Publ. Math. Debrecen
82/1 (2013), 11–20
DOI: 10.5486/PMD.2013.5119

Oscillation of second-order differential equations

By JÁN OHRISKA (Ružomberok)

Abstract. The aim of this paper is to present sufficient conditions for the non-linear differential equation $(r(t)y'(t))' + p(t)f(y(g(t))) = 0$ with deviating argument, and for the ordinary or advanced linear differential equation $(r(t)y'(t))' + p(t)y(\sigma(t)) = 0$ to be oscillatory. Obtained results replenish and extend some known results. The technique used in the paper is established on the notion of the v-derivative of a function.

1. Introduction

We consider the non-linear differential equation with deviating argument

$$(r(t)y'(t))' + p(t)f(y(g(t))) = 0 \tag{1}$$

and its special case, the linear ordinary or advanced differential equation

$$(r(t)y'(t))' + p(t)y(\sigma(t)) = 0 \tag{2}$$

on the interval $[t_0, \infty)$, where

(i) $r \in C([t_0, \infty))$, $r(t) > 0$;
(ii) $p \in C([t_0, \infty))$, $p(t) \geq 0$, $p(t) \not\equiv 0$ in any neighborhood of infinity;
(iii) $f \in C(\mathbb{R})$, $xf(x) > 0$ for $x \neq 0$ and f is a non-decreasing function;
(iv) $g \in C([t_0, \infty))$, $\lim_{t\to\infty} g(t) = \infty$;
(v) $\sigma \in C([t_0, \infty))$, $\sigma(t) \geq t$.

Mathematics Subject Classification: 34K11, 34K06, 26A24.
Key words and phrases: oscillation theory, functional differential equations, generalized derivatives.

We call a function u a solution of the equation of the type (1) for $t \geq t_0$ if $u \in C^1[t_0, \infty)$, $ru' \in C^1[t_0, \infty)$ and it satisfies the equation considered on the interval $[t_0, \infty)$. As usual, we restrict our attention to solutions of (1) which exist, and are nontrivial, on some ray $[t_0, \infty)$, where $t_0 \geq 0$ may depend on the particular solution. Such a solution is said to be oscillatory if it has arbitrarily large zeros and non-oscillatory otherwise. An equation is said to be oscillatory if all of its solutions are oscillatory.

Here we shall supplement the results of the paper [9] with results concerning the differential equations (1) and (2). As it will be noted, obtained results replenish, extend and improve several known results. The technique used in the paper, analogous to the paper [9], is established on the notion of the v-derivative of a function.

In the study of oscillatory nature of differential equations of the type (1) the following two possibilities are mostly considered. One of them is $\int^\infty \frac{\mathrm{d}t}{r(t)} = \infty$, then the equation (1) has, as it were, the canonical form. Other possibility is $\int^\infty \frac{\mathrm{d}t}{r(t)} < \infty$ and then the equation (1) has the non-canonical form. In this paper the condition $\int^\infty \frac{\mathrm{d}t}{r(t)} < \infty$ we substitute by the monotonicity of the function $r(t)$ when both above mentioned cases are possible. It seems the monotonicity of the function $r(t)$ is a new phenomenon in the study of oscillatory nature of differential equations.

2. Preliminaries and auxiliary results

We start with the following definition introduced in [6].

Definition 2.1. Let functions f and v be defined in a neighborhood $O(t)$ of a point $t \in \mathbb{R}$ and let the conditions $x \in O(t)$, $x \neq t$ imply $v(x) \neq v(t)$. If the limit

$$\lim_{x \to t} \frac{f(x) - f(t)}{v(x) - v(t)}$$

exists, then it is called the v-derivative of the function f at the point t and is denoted by $f'_v(t)$ or $\frac{\mathrm{d}f(t)}{\mathrm{d}v}$.

Remark 2.1. One can easily find out that if $c \in \mathbb{R}$ and f and v are functions then for $t \in \mathbb{R}$ the v-derivative $\frac{\mathrm{d}f(t)}{\mathrm{d}v}$ exists if and only if the v-derivative $\frac{\mathrm{d}f(t)}{\mathrm{d}(v+c)}$ exists. Moreover $\frac{\mathrm{d}f(t)}{\mathrm{d}(v+c)} = \frac{\mathrm{d}f(t)}{\mathrm{d}v}$.

In the sequel we will use the following result concerned with the v-derivative.

Theorem 2.1 (Theorem 1.2 in [6]). *Let there exist $v'(t) \neq 0$ on an interval $I \subset \mathbb{R}$. Then for $t \in I$ the v-derivative $f'_v(t)$ exists if and only if the derivative $f'(t)$ exists. Moreover,*

$$f'_v(t) = \frac{f'(t)}{v'(t)}.$$

We give a definition of v-derivatives of higher orders introduced in [6].

Definition 2.2. Let $n > 1$ be a natural number. Let functions v_n and $f^{(n-1)}_{v_1,v_2,\dots,v_{n-1}}$ be defined on a neighborhood $O(t)$ of a point $t \in \mathbb{R}$ and let the conditions $x \in O(t)$, $x \neq t$ imply $v_n(x) \neq v_n(t)$. If the limit

$$\lim_{x \to t} \frac{f^{(n-1)}_{v_1,v_2,\dots,v_{n-1}}(x) - f^{(n-1)}_{v_1,v_2,\dots,v_{n-1}}(t)}{v_n(x) - v_n(t)}$$

exists, then it is called the n-th v-derivative of the function f at the point t and is denoted by

$$f^{(n)}_{v_1,v_2,\dots,v_n}(t) \quad \text{or} \quad \frac{\mathrm{d}^n f(t)}{\mathrm{d}v_n \dots \mathrm{d}v_2 \mathrm{d}v_1}.$$

Now we present auxiliary results needed in next section. We start with an extension of Lemma 2 from [5] which is proved in [9] .

We put

$$R(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r(s)}.$$

Lemma 2.1 (Lemma 2.1 in [9]). *Suppose $u \in C^1[t_0, \infty)$, $u(t) \neq 0$ and assume that $\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R}$ exists for $t \geq T \geq t_0$. Let*

$$u(t)\frac{\mathrm{d}u(t)}{\mathrm{d}R} \geq 0, \quad u(t)\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R} \leq 0 \quad \text{for} \quad t \geq T, \tag{3}$$

where the equalities cannot hold in any neighborhood of infinity. Then for each $k \in (0,1)$ there is a point $T_1 \geq T$ such that

(a) $|u(t)| \geq kR(t)\left|\frac{\mathrm{d}u(t)}{\mathrm{d}R}\right|$ *for $t \geq T_1$, if $\int^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty$,*

(b) $|u(t)| \geq k\frac{t}{r(t)}\left|\frac{\mathrm{d}u(t)}{\mathrm{d}R}\right|$ *for $t \geq T_1$, if the function r is non-decreasing in a neighborhood of infinity.*

Lemma 2.2. *Suppose $u \in C^1[t_0, \infty)$, $u(t) \neq 0$ and assume that $\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R}$ exists for $t \geq T \geq t_0$. If the function $r(t)$ is non-decreasing and*

$$u(t)\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R} \leq 0 \quad \text{for } t \geq T,$$

where the equality cannot hold in any neighborhood of infinity, then

$$u(t)\frac{\mathrm{d}u(t)}{\mathrm{d}R} > 0 \quad \text{for } t \geq T.$$

Proof. First we consider the case

$$u(t) > 0, \quad \frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u(t)}{\mathrm{d}R}\right) \le 0 \quad \text{for } t \ge T.$$

Using the monotonicity of functions $\frac{\mathrm{d}u(t)}{\mathrm{d}R}(= r(t)u'(t))$ and $r(t)$ we have

$$u'(t_1) \ge \frac{r(t_2)}{r(t_1)} u'(t_2) \ge u'(t_2)$$

for any $t_2 > t_1 \ge T$, i.e. the function $u'(t)$ is non-increasing on the interval $[T, \infty)$. This implies $u'(t) > 0$ for $t \ge T$. Really, because if there exists a point $t_3 \ge T$ such that $u'(t_3) = 0$, then due to assumptions there exists $t_4 > t_3$ such that $u'(t_4) < 0$ and the monotonicity of $u'(t)$ implies that $u(t) \to -\infty$ as $t \to \infty$ which contradicts the assumption $u(t) > 0$ for $t \ge T$. Then also $u(t)\frac{\mathrm{d}u(t)}{\mathrm{d}R} > 0$ for $t \ge T$.

In the opposite case, i.e. when

$$u(t) < 0, \quad \frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R} \ge 0 \quad \text{for } t \ge T$$

we put $v(t) = -u(t)$. Then

$$v(t) > 0, \quad \frac{\mathrm{d}^2 v(t)}{\mathrm{d}t\mathrm{d}R} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}v(t)}{\mathrm{d}R}\right) \le 0 \quad \text{for } t \ge T$$

and according to what was just proved we have $v(t)\frac{\mathrm{d}v(t)}{\mathrm{d}R} > 0$ for $t \ge T$ and so

$$u(t)\frac{\mathrm{d}u(t)}{\mathrm{d}R} = -v(t)\frac{\mathrm{d}(-v(t))}{\mathrm{d}R} = v(t)\frac{\mathrm{d}v(t)}{\mathrm{d}R} > 0 \quad \text{for} \quad t \ge T. \qquad \square$$

Note that if $\int^\infty \frac{\mathrm{d}t}{r(t)} = \infty$, the assertion of Lemma 2.2 follows from Lemma 2.3 without assumption of monotonicity of the function $r(t)$. For all that Lemma 2.2 is interesting especially when $\int^\infty \frac{\mathrm{d}t}{r(t)} < \infty$. Now we give one example of such type.

Example 1. Consider the functions $r(t) = t^2$ and $u(t) = 10 - t^{-3}$ on the interval $[1, \infty)$. Then the function $r(t)$ is increasing, $\int^\infty \frac{\mathrm{d}t}{r(t)} < \infty$ and the function $u(t)$ is positive for $t \ge 1$. Moreover $\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R} = (r(t)u'(t))' = -6t^{-3} < 0$ and $\frac{\mathrm{d}u(t)}{\mathrm{d}R} = r(t)u'(t) = 3t^{-2} > 0$.

Lemma 2.3 (Lemma 2.3 in [9])**.** *Suppose $u \in C^1[t_0, \infty)$, $u(t) \ne 0$ and assume that $\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R}$ exists for $t \ge T \ge t_0$. Let*

$$u(t)\frac{\mathrm{d}u(t)}{\mathrm{d}R} \le 0, \quad u(t)\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t\mathrm{d}R} \le 0 \quad \text{for } t \ge T \ge t_0, \tag{4}$$

where the equalities cannot hold in any neighborhood of infinity. Then $\int^\infty \frac{\mathrm{d}t}{r(t)} < \infty$.

3. Main results

It is known that fulfilment of the conditions $\int^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty$, $\int^{\infty} p(t)\mathrm{d}t = \infty$ suffices for the oscillatory nature of the self-adjoint differential equation $(r(t)y'(t))' + p(t)y(t) = 0$ (see e.g. [3], [4], or [7]). Now we show that the same result holds true for a non-linear differential equation with any deviating argument. Hence, first we consider the differential equation (1) and give the following result.

Theorem 3.1. *Assume the conditions* (i), (ii), (iii) *and* (iv) *are satisfied. Let*

$$\int^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty \quad \text{and} \quad \int^{\infty} p(t)\mathrm{d}t = \infty.$$

Then equation (1) *is oscillatory.*

Proof. Suppose there exists a non-oscillatory solution $u(t)$ of (1). Then it is eventually of one sign and we assume that $u(t) > 0$ for $t \geq t_1 (\geq t_0)$. In view of the condition (iv) there exists $t_2 \geq t_1$ such that $g(t) \geq t_1$ for $t \geq t_2$ and in regard to the equation (1) we see that

$$(r(t)u'(t))' = -p(t)f(u(g(t))) \leq 0 \quad \text{for } t \geq t_2.$$

Now using Lemma 2.3 and the first integral condition we know that $r(t)u'(t) > 0$ (therefore also $u'(t) > 0$) for t large enough, e.g. for $t \geq t_3 (\geq t_2)$. Then, in view of the monotonicity of u and f we have

$$(r(t)u'(t))' + f(u(t_1))p(t) \leq 0, \quad t \geq t_3.$$

The integration of this inequality from t_3 to t $(t \geq t_3)$ yields

$$r(t)u'(t) - r(t_3)u'(t_3) + f(u(t_1)) \int_{t_3}^{t} p(s)\mathrm{d}s \leq 0.$$

As the function $r(t)u'(t)$ is positive and non-increasing and $f(u(t_1)) > 0$, the last inequality, in view of the second integral condition, produces a contradiction.
A similar proof follows if we assume $u(t) < 0$ for $t \geq t_1 (\geq t_0)$. □

Here we give the following example of a linear differential equation with advanced argument as illustration of the above Theorem.

Example 2. Consider the differential equation

$$(tu'(t))' + \sqrt{t^2 + 4\pi} \left(t^2 - \frac{1}{t^2} \right) u\big(\sqrt{t^2 + 4\pi}\,\big) = 0, \quad t \geq 1. \tag{5}$$

Then $r(t) = t$, $p(t) = \sqrt{t^2 + 4\pi}\big(t^2 - \frac{1}{t^2}\big)$. It is evident that $\int^\infty \frac{\mathrm{d}t}{r(t)} = \infty$ and also $\int^\infty p(t)\mathrm{d}t = \infty$ so by Theorem 3.1 the equation (5) is oscillatory. Note that $u(t) = \frac{1}{t}\sin\frac{t^2}{2}$ is a solution of (5).

Using the monotone nature of the function r we obtain the following result.

Theorem 3.2. *Assume the conditions* (i), (ii), (iii) *and* (iv) *are satisfied. Let*

(α) *the function r be non-decreasing in a neighborhood of infinity,*

(β) $\int^\infty p(t)\mathrm{d}t = \infty$.

Then equation (1) *is oscillatory.*

One can prove this Theorem in a similar way as the Theorem 3.1 was proved where the Lemma 2.2 will be used instead of Lemma 2.3.

We illustrate the Theorem 3.2 in the following example.

Example 3. Consider the advanced differential equation

$$(t\sqrt{t}u'(t))' + \left(25t^3 - \frac{3}{2t^2}\right)\sqrt[5]{(t^2\sqrt{t} + 10\pi)^3}\,u\left(\sqrt[5]{(t^2\sqrt{t} + 10\pi)^2}\right) = 0,\ t \geq 1. \tag{6}$$

Then the function $r(t) = t\sqrt{t}$ is non-decreasing and the function

$$p(t) = \left(25t^3 - \frac{3}{2t^2}\right)\sqrt[5]{\left(t^2\sqrt{t} + 10\pi\right)^3}$$

has the property $\int^\infty p(t)\mathrm{d}t = \infty$. Moreover, it is clear that for function $\sigma(t) = \sqrt[5]{(t^2\sqrt{t} + 10\pi)^2}$ the inequality $\sigma(t) \geq t$ holds true. Therefore, by Theorem 3.2 the equation (6) is oscillatory. Note that $u(t) = \frac{2}{t\sqrt{t}}\cos(2t^2\sqrt{t}\,)$ is a solution of (6). Mention that $\int_1^\infty \frac{1}{s\sqrt{s}}\,\mathrm{d}s = 2$.

Now we consider the equation (2) and give further sufficient conditions for oscillation of this equation.

Theorem 3.3. *Let* $\int^\infty \frac{\mathrm{d}t}{r(t)} = \infty$. *Assume the conditions* (i), (ii), (v) *are satisfied and*

$$\limsup_{t\to\infty} R(t)\int_t^\infty p(s)\mathrm{d}s > 1. \tag{7}$$

Then equation (2) *is oscillatory.*

Proof. Suppose there exists a non-oscillatory solution u of equation (2). Without loss of generality ($-u$ is a solution of (2) too) we may assume that $u(t) > 0$ and $u(\sigma(t)) > 0$ for $t \geq T \geq t_0$. Then from (2) we see that $(r(t)u'(t))' \leq 0$ for $t \geq T$ and by Lemma 2.3 we have $r(t)u'(t) > 0$ for $t \geq T$. Therefore we can integrate equation (2) from t to ∞ and the monotonicity of the function $r(t)u'(t)$ yields the inequality

$$\frac{\mathrm{d}u(t)}{\mathrm{d}R} \left(= r(t)u'(t)\right) \geq \int_t^\infty p(s)u(\sigma(s))\mathrm{d}s. \tag{8}$$

If we multiply this inequality by $kR(t)$, where $k \in (0,1)$ and use Lemma 2.1 (part (a)) then we have

$$u(t) \geq kR(t) \int_t^\infty p(s)u(\sigma(s))\mathrm{d}s, \quad t \geq T_1 \geq T \tag{9}$$

and with reference the monotonicity of the function u we obtain

$$1 \geq kR(t) \int_t^\infty p(s)\mathrm{d}s, \quad t \geq T_1,\ k \in (0,1), \tag{10}$$

which means that

$$\limsup_{t\to\infty} R(t) \int_t^\infty p(s)\mathrm{d}s < \infty.$$

Now we put $a = \limsup_{t\to\infty} R(t) \int_t^\infty p(s)\mathrm{d}s$ and take into account the assumption (7). Then there exists a sequence $\{t_n\}$ with the properties

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} R(t_n) \int_{t_n}^\infty p(s)\mathrm{d}s = a > 1.$$

But then for $\epsilon = \frac{a-1}{2}$ there exists a number n_0 such that for every $n \in \mathbb{N},\ n \geq n_0$ we have

$$\frac{a+1}{2} = a - \epsilon < R(t_n) \int_{t_n}^\infty p(s)\mathrm{d}s.$$

If we choose $n > n_0$ such that $t_n \geq T_1$ and take $k \in (\frac{2}{a+1}, 1)$ then

$$kR(t_n) \int_{t_n}^\infty p(s)\mathrm{d}s > k\frac{a+1}{2} > \frac{2}{a+1}\frac{a+1}{2} = 1$$

which contradicts (10) and the proof is complete. □

Here we can compare Theorem 3.3 with several known results. So we note that our Theorem 3.3 is an extension of Theorem 1.5 in [1] to equations with advanced argument and of Corollary 1 in [5] to equation with a quasi derivative. And next, our Theorem replenishes and extends Theorem 2.3 in [2], where the author has proved the following result: *If* $p(t) \geq 0$ *and* $\limsup_{t\to\infty} t \int_t^\infty s^{n-2} p(s)\mathrm{d}s > (n-1)!$, *then all solutions of the equation* $u^{(n)}(t) + p(t)u(t) = 0$, $n \geq 3$ *are oscillatory for* n *even, and ... for* n *odd.*

The following example illustrates application of Theorem 3.3.

Example 4. Consider the linear differential equation with advanced argument

$$\left(\frac{1}{\sqrt{t}}u'(t)\right)' + \frac{1+2t}{2t^2\left(\sqrt{t}+\pi\right)}u\left((\sqrt{t}+\pi)^2\right) = 0, \quad t \geq 1. \tag{11}$$

Then $r(t) = \frac{1}{\sqrt{t}}$ and

$$p(t) = \frac{1+2t}{2t^2(\sqrt{t}+\pi)}.$$

Simple computation shows that $\int^\infty \frac{\mathrm{d}t}{r(t)} = \infty$. Using the Remark 2.1 we can put $R(t) = \frac{2}{3}t^{3/2}$. Then we can write

$$\limsup_{t\to\infty} R(t)\int_t^\infty p(s)\mathrm{d}s \geq \limsup_{t\to\infty} \frac{2}{3}t^{3/2}\int_t^\infty \frac{2s}{4s^{5/2}}\mathrm{d}s = \limsup_{t\to\infty} \frac{2}{3}t = \infty > 1$$

and thus, by Theorem 3.3 the equation (11) is oscillatory. Note that $u(t) = \sqrt{t}\sin(2\sqrt{t}\,)$ is a solution of (11).

Theorem 3.4. *Assume the conditions* (i), (ii), *and* (v) *are satisfied. Let the function* r *be non-decreasing in a neighborhood of infinity, and*

$$\limsup_{t\to\infty} \frac{t}{r(t)}\int_t^\infty p(s)\mathrm{d}s > 1.$$

Then equation (2) *is oscillatory.*

Proof. The proof of this Theorem is similar to the previous one. Thereat we suppose there exists a non-oscillatory solution u of equation (2). We assume that $u(t) > 0$ and $u(\sigma(t)) > 0$ for $t \geq T \geq t_0$. Then from (2) we see that $(r(t)u'(t))' \leq 0$ for $t \geq T$ and by Lemma 2.2 we have $r(t)u'(t) > 0$ for $t \geq T$. Therefore we can integrate equation (2) from t to ∞ and the monotonicity of the function $r(t)u'(t)$ yields the inequality

$$\frac{\mathrm{d}u(t)}{\mathrm{d}R}\,(= r(t)u'(t)) \geq \int_t^\infty p(s)u(\sigma(s))\mathrm{d}s. \tag{12}$$

If we multiply this inequality with $k\frac{t}{r(t)}$, where $k \in (0,1)$ and use Lemma 2.1 (part (b)) then we have

$$u(t) \geq k\frac{t}{r(t)}\int_t^\infty p(s)u(\sigma(s))\mathrm{d}s, \quad t \geq T_k \geq T \tag{13}$$

and with reference the monotonicity of the function u we obtain

$$1 \geq k\frac{t}{r(t)}\int_t^\infty p(s)\mathrm{d}s, \quad t \geq T^* \geq T_k,\ k \in (0,1), \tag{14}$$

which means that

$$\limsup_{t\to\infty} \frac{t}{r(t)}\int_t^\infty p(s)\mathrm{d}s < \infty.$$

Continue likewise as in the proof of Theorem 3.3 we obtain a contradiction which finishes the proof. □

As demonstration of using the above Theorem 3.4 we present the following example.

Example 5. We consider the ordinary linear differential equation

$$\left(\sqrt{t}u'(t)\right)' + \frac{1}{t\sqrt{t}}u(t) = 0, \quad t \geq 1. \tag{15}$$

Then $r(t) = \sqrt{t}$ is the non-decreasing function. Moreover $p(t) = \frac{1}{t\sqrt{t}}$ and thus

$$\limsup_{t\to\infty} \frac{t}{r(t)}\int_t^\infty p(s)\mathrm{d}s = 2 > 1$$

what, according Theorem 3.4, means that the equation (15) is oscillatory. One solution of this equation is the function $u(t) = \sqrt[4]{t}\cos\left(\frac{\sqrt{15}}{2}\ln(2\sqrt{t}\,)\right)$.
Note that the equation (15) is so-called generalized Euler equation. Some information about such equations can be found in [8].

References

[1] J. H. Barrett, Oscillation theory of ordinary linear differential equations, *Adv. Math.* **3** (1969), 415–509.

[2] T. A. Canturija, Integral critria of oscillation of solutions of linear differential equations of higher order, *Differ. Uravn.* **XVI**, no. 3 (1980), 470–482 (in *Russian*).

[3] B. J. Harris, On the oscillation of solution of linear differential equations, *Mathematika* **31** (1984), 214–226.

[4] W. Leighton, Principal quadratic functionals and self-adjoint second-order differential equations, *Proc. Nat. Acad. Sci.* **35** (1949), 192–193.

[5] J. Ohriska, Oscillation of second order delay and ordinary differential equation, *Czechoslovak Math. J.* **34**(109) (1984), 107–112.

[6] J. Ohriska, Oscillation of differential equations and v-derivatives, *Czechoslovak Math. J.* **39**(114) (1989), 24–44.

[7] J. Ohriska, Second order self-adjoint differential equation and oscillation, *Proc. Int. Sci. Conf. Math., Zilina* (1998), 209–216.

[8] J. Ohriska, Problems with one quarter, *Czechoslovak Math. J.* **55**(130) (2005), 349–363.

[9] J. Ohriska, Oscillation of second-order linear delay differential equations, *Cent. Eur. J. Math.* **6**, no. 3 (2008), 439–452.

JÁN OHRISKA
DEPARTMENT OF MATHEMATICS
CATHOLIC UNIVERSITY
034 01 RUŽOMBEROK
SLOVAK REPUBLIC

E-mail: jan.ohriska@gmail.com, jan.ohriska@ku.sk

(Received January 31, 2011; revised February 8, 2012)