Publ. Math. Debrecen
82/1 (2013), 21–30
DOI: 10.5486/PMD.2013.5124

Characterizing injective operator space V for which $I_{11}(V) \cong B(H)$

By ALI REZA MEDGHALCHI (Tehran) and HAMED NIKPEY (Tehran)

Abstract. Let $V \cong B(K,H)$ where H and K are Hilbert spaces. Then we know that $I_{11}(V) \cong B(H)$. Let V be an injective operator space. In this paper we recover the above result and show that $I_{11}(V) \cong \oplus_{i=1}^{n} B(H_i)$ where $H_1, \dots, H_n$ are Hilbert spaces if and only if there are Hilbert spaces $K_1, \dots, K_n$ such that $V \cong \oplus_{i=1}^{n} B(K_i, H_i)$.

1. Introduction

Let $B(H)$ be the set of all bounded linear operators on the Hilbert space H. Operator spaces are the concrete closed subspaces of $B(H)$ as formulated in [3]. Given operator spaces V and W and a linear mapping $\varphi : V \to W$, for each $n \in \mathbb{N}$, there is a corresponding linear mapping $\varphi_n : M_n(V) \to M_n(W)$ defined by $\varphi_n(T) = [\varphi(T_{i,j})]$ for all $T = [T_{i,j}] \in M_n(V)$. The completely bounded norm of φ is defined by

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\},$$

(this might be infinite). It is evident that the norms $\|\varphi_n\|$ form an increasing sequence

$$\|\varphi\| \le \|\varphi_2\| \le \cdots \le \|\varphi_n\| \le \cdots \le \|\varphi\|_{cb}.$$

It is said that φ is completely bounded (respectively, completely contractive) if $\|\varphi\|_{cb} < \infty$ (respectively, $\|\varphi\|_{cb} \le 1$). We say that the operator spaces V and W are completely isometrically isomorphic if there is an onto linear map $\varphi : V \to W$

Mathematics Subject Classification: 47L25, 47L05.
Key words and phrases: operator space, Paulsen operator system, injective operator space, left multiplier, Haagerup tensor product.

such that each mapping $\varphi_n : M_n(V) \to M_n(W)$ is an isometry. This notion is indicated by $V \cong W$.

Recall that an operator space V is injective, if for given operator spaces $W_1 \subseteq W_2$, any completely bounded linear map $\varphi_1 : W_1 \to V$ can be extended to a completely bounded map $\varphi_2 : W_2 \to V$ with $\|\varphi_2\|_{cb} = \|\varphi_1\|_{cb}$. It has been known for a long time that $B(H)$ is an injective operator space for any Hilbert space H (see [7]). If we are given a linear space V, then we say that a linear mapping $\varphi : V \to V$ is a projection if $\varphi^2 = \varphi$.

Lemma 1.1. *Let V be an operator subspace of $B(H)$. Then V is an injective operator space if and only if there is a completely contractive projection of $B(H)$ onto V.*

Let V be an operator subspace of $B(H)$. From Wittstock Theorem [7], $B(H)$ is an injective operator space contains V. Hamana [5], [6] and Ruan [4] independently have shown that for any operator space V in $B(H)$ there is a minimal injective operator subspace of $B(H)$ contains V, called injective envelope of V and denoted by $I(V)$.

Let A be a unital C^*-algebra with the unit $I = I_A$. The operator space $V \subseteq A$ is called an operator system if $I \in V$ and $V^* = V$ such that V^* is the space of all adjoint of members of V. If V is an operator system then $M_n(V)$ is an operator system. Given operator systems V and W, a linear mapping $\varphi : V \to W$ is called completely positive if $\varphi_n \geq 0$ for all $n \in \mathbb{N}$, and we then write $\varphi \geq_{cb} 0$. We need the following theorem which proof is found in [3, Theorem 6.1.3].

Theorem 1.1. *If $V \subseteq B(H)$ is an injective operator system, then there is a unique multiplication*

$$\circ : V \times V \to V$$

for which, together with its given $*$-operation and norm, is a C^*-algebra with the multiplication identity I.

Suppose V is an injective operator system, we may fix a completely contractive onto projection $\varphi : B(H) \to V$. Given $T, S \in V$, we define an operation $\circ_\varphi$ on V by

$$T \circ_\varphi S = \varphi(TS).$$

With this definition $(V, \circ_\varphi)$ is a unital C^*-algebra. For more detail see [3, Section 6].

Lemma 1.2. *Let V be a unital operator space. Then $I(V)$ is a unital injective C^*-algebra.*

PROOF. From Hamana theorem, V has an injective envelope $I(V)$ in $B(H)$, thus there is a completely contractive onto projection $\varphi : B(H) \to I(V)$. Since V is a unital operator space, φ is a unital map and so φ is completely positive [3, Corollary 5.1.2] therefore, $I(V)$ is a unital operator system. Now by Theorem 1.1, $I(V)$ is a unital injective C^*-algebra. □

Let V be a subspace of $B(H)$, the Paulsen operator system $\mathcal{S}(V)$ is defined by

$$\mathcal{S}(V) = \begin{bmatrix} \mathbb{C}I_H & V \\ V^* & \mathbb{C}I_H \end{bmatrix} = \left\{ \begin{bmatrix} \alpha & T \\ S^* & \beta \end{bmatrix} : T, S \in V, \alpha, \beta \in \mathbb{C} \right\}$$

in $M_2(B(H))$, where the entries α and β stand for αI_H and βI_H and S^* means the adjoint of S in $B(H)$. Hence $\mathcal{S}(V)$ is an operator subspace of $B(H^2)$. From Hamana and Ruan Theorems, $\mathcal{S}(V)$ has an injective envelope in $B(H^2)$ which we denote by $I(\mathcal{S}(V))$. So there is a unital completely contractive onto projection $\Phi : B(H^2) \to I(\mathcal{S}(V))$. Therefore $I(\mathcal{S}(V))$ is a unital C^*-algebra with the new product $T \circ_\Phi S = \Phi(TS)$ where $T, S \in I(\mathcal{S}(V))$. Indeed, since $\Phi : B(H \oplus K) \to B(H \oplus K)$ fixes the C^*-algebra $\mathbb{C} \oplus \mathbb{C}$ which is the diagonal of $\mathcal{S}(V)$, it follows immediately that the following elements of $\mathcal{S}(V)$ are two self adjoint projections with sum I in the C^*-algebra $I(\mathcal{S}(V))$:

$$p = \begin{bmatrix} I_H & 0 \\ 0 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 & 0 \\ 0 & I_H \end{bmatrix}.$$

Since $\Phi(p) = p$ and $\Phi(q) = q$, it follows from [1, 2.6.16] that Φ is 'corner-preserving', thus we can write $\Phi = \left[\begin{smallmatrix} \varphi_1 & \varphi \\ \varphi^* & \varphi_2 \end{smallmatrix}\right]$, such that $\varphi^*(T) = \varphi(T^*)^*$ for any $T \in B(H)$. Therefore, we may decompose $I(\mathcal{S}(V))$ to write it as consisting of 2×2 matrices. Hamana has shown that $pI(\mathcal{S}(V))q$, the 1-2 corner of $I(\mathcal{S}(V))$, is the injective envelope of V. The four corners of $I(\mathcal{S}(V))$ we will name:

$$I(\mathcal{S}(V)) = \begin{bmatrix} I_{11}(V) & I(V) \\ I(V)^* & I_{22}(V) \end{bmatrix}.$$

It is clear that $I_{11}(V)$ and $I_{22}(V)$ are also injective C^*-algebras with the new product. We define

$$IM_l(V) = \{T \in I_{11}(V) : T \circ_\varphi V \subseteq V\},$$

where $T \circ_\varphi S = \varphi(TS)$ for any $T \in I_{11}(V)$ and $S \in V$. From [1, Theorem 4.5.5], V is a left operator $IM_l(V)$-module. Now we define the left multiplier space of V

to be the family of all linear maps $\psi : V \to V$ such that there exists a Hilbert space K, $T \in B(K)$, and a linear completely isometry $\pi : V \to B(K)$ such that $\pi(\psi(S)) = T\pi(S)$ for any $S \in V$. We define the multiplier norm of ψ to be the infimum of $\|T\|$ over all possible K, T, π as above and denote that with $\mathcal{M}_l(V)$. From [1, Theorem 4.5.5] we have $\mathcal{M}_l(V)$ and $IM_l(V)$ are completely isometrically isomorphic, i.e. $\mathcal{M}_l(V) \cong IM_l(V)$. Let H and K be Hilbert spaces. Then from [1] 4.5.1 and 4.12 we have

$$\mathcal{M}_l(B(K,H)) \cong IM_l(B(K,H)) = \{T \in I_{11}(B(K,H)) : T \circ B(K,H) \subseteq B(K,H)\}$$

(with the new product) is a subspace of $B(H)$. From the definition of $\mathcal{M}_l(B(K,H))$ we have $B(H) \subseteq \mathcal{M}_l(B(K,H))$. Thus

$$\mathcal{M}_l(B(K,H)) \cong IM_l(B(K,H)) = B(H).$$

2. Main results

Let $V = B(K,H)$ be such that H and K are Hilbert spaces. Then from the above notation we have $I_{11}(V) \cong B(H)$. By using this, our aim is to characterize injective operator space V for which $I_{11}(V) \cong B(H)$. We will show that this is the case if and only if $V \cong B(K,H)$ where K is a Hilbert space. We then characterize the operator spaces V for which $I_{11}(V) \cong \oplus_{i=1}^{n} B(H_i)$.

Theorem 2.1. *Let V be an injective operator space. Then $I_{11}(V) \cong B(H)$ if and only if V is completely isometrically isomorphic to $B(K,H)$ for some Hilbert space K.*

Proof. ($\Leftarrow$) Let V be completely isometrically isomorphic to $B(K,H)$. Then from the definition of the left multiplier algebra and [1, Theorem 4.5.5] we have

$$I_{11}(V) \cong I_{11}(B(K,H)) = IM_l(B(K,H)) \cong \mathcal{M}_l(B(K,H)) = B(H).$$

($\Rightarrow$) Let V be an injective operator subspace of $B(L)$ for some Hilbert space L. By the injectivity of $I(\mathcal{S}(V)) \subseteq B(L^2)$ there is a completely contractive onto projection

$$\Phi = \begin{bmatrix} \varphi_1 & \varphi \\ \varphi^* & \varphi_2 \end{bmatrix} : B(L^2) \to I(\mathcal{S}(V)) = \begin{bmatrix} I_{11}(V) & V \\ V^* & I_{22}(V) \end{bmatrix}.$$

Thus by Theorem 1.1, $(I(\mathcal{S}(V)), \circ_\Phi)$ is a unital injective C^*-algebra with the new product. From assumption, there is some unital completely isometric isomorphism mapping $\psi : I_{11}(V) \to B(H)$. Now we define the map $m : V \otimes V^* \to B(H)$ by $m(T \otimes S^*) = \psi(\varphi_1(TS^*))$ for any $T, S \in V$ which infact comes from

$$\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \circ_\Phi \begin{bmatrix} 0 & 0 \\ S^* & 0 \end{bmatrix} = \Phi\left(\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ S^* & 0 \end{bmatrix}\right) = \begin{bmatrix} \varphi_1(TS^*) & 0 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} I_{11}(V) & 0 \\ 0 & 0 \end{bmatrix}.$$

We have $I(\mathcal{S}(V))$ is a C^*-algebra and also an operator algebra with the new product. Thus from [1, Theorem 2.3.2], m can be extended to Haagerup tensor product $V \otimes_h V^*$ and trivially we have $\|m\|_{cb} = \|\psi \circ \varphi_1\|_{cb} \leq 1$. From [3, Theorem 9.4.3] there exists a Hilbert space K and completely contractive mappings

$$\psi_1 : V \to B(K, H) \quad \text{and} \quad \psi_2 : V^* \to B(H, K)$$

such that for any $T, S \in V$ we have

$$\psi(\varphi_1(TS^*)) = m(T \otimes S^*) = \psi_1(T)\psi_2(S^*).$$

Let p be a projection in K onto the closure of $\psi_2(V^*)H$. Then we have

$$\psi(\varphi_1(TS^*)) = m(T \otimes S^*) = [\psi_1(T)p][p\psi_2(S^*)],$$

thus we can assume that $\psi_2(V^*)H$ is a dense subspace of Hilbert space K. For any $T \in V$ we have

$$\|T\|^2 = \left\|\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}\right\|^2 = \left\|\begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \circ_\Phi \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}^*\right\|$$
$$= \|\varphi_1(TT^*)\| = \|\psi(\varphi_1(TT^*))\| = \|\psi_1(T)\psi_2(T^*)\| \leq \|\psi_1(T)\|\|T\| \leq \|T\|^2.$$

So ψ_1 is a completely isometry and $V \cong \psi_1(V)$. Let I_H and I_K be identities for $B(H)$ and $B(K)$. Then $\mathcal{S}(\psi_1(V))$ and $\begin{bmatrix} \mathbb{C}I_H & \psi_1(V) \\ \psi_1(V)^* & \mathbb{C}I_K \end{bmatrix}$ are completely isometrically isomorphic together, so are $I(\mathcal{S}(\psi_1(V)))$ and $I\left(\begin{bmatrix} \mathbb{C}I_H & \psi_1(V) \\ \psi_1(V)^* & \mathbb{C}I_K \end{bmatrix}\right)$. Therefore we have

$$\begin{bmatrix} I_{11}(\psi_1(V)) & \psi_1(V) \\ \psi_1(V)^* & I_{22}(\psi_1(V)) \end{bmatrix} = I\left(\begin{bmatrix} \mathbb{C}I_H & \psi_1(V) \\ \psi_1(V)^* & \mathbb{C}I_K \end{bmatrix}\right) \subseteq B(H \oplus K).$$

Now we want to prove $B(H)\psi_1(V) \subseteq \psi_1(V)$ and then $I_{11}(\psi_1(V)) = B(H)$. Let $T \in B(H)$. Then there is some $T' \in I_{11}(V)$ such that $T = \psi(T')$. Therefore for any $S_1, S_2 \in V$ and $h \in H$ we have

$$\begin{aligned}[T\psi_1(S_1)]\psi_2(S_2^*)h &= \psi(T')[\psi_1(S_1)\psi_2(S_2^*)]h = \psi(T')\psi(\varphi_1(S_1S_2^*))h \\ &= \psi(T' \circ_{\varphi_1} \varphi_1(S_1S_2^*))h = \psi(\varphi(T'S_1) \circ_{\varphi_1} S_2^*))h \\ &= \psi(\varphi_1(\varphi(T'S_1)S_2^*))h = [\psi_1(\varphi(T'S_1))]\psi_2(S_2^*)h.\end{aligned}$$

Thus for any $S \in V$ we have $T\psi_1(S) = \psi_1(\varphi(T'S))$ on $\psi_2(V^*)H$. Therefore $T\psi_1(S) = \psi_1(\varphi(T'S)) \in \psi_1(V)$. That means, $B(H)\psi_1(V) \subseteq \psi_1(V)$ and therefore the new product of $I_{11}(\psi_1(V))$ on $\psi_1(V)$ comes from the usual product of $B(H)$ on $B(K,H)$.

By [2, Corollary 1.2], $\varphi_1(VV^*) = V \circ_{\varphi_1} V^*$ is an essential ideal of the C^*-algebra $(I_{11}(V), \circ_{\varphi_1})$. By assumption, since $\psi : (I_{11}(V), \circ_{\varphi_1}) \to B(H)$ is a unital completely isometric surjection between two C^*-algebras, by [1, Corollary 1.3.10], ψ is an $*$-homomorphism. Thus $\psi(\varphi_1(VV^*)) = \psi_1(V)\psi_2(V^*)$ is an essential ideal of $B(H)$. Let $T \in B(H)$ such that $T\psi_1(V) = 0$. Then $T\psi_1(V)\psi_2(V^*) = 0$. Since $\psi_1(V)\psi_2(V^*)$ is an essential ideal of $B(H)$, $T = 0$. Therefore by definition of the left multiplier algebra of an operator space, for any $T \in B(H)$, $\varphi_T \in \mathcal{M}_l(\psi_1(V))$ by definition $\varphi_T(\psi_1(S)) = T\psi_1(S)$ for any $S \in V$. By [1, Theorem 4.5.2] there is some $T'' \in I_{11}(\psi_1(V)) \subseteq B(H)$ such that $\varphi_T(\psi_1(S)) = T'' \circ \psi_1(S)$ (with the new product). On the other hand, the product of $I_{11}(\psi_1(V))$ on $\psi_1(V)$ is the usual product. So, for any $S \in V$,

$$T\psi_1(S) = \varphi_T(\psi_1(S)) = T'' \circ \psi_1(S) = T''\psi_1(S),$$

i.e. $T'' = T$. Therefore $I_{11}(\psi_1(V)) = B(H)$ such that $B(H)\psi_1(V) \subseteq \psi_1(V)$.

Let $W = \psi_1(V)$. Then $\mathcal{S}(W)$ is completely isometrically isomorphic to $\begin{bmatrix} \mathbb{C}I_H & W \\ W^* & \mathbb{C}I_K \end{bmatrix}$, so are $I(\mathcal{S}(W))$ and $\begin{bmatrix} B(H) & W \\ W^* & I_{22}(W) \end{bmatrix}$. Thus there is some completely contractive onto projection

$$\Phi : B(H \oplus K) \to \begin{bmatrix} B(H) & W \\ W^* & I_{22}(W) \end{bmatrix} \subseteq B(H \oplus K)$$

such that $\begin{bmatrix} B(H) & W \\ W^* & I_{22}(W) \end{bmatrix}$ is a C^*-algebra with the new product. Φ is the identity on $B(H)$. Thus from [3, Corollary 5.2.2] we have

$$\Phi\left(\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}\right)\Phi\left(\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix},$$

for any $T \in B(H)$ and $S \in W$. That means the new product of $B(H)$ on W comes from the usual product of $B(H)$ on $B(K, H)$ and therefore $B(H)W \subseteq W$.

Let $\{e_\alpha\}_{\alpha\in\Gamma}$ be an orthogonal basis for the Hilbert space H. Then $e_\alpha \otimes e_\alpha$'s are projections in $B(H)$, where $(e_\alpha \otimes e_\alpha)(h) = \langle h, e_\alpha\rangle e_\alpha$ for any $h \in H$. $(e_\alpha \otimes e_\alpha)W \subseteq W$. Since, the product of $B(H)$ on W is the usual product. For any $T \in (e_\alpha \otimes e_\alpha)W$ there is a $h \in K$ such that $T(g) = \langle g, h\rangle e_\alpha$ for each $g \in K$, we then denote T with T_{h,e_α}. Let $K_\alpha = \{h \in K : T_{h,e_\alpha} \in (e_\alpha \otimes e_\alpha)W\}^{-\|\cdot\|}$. We have $(e_\alpha \otimes e_\beta)W \subseteq W$ for each $\alpha, \beta \in \Gamma$, thus $K_\alpha = K_\beta$ for each $\alpha, \beta \in \Gamma$. So we can assume that $W \subseteq B(K', H)$ where $K' = K_\alpha$ for some $\alpha \in \Gamma$.

Also $\Sigma_{\alpha\in F}(e_\alpha \otimes e_\alpha)W \subseteq W$ for any finite set F. Therefore $K(K', H)$, the space of compact operators in $B(K', H)$ is a subspace of W too. We have

$$K(H \oplus K') = \begin{bmatrix} K(H) & K(K', H) \\ K(H, K') & K(K') \end{bmatrix}$$

is an essential ideal of

$$B(H \oplus K') = \begin{bmatrix} B(H) & B(K', H) \\ B(H, K') & B(K') \end{bmatrix}.$$

Therefore by [2] we have $I(K(H \oplus K')) = B(H \oplus K')$, so $W = B(K', H)$. Thus we have $V \cong B(K', H)$ □

Corollary 2.1. *Let V be an injective operator space. Then V is completely isometrically isomorphic to some row Hilbert operator space if and only if $I_{11}(V) \cong C$.*

Proof. By Theorem 2.1 we have $I_{11}(V) \cong \mathbb{C}$ if and only if $V \cong B(K, \mathbb{C})$ for some Hilbert space K, such that $B(K, \mathbb{C})$ is a row Hilbert space. □

Corollary 2.2. *Let V be an operator space. Then $I_{11}(V) = I_{22}(V) \cong \mathbb{C}$ if and only if $V \cong \mathbb{C}$.*

Proof. If $V \cong \mathbb{C}$, then obviously we have $I_{11}(V) = I_{22}(V) \cong \mathbb{C}$. On the other hand, if $I_{11}(V) = I_{22}(V) \cong \mathbb{C}$ then V is completely isomorphic to row and column Hilbert operator spaces. Therefore V is one dimensional. □

Note that for any operator space V, we have $I(\mathcal{S}(V)) = I(\mathcal{S}(I(V)))$ and so

$$I_{11}(V) = I_{11}(I(V)) = IM_l(I(V)) \cong M_l(I(V))$$

and

$$I_{22}(V) = I_{22}(I(V)) = IM_r(I(V)) \cong M_r(I(V)).$$

Lemma 2.1. *Let V and W be two operator spaces. Then $I_{11}(V \oplus W) = I_{11}(V) \oplus I_{11}(W)$ and $I_{22}(V \oplus W) = I_{22}(V) \oplus I_{22}(W)$.*

Proof. For any operator spaces V and W, we have $I(V \oplus W) = I(V) \oplus I(W)$. So

$$\mathcal{S}(V \oplus W) \subseteq \begin{bmatrix} I_{11}(V) \oplus I_{11}(W) & I(V) \oplus I(W) \\ I(V)^* \oplus I(W)^* & I_{22}(V) \oplus I_{22}(W) \end{bmatrix} \cong I(\mathcal{S}(V)) \oplus I(\mathcal{S}(W)),$$

such that $I(\mathcal{S}(V)) \oplus I(\mathcal{S}(W))$ is an injective operator space. So $I_{11}(V \oplus W) \subseteq I_{11}(V) \oplus I_{11}(W)$. On the other hand, let $T \in I_{11}(V) = IM_l(I(V)) \cong M_l(I(V))$ and $S \in I_{11}(W) = IM_l(I(W)) \cong M_l(I(W))$. Then

$$\begin{aligned} \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \in M_l(I(V) \oplus I(W)) &\cong IM_l(I(V) \oplus I(W)) = IM_l(I(V \oplus W)) \\ &= I_{11}(I(V \oplus W)) = I_{11}(V \oplus W). \end{aligned}$$

This means that $I_{11}(V) \oplus I_{11}(W) \subseteq I_{11}(V \oplus W)$, so the proof is completed. For I_{22} one can use the same argument. □

Theorem 2.2. *Let V be an injective operator space. Then $I_{11}(V) \cong \oplus_{i=1}^n B(H_i)$ if and only if $V \cong \oplus_{i=1}^n B(K_i, H_i)$ where H_i and K_i are Hilbert spaces $(1 \leq i \leq n)$.*

Proof. ($\Leftarrow$) Let $V \cong \oplus_{i=1}^n B(K_i, H_i)$. Then by Lemma 2.1 and Theorem 2.1, we have

$$I_{11}(V) \cong I_{11}(\oplus_{i=1}^n B(K_i, H_i)) = \oplus_{i=1}^n I_{11}(B(K_i, H_i)) \cong \oplus_{i=1}^n B(H_i).$$

($\Rightarrow$) Let V be an injective operator subspace of $B(L)$ for some Hilbert space L. From the injectivity of $I(\mathcal{S}(V))$ there is a completely contractive onto projection

$$\Phi = \begin{bmatrix} \varphi_1 & \varphi \\ \varphi^* & \varphi_2 \end{bmatrix} : B(L^2) \to I(\mathcal{S}(V)) = \begin{bmatrix} I_{11}(V) & V \\ V^* & I_{22}(V) \end{bmatrix}.$$

Thus by Theorem 1.1, $(I(\mathcal{S}(V)), \circ_\Phi)$ is a unital injective C^*-algebra with the new product. Let $\psi : I_{11}(V) \to \oplus_{i=1}^n B(H_i)$ be a unital completely isometric isomorphism. Thus by the same technique of the above theorem, we can define a completely contractive map $m : V \otimes_h V^* \to \oplus_{i=1}^n B(H_i)$ by $m(T \otimes S^*) =$

$\psi(\varphi_1(TS^*))$. Suppose $\pi_i : \oplus_{i=1}^n B(H_i) \to B(H_i)$ is a completely contractive onto projection, then there are Hilbert spaces K_i and completely contractive mappings

$$\psi_i : V \to B(K_i, H_i) \quad \text{and} \quad \psi_i' : V^* \to B(H_i, K_i)$$

such that for any $T, S \in V$ we have

$$\pi_i(m(T \otimes S^*)) = \psi_i(T)\psi_i'(S^*).$$

For each $T \in V$ we have

$$\begin{aligned}\|T\|^2 &= \left\| \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \circ_\Phi \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}^* \right\| = \|\varphi_1(TT^*)\| = \|\psi(\varphi_1(TT^*))\| = \|m(T \otimes T^*)\| \\ &= \max_{1 \le i \le n} \|\pi_i(m(T \otimes T^*))\| = \max_{1 \le i \le n} \|\psi_i(T)\psi_i'(T^*)\| \\ &\le \max_{1 \le i \le n} \|\psi_i(T)\|\|T\| \le \|T\|^2.\end{aligned}$$

Therefore there are completely isometric map

$$\Psi = (\psi_i)_{i=1}^n : V \to \oplus_{i=1}^n B(K_i, H_i)$$

and completely contractive map

$$\Psi' = (\psi_i')_{i=1}^n : V^* \to \oplus_{i=1}^n B(H_i, K_i),$$

where

$$\psi(\varphi_1(TS^*)) = m(T \otimes S^*) = \Psi(T)\Psi'(S^*).$$

Thus $V \cong \Psi(V) \subseteq \oplus_{i=1}^n B(K_i, H_i)$, respectively. Let I_1 and I_2 be identities for operator space $\oplus_{i=1}^n B(H_i)$ and $\oplus_{i=1}^n B(K_i)$. Thus $\mathcal{S}(\Psi(V))$ and $\begin{bmatrix} \mathbb{C}I_1 & \Psi(V) \\ \Psi(V)^* & \mathbb{C}I_2 \end{bmatrix}$ are completely isometrically isomorphic, so are $I(\mathcal{S}(\Psi(V)))$ and $I\left(\begin{bmatrix} \mathbb{C}I_1 & \Psi(V) \\ \Psi(V)^* & \mathbb{C}I_2 \end{bmatrix}\right)$. Therefore there is a completely contractive onto projection

$$\begin{bmatrix} \varphi_1' & \varphi' \\ \varphi'^* & \varphi_2' \end{bmatrix} : \begin{bmatrix} \oplus_{i=1}^n B(H_i) & \oplus_{i=1}^n B(K_i, H_i) \\ \oplus_{i=1}^n B(H_i, K_i) & \oplus_{i=1}^n B(K_i) \end{bmatrix} \to \begin{bmatrix} I_{11}(\Psi(V)) & \Psi(V) \\ \Psi(V)^* & I_{22}(\Psi(V)) \end{bmatrix},$$

such that the first operator space is completely isometrically isomorphic to the injective operator space $\oplus_{i=1}^n B(H_i \oplus K_i)$.

Similar to Theorem 2.1 we have $I_{11}(\Psi(V)) = \oplus_{i=1}^n B(H_i)$ and the new product of $\oplus_{i=1}^n B(H_i)$ on $\Psi(V)$ is the usual product on $\oplus_{i=1}^n B(K_i, H_i)$. Let P_i be a projection on $\oplus_{i=1}^n B(H_i)$ which is the identity on $B(H_i)$ and zero else. Thus $P_i V \subseteq V$ for $1 \le i \le n$. Then we can write $V = \oplus_{i=1}^n V_i$ such that V_i are injective operator spaces in $B(K_i, H_i)$ and $I_{11}(V_i) = B(H_i)$ for $1 \le i \le n$. Hence by Theorem 2.1 there are Hilbert spaces $K_i' \subseteq K_i$ such that $V_i \cong B(K_i', H_i)$ for $1 \le i \le n$, i.e. $V \cong \oplus_{i=1}^n B(K_i', H_i)$. □

Corollary 2.3. *Let V be an injective operator space. Then V is a direct sum of row Hilbert spaces if and only if $I_{11}(V) = \oplus_{i=1}^{n}\mathbb{C}$.*

PROOF. Put $H_i = \mathbb{C}$ in the above theorem. □

Note that all of above statements are true for $I_{22}(V)$ too.

References

[1] D. BLECHER and C. LE MERDY, Operator algebras and their modules-an operator space approach, London Mathematical Society Monographs, New Series 30, *Oxford Science Publications*, *Oxford*, 2004.

[2] D. BLECHER and V. I. PAULSEN, Multipliers of operator spaces and the injective envelope, *Pacific J. Math.* **200**, no. 1 (2001), 1–17.

[3] E. G. EFFROS and Z. J. RUAN, Operator Space, London Math. Soc. Monographs, New Series 23, *Oxford University Press*, *New York*, 2000.

[4] Z. J. RUAN, Injectivity of Operator Spaces, *Trans. Amer. Math. Soc.* **315**, no. 1 (1989), 89–104.

[5] M. HAMANA, Injective envelopes of operator systems, *Publ. RIMS, Kyoto Univ.* **15** (1979), 773–785.

[6] M. HAMANA, Injective envelopes of dynamical systems, in "Operator Algebras and Operator Theory",, *Pitman Research Notes* **271** (1992), 69–77.

[7] G. WITTSTOCK, Extension of completely bounded C^*-module homomorphisms, Proceedings, Conf. on Operator Algebras and Group Representations, 1983, 238–250.

ALI REZA MEDGHALCHI
DEPARTMENT OF MATHEMATICS
TARBIAT MOALLEM UNIVERSITY
50 TALEGHANI AVENUE
TEHRAN 15618
IRAN

E-mail: a_medghalchi@saba.tmu.ac.ir

HAMED NIKPEY
DEPARTMENT OF MATHEMATICS
TARBIAT MOALLEM UNIVERSITY
50 TALEGHANI AVENUE
TEHRAN 15618
IRAN

E-mail: hamednikpey@gmail.com

(Received February 4, 2011; revised November 22, 2011)