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On the Diophantine equation $f(x)f(y) = f(z^2)$

By YONG ZHANG (Hangzhou) and TIANXIN CAI (Hangzhou)

Abstract. Let $f \in \mathbb{Q}[X]$, $\deg(f) \geq 2$, in this paper we extend the Diophantine equation $f(x)f(y) = f(z)^2$ for $f(X) = X^2 - tX$ from $t = 2k$ to $t = 2k+1$, then we mainly consider the Diophantine equation $f(x)f(y) = f(z^2)$, and prove that there are infinitely many nontrivial positive integer solutions for some special cases.

1. Introduction

Let $f \in \mathbb{Q}[X]$, $\deg(f) \geq 2$, the Diophantine equation

$$f(x)f(y) = f(z)^2 \tag{1}$$

has been studied by several authors.

In 1963, Schinzel and Sierpinski [5] investigated (1) for $f(X) = X^2 - 1$, they showed that there are infinitely many solutions in integers. In 1967, Szymiczek [6] got the same result for $f(X) = X^2 - k^2$. In 2007, Bennett [1] showed that (1) has no nontrivial integer solution for $f(X) = X^k - 1$, $k \geq 4$.

In 2006, Katayama [3] studied (1) for $f(X) = X^2 + 1$, he showed that there are infinitely many solutions in integers. In 2008, Ulas [8] got the same result for $f(X) = X^2 + k$, $k = \pm(a^2 - 2b^2)$. Some related information on this problem can be found in [2]: *D23 Some quartic equations.*

Note that if we take $X = Y - k$ for $f(X) = X^2 - k^2$, then $f(Y-k) = Y^2 - 2kY$, i.e., (1) has infinitely many nontrivial integer solutions for $f(X) = X^2 - 2kX$.

In this paper, firstly, we extend $2k$ to $2k+1$, and have the following theorem.

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Theorem 1. *For* $f(X) = X^2 - (2k+1)X$, $k \in \mathbb{Z}$, (1) *has infinitely many nontrivial integer solutions.*

Corollary. *For* $f(X) = X^2 - kX$, $k \in \mathbb{Z}$, (1) *has infinitely many nontrivial integer solutions.*

Secondly, we consider an analogous Diophantine equation

$$f(x)f(y) = f(z^2). \tag{2}$$

The motivation of studying (2) is that the same degree of the left and right hand, it seems that (2) is more difficult than (1). The transformation

$$z = \frac{x-y}{2}$$

is powerful to solve (1) for some special cases, from some numerical calculations, we find that there exists analogous transformation to solve (2) for some special cases.

When $\deg(f) = 2$, i.e., $f(X)$ is a quadratic, we consider the reducible and irreducible cases, respectively, and get the following theorems.

Theorem 2. *For* $f(X) = X^2 - k^2X$, $k \in \mathbb{Z}^+$, (2) *has infinitely many nontrivial positive integer solutions.*

Theorem 3. *For* $f(X) = X^2 + X + 1$, (2) *has infinitely many nontrivial positive integer solutions.*

When $\deg(f) = 3$, i.e., $f(X)$ is a cubic, we just get the following result, and it is related to the other paper of the authors [9], which leads to the initial mind to consider (2). For the general case of cubic, there is few results just now, maybe there are some other special cases could be solved for (2).

Theorem 4. *For* $f(X) = X^3 - X$, *(2) has infinitely many nontrivial positive integer solutions.*

In Section 2, we shall prove the theorems from 1 to 4. Section 3 is devoted to considering the rational solutions of (2) and we have the following theorem.

Theorem 5. *For* $f(X) = X^2 - kX$, $k \in \mathbb{Z} - \{0\}$, (2) *has infinitely many nontrivial positive rational solutions.*

In the last section, we raise some related questions for (2).

2. Proofs of the theorems

Noting that the transformation

$$z = \frac{x - y}{2}$$

is not suitable to solve (1) for $f(X) = X^2 - (2k+1)X$, by numerical calculation, we find that the transformation

$$z = \frac{x + y}{2}$$

is working, but it leads to negative integers for x.

PROOF OF THEOREM 1. For $f(X) = X^2 - (2k+1)X$, let

$$z = \frac{x + y}{2},$$

by using *Maple*, we have

$$\begin{aligned} &f(x)f(y) - f(z)^2 \\ &\quad = -\frac{(x-y)^2}{16}\big(x^2 + 6xy + y^2 - (8k+4)x - (8k+4)y + (4k+2)^2\big) = 0. \end{aligned}$$

Let $x^2 + 6xy + y^2 - (8k+4)x - (8k+4)y + (4k+2)^2 = 0$, then

$$\big(x + 3y - (4k+2)\big)^2 - 2\big(2y - (2k+1)\big)^2 = 2(2k+1)^2.$$

Taking $X = x + 3y - (4k+2), Y = 2y - (2k+1)$, we get

$$X^2 - 2Y^2 = 2(2k+1)^2.$$

An infinity of positive integer solutions are given by

$$X_n + Y_n\sqrt{2} = \Big(2(2k+1) + 2(k+1)\sqrt{2}\Big)\Big(3 + 2\sqrt{2}\Big)^n, \quad n \geq 0,$$

leading to $X_n = 3X_{n-1} + 4Y_{n-1}$, $Y_n = 2X_{n-1} + 3Y_{n-1}$. Then

$$\begin{cases} X_n = 6X_{n-1} - X_{n-2}, & X_0 = 2(2k+1), X_1 = 10(2k+1); \\ Y_n = 6Y_{n-1} - Y_{n-2}, & Y_0 = 2k+1, Y_1 = 7(2k+1). \end{cases}$$

It is easy to prove that

$$\frac{X_n}{2k+1} \equiv 0 \pmod{2}, \qquad \frac{Y_n}{2k+1} \equiv 1 \pmod{2},$$

then $y_n \in \mathbb{Z}$, $x_n \in \mathbb{Z}$. Hence, we have

$$\begin{cases} x_n = 6x_{n-1} - x_{n-2} - 8(2k+1), & x_0 = 2k+1, x_1 = 0; \\ y_n = 6y_{n-1} - y_{n-2} - 2(2k+1), & y_0 = 2k+1, y_1 = 4(2k+1). \end{cases}$$

Then

$$z_n = \frac{x_n + y_n}{2} = 6z_{n-1} - z_{n-2} - 5(2k+1), \quad z_0 = 2k+1, z_1 = 2(2k+1).$$

Therefore, we have infinitely many nontrivial integer solutions (x_n, y_n, z_n) for (1). □

For example, when $k = 0$, we have $f(X) = X^2 - X$, then

$$\begin{cases} x_n = 6x_{n-1} - x_{n-2} - 8, & x_0 = 1, x_1 = 0; \\ y_n = 6y_{n-1} - y_{n-2} - 2, & y_0 = 1, y_1 = 4; \\ z_n = \dfrac{x_n + y_n}{2} = 6z_{n-1} - z_{n-2} - 5, & z_0 = 1, z_1 = 2. \end{cases}$$

Solving (2) seems difficult for the general case, fortunately, we find that for $f(X) = X^2 - k^2X$, $k \in \mathbb{Z}^+$, (2) has infinitely many nontrivial positive integer solutions, which mainly due to the transformation

$$z = k\frac{y-x}{2} \Longleftrightarrow y = x + \frac{2z}{k},$$

which is similar to the transformation

$$z = \frac{x-y}{2}$$

for (1). Perhaps, the transformation is essential, since from the viewpoint of geometry, (1) and (2) denote surfaces in the 3-dimensional space, the transformation $z = \frac{x-y}{2}$ and $z = k\frac{y-x}{2}$ denote planes, the solutions of (1) and (2) mean that there are infinitely many nontrivial positive integer points (x, y, z) on the planes $z = \frac{x-y}{2}$ and $z = k\frac{y-x}{2}$. However, it is difficult to finding more cases with this transformation. In the following, we give the proof of theorem 2.

Proof of Theorem 2. For $f(X) = X^2 - k^2X$, let

$$y = x + \frac{2z}{k},$$

by using *Maple*, we have

$$f(x)f(y) - f(z^2) = \frac{(kx^2 - k^3x + 2xz - k^2z + kz^2)(kx^2 - k^3x + 2xz - k^2z - kz^2)}{k^2} = 0.$$

Let $kx^2 - k^3x + 2xz - k^2z - kz^2 = 0$, then

$$(2kx + 2z - k^3)^2 - (k^2 + 1)(2z)^2 = k^6.$$

Taking $X = 2kx + 2z - k^3, Y = 2z$, we have

$$X^2 - (k^2 + 1)Y^2 = k^6.$$

Since $k^2 + 1$ is not a square, so we have infinitely integer solutions for this Pell equation. An infinity of positive integer solutions are given by

$$X_n + Y_n\sqrt{k^2+1} = \left(k(k^2+2) + 2k\sqrt{k^2+1}\right)\left(2k^2+1+2k\sqrt{k^2+1}\right)^n, \quad n \geq 0,$$

leading to

$$X_n = (2k^2 + 1)X_{n-1} + 2k(k^2 + 1)Y_{n-1},\ Y_n = 2kX_{n-1} + (2k^2 + 1)Y_{n-1}.$$

Then

$$\begin{cases} X_n = 2(2k^2+1)X_{n-1} - X_{n-2}, & X_0 = k(k^2+2), \\ & X_1 = k(2k^4 + 4k^3 + 5k^2 + 4k + 2); \\ Y_n = 2(2k^2+1)Y_{n-1} - Y_{n-2}, & Y_0 = 2k, \\ & Y_1 = 2k(k+1)(k^2+k+1). \end{cases}$$

It is easy to prove that

$$X_n \equiv 0 \pmod{k}, \quad Y_n \equiv 0 \pmod{2k},$$

then $x_n \in \mathbb{Z},\ z_n \in \mathbb{Z}$. Hence, we have

$$\begin{cases} x_n = 2(2k^2+1)x_{n-1} - x_{n-2} - 2k^4, & x_0 = k^2, x_1 = k^2(k^2+k+1); \\ z_n = 2(2k^2+1)z_{n-1} - z_{n-2}, & z_0 = k, z_1 = k(k+1)(k^2+k+1). \end{cases}$$

Then

$$y_n = x_n + \frac{2z_n}{k} = 2(2k^2+1)y_{n-1} - y_{n-2} - 2k^4,$$
$$y_0 = k^2 + 2, y_1 = (k^2 + k + 1)(k^2 + 2k + 2).$$

Therefore, we have infinitely many nontrivial positive integer solutions (x_n, y_n, z_n) for (2) when $f(X) = X^2 - k^2X$. □

For example, when $k=1$, we have $f(X)=X^2-X$, then

$$\begin{cases} x_n = 6x_{n-1} - x_{n-2} - 2, & x_0 = 1, x_1 = 3; \\ z_n = 6z_{n-1} - z_{n-2}, & z_0 = 1, z_1 = 6; \\ y_n = x_n + 2z_n = 6y_{n-1} - y_{n-2} - 2, & y_0 = 3, y_1 = 15. \end{cases}$$

Remark 2.1. For $f(X) \in \mathbb{Q}[X]$ we give another example of (2), i.e.,

$$f(X) = \frac{X(X-1)}{2} = \binom{X}{2},$$

let

$$y = x + 3z \Longleftrightarrow z = \frac{y-x}{3},$$

this transformation is similar to the transformation for (1), then

$$f(x)f(y) - f(z^2) = \frac{(x^2 - x + 4xz - 2z + 2z^2)(x^2 - x + 2xz - z - z^2)}{4} = 0.$$

It's easy to show that $x^2 - x + 4xz - 2z + 2z^2 = 0$ has infinitely many nontrivial positive integer solutions, then (2) has infinitely many nontrivial positive integer solutions for $f(X) = \binom{X}{2}$.

Motivated by this, we consider a generalized case for $f(X) \in \mathbb{Q}[X]$, let

$$f(X) = \frac{X^2 - k^2 X}{l}, \quad k, l \in \mathbb{Z}^+,$$

when $l = 1$, it is the case of theorem 2. Let $y = kz + k^2$ or $y = kz$, we have

$$f(x)f(y) - f(z^2) = \frac{z(z+k)(k^2x^2 - xk^4 + lkz - lz^2)}{l^2} = 0,$$

or

$$f(x)f(y) - f(z^2) = \frac{z(z-k)(k^2x^2 - xk^4 - lkz - lz^2)}{l^2} = 0.$$

Consider $k^2x^2 - xk^4 + lkz - lz^2 = 0$ or $k^2x^2 - xk^4 - lkz - lz^2 = 0$, when l is not a square, we can get infinitely many nontrivial positive integer solutions of (2) for suitable l with $k \in \mathbb{Z}^+$.

We have considered some cases for the reducible case, in the following we study the irreducible case, by the help of the computer we just get the case for $f(X) = X^2 + X + 1$, and the transformation is the same as for (1).

Proof of Theorem 3. For $f(X) = X^2 + X + 1$, let

$$y = x + 2z \Longleftrightarrow z = \frac{y - x}{2},$$

by using *Maple*, we have

$$f(x)f(y) - f(z^2) = (x + z)(x + z + 1)(x^2 + 2xz + x + z + 2 - z^2) = 0.$$

Let $x^2 + 2xz + x + z + 2 - z^2 = 0$, then

$$(2x + 2z + 1)^2 - 2(2z)^2 = -7.$$

Take $X = 2x + 2z + 1, Z = 2z$, we have

$$X^2 - 2Z^2 = -7.$$

An infinity of positive integer solutions are given by

$$X_n + Z_n\sqrt{2} = \left(1 + 2\sqrt{2}\right)\left(3 + 2\sqrt{2}\right)^n, n \geq 0,$$

leading to $X_n = 3X_{n-1} + 4Z_{n-1},\ Z_n = 2X_{n-1} + 3Z_{n-1}$. Then

$$\begin{cases} X_n = 6X_{n-1} - X_{n-2}, & X_0 = 1, X_1 = 11; \\ Z_n = 6Z_{n-1} - Z_{n-2}, & Z_0 = 2, Z_1 = 8. \end{cases}$$

It is easy to prove that $X_n \equiv 1 \pmod 2$, $Z_n \equiv 0 \pmod 2$, then $x_n \in \mathbb{Z}$, $z_n \in \mathbb{Z}$. Hence, we have

$$\begin{cases} x_n = 6x_{n-1} - x_{n-2}, & x_0 = -1, x_1 = 1; \\ z_n = 6z_{n-1} - z_{n-2} - 2, & z_0 = 1, z_1 = 4. \end{cases}$$

Then

$$y_n = x_n + 2z_n = 6y_{n-1} - y_{n-2} - 4, y_0 = 1, y_1 = 9.$$

Therefore, we have infinitely many nontrivial positive integer solutions (x_n, y_n, z_n) for (2). □

Remark 2.2. We search the integer solutions for (2) by computer for some simple cases when $f(X)$ is irreducible, such as X^2+1, X^2-X+1, X^2-4X+1, and just find a few solutions, but for $X^2 - 3X + 1$ we can get infinitely many trivial positive integer solutions, which is not interest. Maybe, (2) has infinitely many nontrivial positive integer solutions for some irreducible polynomials with parametric solutions, but it's too complicated to find them.

For the quadratic, the results for (2) is not rich, and for the cubic we just have the theorem for $f(X) = X^3 - X$. In fact, this result is a particular case of the authors' other paper, which is the initial mind which related to this problem.

Proof of Theorem 4. For $f(X) = X^3 - X$, let

$$y = x + z \Longleftrightarrow z = y - x,$$

this is also a similar transformation, by using *Maple*, we get

$$f(x)f(y)-f(z^2) = (x^2-z^2+xz-1)(x^4+2x^3z-x^2+2x^2z^2-xz+xz^3-z^2+z^4) = 0.$$

We only consider $x^2 - z^2 + xz - 1 = 0$, then

$$(2x + z)^2 - 5z^2 = 4.$$

Let $X = 2x + z, Z = z$, we have

$$X^2 - 5Z^2 = 4.$$

An infinity of positive integer solutions are given by

$$X_n + Y_n\sqrt{5} = \big(3 + \sqrt{5}\big)\big(9 + 4\sqrt{5}\big)^n, \quad n \geq 0,$$

leading to $X_n = 9X_{n-1} + 20Z_{n-1}$, $Z_n = 4X_{n-1} + 9Z_{n-1}$. Then

$$\begin{cases} X_n = 18X_{n-1} - X_{n-2}, & X_0 = 3, X_1 = 47; \\ Z_n = 18Z_{n-1} - Z_{n-2}, & Z_0 = 1, Z_1 = 21. \end{cases}$$

It ia easy to prove that $X_n \equiv 1 \pmod 2$, $Z_n \equiv 1 \pmod 2$, $x_n \in \mathbb{Z}$. Hence, we have

$$\begin{cases} x_n = 18x_{n-1} - x_{n-2}, & x_0 = 1, x_1 = 13; \\ z_n = 18z_{n-1} - z_{n-2} - 2, & z_0 = 1, z_1 = 21. \end{cases}$$

Then

$$y_n = x_n + z_n = 18y_{n-1} - y_{n-2} - 2, \quad y_0 = 2, y_1 = 34.$$

Therefore, we have infinitely many nontrivial positive integer solutions (x_n, y_n, z_n) for (2). □

In fact, the Pell equation $X^2 - 5Z^2 = 4$ is related to the Fibonacci numbers and Lucas numbers. We have the fundamental identity (c.f. [4] p. 61)

$$L_n^2 = 5F_n^2 + 4(-1)^n.$$

Let $n = 2k$, we have $L_{2k}^2 - 5F_{2k}^2 = 4$, then

$$2x_k + z_k = L_{2k}, z_k = F_{2k}.$$

Hence

$$\begin{cases} x_k = (L_{2k} - F_{2k})/2 = F_{2k-1}, \\ y_k = F_{2k-1} + F_{2k} = F_{2k+1}. \end{cases}$$

Remark 2.3. For $f(X) = X^k - 1$, $k \geq 3$, it's easy to see that (2) has no nontrivial integer solutions, since BENNETT [1] showed that: *The equation* $(x^k-1)(y^k-1) = z^k - 1$ *has only the integer solutions* $(x, y, z, k) = (-1, 4, -5, 3)$ *and* $(4, -1, -5, 3)$ *with* $|z| \geq 2$ *and* $k \geq 3$. However, we don't find an example which has infinitely many nontrivial positive integer solutions for (2) with $\deg(f) \geq 4$.

3. Rational solutions for (2)

In 2007, ULAS [7] consider (1) for the solutions in $\mathbb{Q}(t)$, where t is a parameter. Similarly, we can study (2) for the solutions in $\mathbb{Q}(t)$, but we don't get the similar results as in [7], and just get some results for the reducible case when $f(X)$ is a quadratic. In this section, we give the proof of Theorem 5.

PROOF OF THEOREM 5. For $f(X) = X^2 - kX$, $k \in \mathbb{Z} - \{0\}$, form (2) we have

$$x(x-k)y(y-k) = z^2(z^2-k),$$

let

$$z^2 = xy, z^2 - k = (x-k)(y-k),$$

leading to $x + y = k + 1$, then

$$z^2 + x^2 - (k+1)x = 0.$$

Taking $z = tx$, we get

$$x = \frac{k+1}{t^2+1},$$

leading to

$$y = \frac{t^2(k+1)}{t^2+1}, z = \frac{t(k+1)}{t^2+1}.$$

Hence, we have infinitely many nontrivial positive rational solutions

$$(x, y, z) = \left(\frac{k+1}{t^2+1}, \frac{t^2(k+1)}{t^2+1}, \frac{t(k+1)}{t^2+1}\right)$$

for (2) where t is a parameter. □

Remark 3.1. In fact, we can get the same result for $f(X) = (X-a)(X-b)$ for suitable a and b. But for the irreducible case with $\deg(f) = 2$ and the polynomials with $\deg(f) \geq 3$ except $f(X) = X^3 - X$, we can't say anything. We hope that there are some special cases which have infinitely many nontrivial positive rational solutions for (2).

4. Some related questions

The results presented in the previous sections are very limited, maybe the method is powerful but not suitable for as many as cases, it seems this problem related to geometry, and some methods of geometry is needed. In the end, we raise some questions for $f(x)f(y) = f(z^2)$.

Question 4.1. Does there exist infinitely many irreducible quadratic $f(X) \in \mathbb{Q}[X]$ such that (2) has infinitely many nontrivial positive integer or rational solutions?

Question 4.2. Does there exist infinitely many $f(X) \in \mathbb{Q}[X]$ with $\deg(f) = 3$ such that (2) has infinitely many nontrivial positive integer or rational solutions?

Question 4.3. Does there exist a polynomial $f(X) \in \mathbb{Q}[X]$ with $\deg(f) \geq 4$ such that (2) has infinitely many nontrivial positive integer or rational solutions?

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YONG ZHANG
DEPARTMENT OF MATHEMATICS
ZHEJIANG UNIVERSITY
HANGZHOU, ZHEJIANG, 310027
CHINA

E-mail: zhangyongzju@163.com

TIANXIN CAI
DEPARTMENT OF MATHEMATICS
ZHEJIANG UNIVERSITY
HANGZHOU, ZHEJIANG, 310027
CHINA

E-mail: txcai@zju.edu.cn

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