= a_N \cdot (2^{m+1} - 1) + \varepsilon_{N+1}(\alpha) \cdot (2^m - 1) + \dots + \varepsilon_{N+m}(\alpha) \end{aligned}$$

as we asserted.

Let now $m = n^3$. In the next step we show that there is an $u \in \mathbb{N}$ for which $(u, n) = 1$ and the congruence

$$2^x - 1 \equiv u \pmod{n} \tag{1.2}$$

has infinitely many solutions. If $n = 2^t$ then for every $x \in \mathbb{N}$ $(2^x-1, n)=1$ and by the pigeonhole principle for some u (1.2) has infinitely many solutions. Now let $n = 2^t \cdot z$, $z = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \dots p_r^{\alpha_r} > 1$. Then for every i

$$2^{s\cdot\phi(z)+1} - 1 \equiv 2 \cdot (2^{\phi(z)})^s - 1 \equiv 2 - 1 \equiv 1 \pmod{p_i}$$

and by $2 \nmid 2^x - 1$ we get $(2^{s\cdot\phi(z)+1} - 1, n) = 1$. Furthermore let us note that if $a > t$ then for every $i \in \mathbb{N}$

$$2^a - 1 \equiv 2^{a+i\cdot\phi(z)} - 1 \pmod{n}.$$

Let $U = \{(t+i)\cdot\phi(z) + 1 \mid i \in \mathbb{N}\} = \{u_1 < u_2 < \dots\}$. Thus if $x \in U$ then x is a solution of (1.2). Clearly $m > \max_{1\le i\le n} \{u_i\}$.

Now we are going to prove that there are m-tuples of digits

$$\varepsilon_{N+1}(\alpha), \varepsilon_{N+2}(\alpha), \dots, \varepsilon_{N+m}(\alpha)$$

for which

$$x_{N+m} - x_{N-1} \equiv r - x_{N-1} - a_N \cdot (2^{m+1} - 1) \pmod{n}.$$

Since $(u, n) = 1$, there is an y, $1 \le y < n$ for which

$$y \cdot u \equiv r - x_{N-1} - a_N \cdot (2^{m+1} - 1) \pmod{n}. \tag{1.3}$$

Now if

$$\varepsilon_{N+h}(\alpha) = \begin{cases} 1 \text{ if } \exists i & \text{for which } h = m - u_i + 1 \\ 0 & \text{otherwise} \end{cases}$$

then by (1.3)

$$\begin{aligned} x_{N+m} &\equiv X_{N-1} + a_N \cdot (2^{m+1} - 1) + 2^{u_1} - 1 + 2^{u_2} - 1 + \dots + 2^{u_y} - 1 \equiv \\ &\equiv x_{N-1} + a_N \cdot (2^{m+1} - 1) + y \cdot u \equiv r \pmod{n}. \end{aligned}$$

Since the digits $\{\varepsilon_{N(n)}(\alpha)\}$ have been choosen without restriction we get that there are continuum many α for which $A_{\alpha\beta}$ is not subcomplete.

Proof of Theorem 2. First we need a lemma which is essentially a quantative form of a results of mine [3].

Lemma 2. *Let m be a positive integer, s be a nonnegative integer. Let us suppose that*

$$\sum_{i=1}^{\infty} \varepsilon_i(\beta) > 2m^2. \tag{2.1}$$

Then there exists $x_{m,s} \in P(A_\beta)$ for which $x_{m,s} \equiv s \pmod{m}$.

Proof of Lemma 2. By (2.1) we can select a sequence of indices $k_1 < k_2 < \cdots < k_{m^2}$ for which $\varepsilon_{k_i+1}(\beta) = 1$ and $k_{i+1} - k_i > 1$ $(i = 1, 2, \dots, m^2)$. Using the pigeonhole principle we conclude that there is a $z \in [0, m-1]$ for which the congruence

$$[2^{k_i}\beta] \equiv z \pmod{m}$$

has at least m solutions. Let these be $b_{k_1}, b_{k_2}, \dots, b_{k_m}$, where $b_{k_i} = [2^{k_i}\beta]$. Let now $t \equiv -2s \pmod{m}$ where $0 \le t < m$. So

$$s = s(2z+1) - 2zs \equiv s(2z+1) + t \cdot z \equiv$$
$$\equiv (b_{k_{i_1}+1} + b_{k_{i_2}+1} + \cdots + b_{k_{i_s}+1}) + (b_{k_{i_1}} + b_{k_{i_2}} + \cdots + b_{k_{i_s}}) \pmod{m}.$$

Let α be FDF. The number of those β's for which β is FDF and $j^*_\beta = j$ is 2^j. Let $A := 2 \cdot [2^{j^*_\alpha}\alpha]$. If

$$\sum_{i=1}^{\infty} \varepsilon_i(\beta) > A$$

then by Lemma 2 we conclude that $A_{\alpha\beta}$ is complete. This implies that the number of those β's for which $j^*_\beta = j$ and $A_{\alpha\beta}$ is not complete is at most

$$\sum_{n=1}^{A} \binom{j}{n} < \sum_{n=1}^{A} j^n < j^{A+1}.$$

Thus

$$g_\alpha(j) > (2^j - j^{A+1})/2^j = 1 - j^{A+1}/2^j$$

which means that

$$\lim_{j\to\infty} g_\alpha(j) = 1.$$

4. Proof of Theorem 3

Lemma 3.1. *Let $P_n(A_\alpha) := P(A_\alpha) \cap [1, a_n]$. The biggest gap in $P_n(A_\alpha)$ is the interval*

$$\mathcal{I}_n := \left[\sum_{i=0}^{n-1} a_i + 1, a_n \right) \tag{3.1}$$

and

$$|\mathcal{I}_n| = \sum_{i=1}^{n} \varepsilon_i(\alpha) + a_0 - 1 \tag{3.2}$$

(if $v < u$ then let $\sum_{i=u}^{v} a_i = 0$).

Proof of Lemma 3.1. We are going to show by induction on n that the assertions of the lemma hold for every $n \geq 0$.

For $n = 0$, $\mathcal{I}_0 = [1, a_0)$ and $|\mathcal{I}_0| = a_0 - 1$.

Assume now that $n \geq 1$ and the assertions hold with $0, 1, \dots, n-1$ in place of n. First let us observe by (1.1) that if $m \notin P_{n-1}(A_\alpha)$ then $m + a_{n-1} \notin P_n(A_\alpha)$. So we conclude that if $\mathcal{J}$ is a gap in $P_{n-1}(A_\alpha)$ then $a_n + \mathcal{J}$ is also a gap in $[a_{n-1}, a_n]$ and conversely if $\mathcal{J}'$ is a gap in $[a_{n-1}, a_n]$ then $\mathcal{J}' - a_{n-1}$ is also one. This implies, using the inductive hypothesis, that the biggest gap in $[1, 2a_{n-1})$ is the interval $\left[\sum_{i=0}^{n-1} a_i + 1, 2a_{n-1} \right)$. Since $2a_{n-1} \in P(A_\alpha)$ if $\varepsilon_n(\alpha) = 0$ and $2a_{n-1} \notin P(A_\alpha)$ otherwise, we get that the biggest gap in $P_n(A_\alpha)$ is the interval $\mathcal{I}_n = \left[\sum_{i=0}^{n-1} a_i + 1, a_n \right)$ and

$$|\mathcal{I}_n| = |\mathcal{I}_{n-1}| + \varepsilon_n(\alpha) = \sum_{i=1}^{n} \varepsilon_i(\alpha) + a_0 - 1.$$

This completes the proof of the lemma.

Now we prove the first point of the theorem.

Let

$$a_n \leq x < a_{n+1}. \tag{3.3}$$

Then

$$2^n \leq [2^n\alpha] = a_n \leq x < a_{n+1} \leq 2^{n+1}\alpha.$$

So

$$\log_2 x - \log_2 \alpha - 1 \leq n \leq \log_2 x. \tag{3.4}$$

By Lemma 3.1 and by (3.3) and (3.4) we get the estimation

$$f_\alpha(x) = |\mathcal{I}_n| = \sum_{i=1}^{n} \varepsilon_i(\alpha) + a_0 - 1 \le n + a_0 - 1 \le \log_2 x + a_0 - 1$$

i.e. $\limsup_{x\to\infty} f_\alpha(x)/\log x \le 1$.

We turn now to the proof of the second point of the theorem.

Lemma 3.2. *Suppose that α is expressed in the scale of r, and the digit b, $0 \le b < r$ occurs n_b times in the first n places. Then for almost all numbers $n_b/n \to 1/r$.*

This is a special case of Th. 148. in [5].

Lemma 3.2 implies that for almost all α $\lim_{n\to\infty} \sum_{i=1}^{n} \varepsilon_i(\alpha)/n = 1/2$. This means that for every $\varepsilon > 0$

$$(1/2 - \varepsilon)\cdot n \le f_\alpha(x) \le (1/2 + \varepsilon)\cdot n$$

if $n > n_0(a_0, \varepsilon)$. By (3.4) we get

$$(1/2 - \varepsilon)\cdot \log_2 x - c_\alpha \le f_\alpha(x) \le (1/2 + \varepsilon)\cdot \log_2 x$$

where c_α depends only on α.

Thus by (3.3) we have that for almost all α

$$f_\alpha(x)/\log_2 x \to 1/2$$

if $x \to \infty$, which proves the second point of the theorem.

Finally we prove the third part of the theorem. (3.3) and (3.4) mean that $n = \log_2 x + O(1)$. The condition

$$f_\alpha(x) \le \log x/2 + \eta\sqrt{\log_2 x}\,/2$$

means that

$$\sum_{i=1}^{n+1} \varepsilon_i(\alpha) + a_0 - 1 \le n/2 + \eta\cdot(n + c')^{1/2}/2 + c \tag{3.5}$$

where c, c' depend only on α.

Let

$$F_{n,A}(\eta,\delta) = F_n(\eta,\delta) = \{\alpha \mid A - 1 \le \alpha < A \text{ and } \sum_{i=1}^{n} \varepsilon_i(\alpha) < n/2 + (\eta+\delta)\cdot\sqrt{n}/2\}.$$

Clearly if $\alpha \in F(\eta, \delta)$ then (3.5) is satisfied if n is large enough. Furthermore

$$\mu(F_n(\eta, \delta)) = \sum{}' \binom{n}{k} \cdot 2^{-n}$$

where the summation in $\sum'$ is taken for those k's for which

$$(k - n/2)/(\sqrt{n}/2) < \eta + \delta.$$

Thus for every $\delta > 0$, using the connection between the binomial and the normal distribution we get

$$\lim_{n\to\infty} \mu(F_n(\eta, \delta)) = \Phi(\eta + \delta)$$

and so

$$\limsup_{x\to\infty} G(x, \eta) \le \Phi(\eta + \delta). \tag{3.6}$$

Using a similar method we have that for every $\delta > 0$

$$\liminf_{x\to\infty} G(x, \eta) \ge \Phi(\eta - \delta). \tag{3.7}$$

(3.6) and (3.7) imply the third part of the theorem.

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