Publ. Math. Debrecen
82/1 (2013), 1–9
DOI: 10.5486/PMD.2013.4855

A normality relationship between two families and its applications

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Abstract. Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k, and there exists $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$ for $f \in \mathcal{F}$. If $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$ is normal, then $\mathcal{F}$ is also normal in D. Some applications of this result are given.

1. Introduction

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sense of Montel, if for any sequence $\{f_n\} \in \mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$, such that $\{f_{n_j}\}$ converges spherically locally uniformly on D, to a meromorphic function or ∞(see [6], [9], [12]).

Let k be a positive integer. Consider the family $\mathcal{F}_k$ consisting of kth derivative functions of all $f \in \mathcal{F}$, that is, $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F},\ z \in D\}$. It is natural to consider the normality relation between these two families. However, the following examples show that there seems no direct relation between $\mathcal{F}$ and $\mathcal{F}_k$.

Example 1. Let $\Delta = \{z : |z| < 1\}$, and $\mathcal{F} = \{f_n(z) = n(z^2 - n^2) : n = 1, 2, \dots\}$. Then $\mathcal{F}_1 = \{f'_n(z) = 2nz : n = 1, 2, \dots\}$. For each $z \in \Delta$,

$$f_n^{\#}(z) = \frac{|2nz|}{1 + |n(z^2 - n^2)|^2} \leq \frac{2n}{1 + (n^3 - n)^2} \to 0$$

Mathematics Subject Classification: 30D45.
Key words and phrases: meromorphic function, derivative function, normal family.
The author is supported by NSFC (Grant Nos. 10871094;11171045).

as $n \to \infty$, where $f_n^{\#}(z) = |f_n'(z)|/(1+|f_n(z)|^2)$ is the spherical derivative of f_n. By Marty's criterion, $\mathcal{F}$ is normal in Δ. But it is easy to see that $\mathcal{F}_1$ is not normal in Δ.

Example 2. Let $\Delta = \{z : |z| < 1\}$, and $\mathcal{F} = \{f_n(z) = nz : n = 1, 2, \dots\}$. Then $\mathcal{F}_1 = \{f_n'(z) = n : n = 1, 2, \dots\}$. Clearly, $\mathcal{F}_1$ is normal in Δ; but $\mathcal{F}$ is not normal in Δ.

In 1996, Chen and Lappan [2] first gave an interesting normality relation between $\mathcal{F}$ and $\mathcal{F}_k$ under an additional condition, as follows.

Theorem A ([2, Corollary 4])**.** *Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least $k+1$. If $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$ is normal, then $\mathcal{F}$ is also normal in D.*

In this paper, by using a different method from that in [2], we first give an extension to the above result, as follows.

Theorem 1. *Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose zeros have multiplicity at least k, and there exists $M > 0$ such that $|f^{(k)}(z)| \le M$ whenever $f(z) = 0$ for $f \in \mathcal{F}$. If $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$ is normal, then $\mathcal{F}$ is also normal in D.*

Remark 1. Theorem 1 is sharp, which can also be shown by Example 2.

The above normality relation between $\mathcal{F}$ and $\mathcal{F}_k$ is indeed useful to study normal families. In section 3, we shall give some applications of Theorem 1.

2. Proof of Theorem 1

We need the following well-known Pang–Zalcman lemma, which is the local version of [8, Lemma 2](cf. [13, pp. 216–217]).

Lemma 1. *Let k be a positive integer and let $\mathcal{F}$ be a family of functions meromorphic in a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists $A \ge 1$ such that $|f^{(k)}(z)| \le A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal at $z_0 \in D$, there exist, for each $\alpha, 0 \le \alpha \le k$,*

(a) *points $z_n \in D$, $z_n \to z_0$,*

(b) *positive numbers $\rho_n \to 0$, and*

(c) *functions $f_n \in \mathcal{F}$*

such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ *locally uniformly with respect to the spherical metric, where* g *is a nonconstant meromorphic function in* $\mathbb{C}$*, all of whose zeros have multiplicity at least* k*, such that* $g^{\#}(\zeta) \le g^{\#}(0) = kA + 1$.

Proof of Theorem 1. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 1, there exist functions $f_n \in \mathcal{F}$, points $z_n \to z_0$ and positive numbers $\rho_n \to 0$, such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \to g(\zeta) \tag{1}$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\zeta)$ is a nonconstant meromorphic function in $\mathbb{C}$, all of whose zeros have multiplicity at least k, and $g^{\#}(\zeta) \le g^{\#}(0) = kM + 1$. (Without loss of generality, we assume that $M > 1$).

From (1), we have

$$g_n^{(k)}(\zeta) = f_n^{(k)}(z_n + \rho_n\zeta) \to g^{(k)}(\zeta) \tag{2}$$

converges uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of g. Suppose that $g(\zeta_0) = 0$, by Hurwitz's theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that $f_n(z_n + \rho_n\zeta_n) = 0$. By the assumption of Theorem 1, we have $|f_n^{(k)}(z_n+\rho_n\zeta_n)| \le M$. Now, it follows from (2) that $|g^{(k)}(\zeta_0)| \le M$. This proves that $|g^{(k)}| \le M$ whenever $g = 0$.

We claim that g can not be a polynomial of degree less than $k+1$. Indeed, g can not be a polynomial of degree less than k since all zeros of g have multiplicity at least k. Now assume that g is a polynomial of degree k. It follows that g has the form

$$g(\zeta) = \frac{A}{k!}(\zeta - \alpha)^k \tag{3}$$

where A, α are complex numbers. Since $g = 0 \Rightarrow |g^{(k)}| \le M$, we see that $|A| \le M$. Calculating $g^{\#}(0)$, we get

$$g^{\#}(0) = \frac{\frac{|A|\,|\alpha|^{k-1}}{(k-1)!}}{1 + \left(\frac{|A|\,|\alpha|^k}{k!}\right)^2} = \frac{k}{|\alpha|} \cdot \frac{\frac{|A|\,|\alpha|^k}{k!}}{1 + \left(\frac{|A|\,|\alpha|^k}{k!}\right)^2}.$$

From the middle expression, we see that $g^{\#}(0) \le |A|$ if $|\alpha| \le 1$, and from the expression on the right we see that $g^{\#}(0) < k/2$ if $|\alpha| > 1$. But these contradict the fact that $g^{\#}(0) = kM + 1$ and $|A| \le M$.

Hence, there exist a point ζ_0 and $M_1 > 0$ such that

$$M_1^{-1} \le |g^{(j)}(\zeta_0)| \le M_1, \quad \text{for } j = k, k+1.$$

It follows that $(2M_1)^{-1} \le |g_n^{(j)}(\zeta_0)| \le 2M_1 (j = k, k+1)$ for sufficiently large n. From (2), $g_n^{(k)}(\zeta_0) = f_n^{(k)}(z_n + \rho_n\zeta_0)$, and then $|f_n^{(k)}(z_n + \rho_n\zeta_0)| \le 2M_1$ for sufficiently large n. So we have

$$\begin{aligned}(2M_1)^{-1} \le |g_n^{(k+1)}(\zeta_0)| &= \rho_n |f_n^{(k+1)}(z_n + \rho_n\zeta_0)| \\ &\le \rho_n(1 + 4M_1^2)\frac{|f_n^{(k+1)}(z_n + \rho_n\zeta_0)|}{1 + |f_n^{(k)}(z_n + \rho_n\zeta_0)|^2}, \end{aligned} \tag{4}$$

for sufficiently large n.

On the other hand, by Marty's criterion, the normality of the family $\mathcal{F}_k$ implies that for each compact subset $K \subset D$, there exists a positive number M_2 such that

$$\frac{|f^{(k+1)}(z)|}{1 + |f^{(k)}(z)|^2} \le M_2$$

for each $f \in \mathcal{F}$ and $z \in K$. Then, for sufficiently large n, we have

$$\frac{|f_n^{(k+1)}(z_n + \rho_n\zeta_0)|}{1 + |f_n^{(k)}(z_n + \rho_n\zeta_0)|^2} \le M_2. \tag{5}$$

Substituting (5) in (4), we obtain

$$(2M_1)^{-1} \le |g_n^{(k+1)}(\zeta_0)| \le \rho_n(1 + 4M_1^2)M_2 \to 0,$$

as $n \to \infty$, a contradiction. Theorem 1 is thus proved. □

3. Some applications of Theorem 1

In this section, we shall give some applications of Theorem 1.

Recently, Chang [1] proved the following result, which improve and generalize the related results due to Pang and Zalcman [8], Fang and Zalcman [5].

Theorem B ([1, Theorem 1]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, let a, b be two nonzero complex numbers such that $a/b \notin \mathbb{N}\backslash\{1\}$. If, for each $f \in \mathcal{F}$, $f = a \Rightarrow f'(z) = a$, and $f'(z) = b \Rightarrow f''(z) = b$ in D, then $\mathcal{F}$ is normal.*

There is an example [1, Example 1], which shows that the condition '$a/b \notin \mathbb{N}\backslash\{1\}$' in Theorem B is necessary. Chang proved another result without the condition '$a/b \notin \mathbb{N}\backslash\{1\}$', as follows.

Theorem C ([1, Theorem 2])**.** *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, let a, b be two nonzero complex numbers. If, for each $f \in \mathcal{F}$, $f = a \Rightarrow f'(z) = a$, $f'(z) \neq b$ and $f''(z) \neq b$ in D, then $\mathcal{F}$ is normal.*

Remark 2. CHANG also gave another example [1, Example 2] to show that the condition '$f''(z) \neq b$' in Theorem C can not be omitted. However, it is easy to see that '$f''(z) \neq b$' in Theorem C is not necessary for the case $a = b (\neq 0)$. Indeed, $f = a \Rightarrow f'(z) = a$ and $f'(z) \neq b$ yield that $f \neq a$ and $f' \neq a$ since $a = b$, then GU's normal criterion [3] implies that $\mathcal{F}$ is normal. We also find that 'a is nonzero' in Theorem C can be removed. In fact, if $a = 0$ and $b \neq 0$, noting that $f' \neq b$ and $f'' \neq b$, Gu's normal criterion asserts that $\mathcal{F}_1 = \{f' : f \in \mathcal{F}\}$ is normal in D. Since $f = 0 \Rightarrow f' = 0$, we conclude from Theorem 1 that $\mathcal{F}$ is also normal in D.

Here, by using Theorem 1 and some known results, we can prove the following results, which improve and generalize Theorem C much more.

Theorem 2. *Let a, b, c be three complex numbers with $c \neq 0$, k, l be two positive integers, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,*

(1) *all zeros of $f - a$ have multiplicity at least k, and there exists $M > 0$ such that $f = a \Rightarrow |f^{(k)}| \leq M$;*

(2) *all zeros of $f^{(k)} - b$ have multiplicity at least $l + 1$, and $f^{(k+l)} \neq c$.*

Then $\mathcal{F}$ is normal in D.

Let $k = l = 1$ and $b = c$ in Theorem 2, we have

Corollary 1. *Let a, b be two complex numbers with $b \neq 0$, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,*

(1) *there exists $M > 0$ such that $f = a \Rightarrow |f'| \leq M$;*

(2) *all zeros of $f' - b$ have multiplicity at least 2, and $f'' \neq b$.*

Then $\mathcal{F}$ is normal in D.

Obviously, the above results improve and generalize Theorem C.

Next we give some more general extensions of Theorem C by extending constants 'a, b, c' in Theorem 2 to functions '$a(z), b(z), c(z)$'.

Theorem 3. *Let k, l be two positive integers, D be a domain in $\mathbb{C}$, let $a(z)$, $b(z)$ be two holomorphic functions in D, and $c(z)$ be a meromorphic function in D such that $c(z) \not\equiv \infty$ and $c(z) \not\equiv b'(z)$, and let $\mathcal{F}$ be a family of meromorphic functions defined in D. Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,*

(i) *all zeros of* $f(z) - a(z)$ *have multiplicity at least* k*, and there exists* $M > 0$ *such that* $f(z) = a(z) \Rightarrow |f^{(k)}| \leq M$*;*

(ii) *all zeros of* $f^{(k)}(z) - b(z)$ *have multiplicity at least* 3*, and* $f^{(k+1)}(z) \neq c(z)$*.*

Then $\mathcal{F}$ *is normal in* D*.*

Theorem 4. *Let* $k, l(\geq 2)$ *be two positive integers,* D *be a domain in* $\mathbb{C}$*, let* $a(z), b(z)$ *be two holomorphic functions in* D*, and* $c(z)$ *be a meromorphic function in* D *such that* $c(z) \not\equiv \infty$ *and* $c(z) \not\equiv b^{(l)}(z)$*, and let* $\mathcal{F}$ *be a family of meromorphic functions defined in* D*. Suppose that, for each* $f \in \mathcal{F}$ *and* $z \in D$*,*

(i) *all zeros of* $f(z) - a(z)$ *have multiplicity at least* k*, and there exists* $M > 0$ *such that* $f(z) = a(z) \Rightarrow |f^{(k)}| \leq M$*;*

(ii) *all zeros of* $f^{(k)}(z) - b(z)$ *have multiplicity at least* $l+1$*, and* $f^{(k+l)}(z) \neq c(z)$*.*

Then $\mathcal{F}$ *is normal in* D*.*

Remark 3. If $k = 1$, the condition 'all zeros of $f - a$ *or* $(f - a(z))$ have multiplicity at least k' in Theorem 2–4 holds naturally, and then can be removed.

Remark 4. The condition $c \neq 0$ in Theorem 2 ($b \neq 0$ in Corollary 1), $c(z) \not\equiv b'(z)$ in Theorem 3, and $c(z) \not\equiv b^{(l)}(z)$ in Theorem 4 can not be omitted, as is shown by the following examples.

Example 3. Let $\Delta = \{z : |z| < 1\}$, $a \neq 0$ and $b = c = 0$, and let $\mathcal{F} = \{f_n(z) = e^{nz} + a : n = 1, 2, \ldots; z \in \Delta\}$. Obviously, $f_n(z) \neq a$, thus $f(z) = a \Rightarrow f'(z) = a$; $f_n'(z) = ne^{nz} \neq 0$, and $f_n''(z) = n^2 e^{nz} \neq 0$. Then all conditions excepting $c \neq 0$ (or $c \neq 0$) of Theorem 2 (Corollary 1) are satisfied. But $\mathcal{F}$ is not normal in Δ.

Example 4. Let $\Delta = \{z : |z| < 1\}$, $a(z) = b(z) = c(z) = e^z$, and let $\mathcal{F} = \{f_n(z) = e^{nz} + e^z : n = 1, 2, \ldots;\ z \in \Delta\}$. It is easy to see that all conditions excepting $c(z) \not\equiv b'(z)$ ($c(z) \not\equiv b^{(l)}(z)$) of Theorem 3-4 are satisfied. But $\mathcal{F}$ is not normal in Δ.

Remark 5. Example 4 also shows that 'nonzero constants a, b' in Theorem B can not be replaced two nonconstant functions (even for non-vanishing holomorphic functions).

To prove the above theorems, we need some known results.

Lemma 2 ([10, Theorem 5])**.** *Let* k *be a positive integer, and let* $\mathcal{F}$ *be a family of meromorphic functions defined in a domain* D*, all of whose poles are multiple and whose zeros all have multiplicity at least* $k+1$*. If, for each* $f \in \mathcal{F}$*,* $f^{(k)}(z) \neq 1$ *in* D*, then* $\mathcal{F}$ *is normal in* D*.*

Lemma 3 ([7, Theorem 1.3], cf. [11, Theorem 2])**.** *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose poles are multiple and whose zeros all have multiplicity at least 3, and let $\psi(z)(\not\equiv 0,\infty)$ be a function meromorphic in D. If, for each $f\in\mathcal{F}$ and for each $z\in D$, $f'(z)\neq\psi(z)$, then $\mathcal{F}$ is normal in D.*

Lemma 4 ([14, Theorem 2])**.** *Let $k\geq 2$ be an integer, $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, all of whose poles are multiple and whose zeros all have multiplicity at least $k+1$, and let $\psi(z)(\not\equiv 0,\infty)$ be a function meromorphic in D. If, for each $f\in\mathcal{F}$ and for each $z\in D$, $f^{(k)}(z)\neq\psi(z)$, then $\mathcal{F}$ is normal in D.*

Proof of Theorem 2. Let $\mathcal{G}=\{g=f^{(k)}-b : f\in\mathcal{F}\}$. Obviously, the poles of g have multiplicity at least $k+1\geq 2$. By the assumptions of theorem, for each $g\in\mathcal{G}$, all zeros of g have multiplicity at least $l+1$, and $g^{(l)}=f^{(k+l)}\neq c$. Lemma 2 implies that $\mathcal{G}$ is normal in D. Hence, the family $\mathcal{H}_k=\{(f-a)^{(k)} : f\in\mathcal{F}, z\in D\}$ is also normal in D, where $\mathcal{H}=\{f-a : f\in\mathcal{F}\}$. Noting condition (1), by Theorem 1, we get that $\mathcal{H}$ is normal, and then $\mathcal{F}$ is normal in D. Theorem 2 is proved. □

Proof of Theorem 3. Since normality is a locally property, we only need to prove $\mathcal{F}$ is normal at each point in D.

Let $z_0\in D$, then there exists $\delta>0$ such that $\bar{D}_\delta(z_0)\subset D$, where $\bar{D}_\delta(z_0)=\{z : |z-z_0|\leq\delta\}$. Let $\mathcal{G}=\{g(z)=f^{(k)}(z)-b(z) : f\in\mathcal{F}\}$. Clearly, all poles of $g\in\mathcal{G}$ are multiple. By the hypotheses of the theorem, for each $g\in\mathcal{G}$, all zeros of g have multiplicity at least 3. Noting that $b(z)$ is holomorphic and $f^{(k+1)}(z)\neq c(z)$, we have $g'=f^{(k+1)}(z)-b'(z)\neq c(z)-b'(z)$ $(\not\equiv 0)$. Then, by Lemma 3, $\mathcal{G}$ is normal in D, and then in $D_\delta(z_0)=\{z : |z-z_0|<\delta\}$. It follows that the family $\mathcal{H}_k=\{(f(z)-a(z))^{(k)} : f\in\mathcal{F}\}$ is normal in $D_\delta(z_0)$, where $\mathcal{H}=\{h=f(z)-a(z) : f\in\mathcal{F}\}$. By the hypotheses of the theorem, for each $h\in\mathcal{H}$, all zeros of h have multiplicity at least k. Moreover, if $h(z)=0$, that is, $f(z)=a(z)$, then $|f^{(k)}(z)|\leq M$, and thus

$$|h^{(k)}(z)|\leq M+|a^{(k)}(z)|.$$

Noting that $a(z)$ is holomorphic in D, there exists $M_1>0$ such that $|a^{(k)}(z)|\leq M_1$ in $\bar{D}_\delta(z_0)$, and then in $D_\delta(z_0)$. We get that $h(z)=0\Rightarrow|h^{(k)}(z)|\leq M_2$ for $z\in D_\delta(z_0)$, where $M_2=M+M_1$. By Theorem 1, $\mathcal{H}$ is normal in $D_\delta(z_0)$. It follows that $\mathcal{F}$ is normal in $D_\delta(z_0)$, and this means that $\mathcal{F}$ is normal at z_0. Theorem 3 is thus proved. □

Proof of Theorem 4. Using the same argument as in Theorem 3 and Lemma 4, we can prove Theorem 4. We here omit the details. □

Next we give another application of Theorem 1. In [4], Fang and Chang gave an extension to Gu's normal criterion in some sense, by allowing $f^{(k)}-1$ have zeros but restricting the zeros of $f^{(k)}$, as follows.

Theorem D ([4, Theorem 1])**.** *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, and let k be a positive integer. If, for each $f\in\mathcal{F}$, $f\neq 0$, $f^{(k)}\neq 0$ and the zeros of $f^{(k)}-1$ have multiplicity at least $(k+2)/k$, then $\mathcal{F}$ is normal.*

Here, we can prove the following extension of Theorem D.

Theorem 5. *Let k, l_1, l_2 be three positive integers (l_1, l_2 can be ∞) with $1/l_1+1/l_2<k/(k+1)$, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. Suppose that, for each $f\in\mathcal{F}$ and $z\in D$,*

(1) *all zeros of f have multiplicity at least k and there exists $M>0$ such that $|f^{(k)}(z)|\leq M$ whenever $f(z)=0$;*

(2) *all zeros of $f^{(k)}$ have multiplicity at least l_1; and*

(3) *all zeros of $f^{(k)}-1$ have multiplicity at least l_2.*

Then $\mathcal{F}$ is normal in D.

Remark 6. We should indicate that Theorem 5 can be followed from [4, Theorem 2] if condition (1) is replaced by a stronger condition "all zeros of f have multiplicity at least $k+1$". However, the method in [4] does not work here, and our proof is very simple.

To prove Theorem 5, we need the following classical result due to Bloch and Valiron, which can be found in [6], [9], [12].

Lemma 5. *Let $a_1,a_2,\dots,a_q$ be q distinct complex numbers, and $l_1,l_2,\dots,l_q$ be positive integers (may equal to ∞) with $\sum_{i=1}^{q}(1-1/l_i)>2$. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D. If, for each $f\in\mathcal{F}$, the zeros of $f-a_i$ have multiplicity at least l_i ($i=1,2,\dots,q$) in D, then $\mathcal{F}$ is normal in D.*

Proof of Theorem 5. Obviously, the poles of $f^{(k)}$ have multiplicity at least $k+1$. Since

$$\frac{1}{l_1}+\frac{1}{l_2}<\frac{k}{k+1},$$

we have

$$\left(1-\frac{1}{l_1}\right)+(1-\frac{1}{l_2})+\left(1-\frac{1}{k+1}\right)>2.$$

Let $q = 3$, $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$, applying Lemma 6 for $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$, we know that $\mathcal{F}_k$ is normal in D. Noting condition (1), Theorem 1 implies that $\mathcal{F}$ is also normal in D. Theorem 5 is proved. $\square$

Acknowledgement. The author thanks Professor H. H. Chen for helpful discussions. The author also thanks the referee for his/her valuable comments and suggestions made to this paper.

References

[1] J. M. Chang, Normality concerning shared values, *Sci. China Ser. A* **52** (2009), 1717–1722.

[2] H. H. Chen and P. Lappan, Spherical derivative, high derivatives and normal families, *Adv. in Math. (China)* **25** (1996), 517–524.

[3] Y. X. Gu, A normal criterion of meromorphic families, *Acta Math. Sinica* **23**, no. 1 (1980), 267–274.

[4] M. L. Fang and J. M. Chang, Normal families and multiple values, *Arch. Math.* **88** (2007), 560–568.

[5] M. L. Fang and L. Zalcman, Normal families and shared values of meromorphic functions III, *Comput. Methods Func. Theory* **2** (2002), 383–395.

[6] W. K. Hayman, Meromorphic Functions, *Clarendon Press, Oxford*, 1964.

[7] X. C. Pang, D. G. Yang and L. Zalcman, Normal families and omitted functions, *Indiana Univ. Math. J.* **54** (2005), 223–235.

[8] X. C. Pang and L. Zalcman, Normal families and shared values, *Bull. London Math. Soc.* **32** (2000), 325–331.

[9] J. Schiff, Normal Families, *Springer-Verlag, New York*, 1993.

[10] Y. F. Wang and M. L. Fang, Picard values and normal families of meromorphic functions with multiple zeros, *Acta Math. Sinica (N.S.)* **14** (1998), 17–26.

[11] Y. Xu, Normal families and exceptional functions, *J. Math. Anal. Appl.* **329** (2007), 1343–1354.

[12] L. Yang, Value Distribution Theory, *Springer-Verlag* & *Science Press, Berlin*, 1993.

[13] L. Zalcman, Normal families: new perspectives, *Bull. Amer. Math. Soc.* **35** (1998), 215–230.

[14] G. M. Zhang, X. C. Pang and L. Zalcman, Normal families and omitted functions II, *Bull. London Math. Soc.* **41** (2009), 63–71.

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(Received February 26, 2010; revised February 24, 2011)