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On double sequences of continuous functions having continuous P-limits

By RICHARD F. PATTERSON (Jacksonville) and EKREM SAVAS (Istanbul)

Abstract. The goal of this paper includes the four-dimensional matrix characterization of double sequence of functions. However the main goal is to present answer to the following question. Is it necessarily the case that if $s_{m,n}(x)$ is a bounded for all (m,n) and x with continuous elements and P-converges to a continuous function there exists an RH-regular matrix transformation that maps $(s_{m,n}(x))$ into a uniformly P-convergent double sequence?

1. Introduction

Let us consider the following double sequence

$$s_{m,n}(x)=\begin{cases} 2^{m+n}x, & \text{if } 0\le x\le \dfrac{1}{2^{m+n}};\\ 2-2^{m+n}x, & \text{if } \dfrac{1}{2^{m+n}}\le x\le \dfrac{1}{2^{m+n-1}};\\ 0, & \text{if } \dfrac{1}{2^{m+n-1}}\le x\le 1. \end{cases}$$

Note $s_{m,n}(x)$ is continuous on $0\le x\le 1$ and $\mathrm{P}-\lim_{m,n} s_{m,n}(x)=0$ because $s_{m,n}(x)=0$ if $x=0$ for all (m,n) and $s_{m,n}(x)$ is also 0 if $0<x\le 1$ for $m+n>1-\frac{\log x}{\log 2}$. However $s_{m,n}\left(\frac{1}{2^{m+n}}\right)=1$. Thus the double sequence is not uniformly P-convergent. Now let us consider the $(C,1,1)$ transformation of the above double sequence. That is

$$\sigma_{m,n}(x)=\frac{1}{mn}\sum_{k,l=1,1}^{m,n} s_{k,l}(x).$$

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This yields the following:

$$\sigma_{m,n}(x) = \begin{cases} \dfrac{(2^{m+1}-2)(2^{n+1}-2)x}{mn}, & \text{if } 0 \leq x \leq \dfrac{1}{2^{m+n}}; \\ \dfrac{(2-4x)}{mn}, & \text{if } \dfrac{1}{2^{m+n}} \leq x \leq 1. \end{cases}$$

Therefore

$$0 \leq \sigma_{m,n}(x) \leq \frac{2}{mn}.$$

Thus $\sigma_{m,n}(x)$ P-converges to 0 uniformly on $0 \leq x \leq 1$. This leads us to pose the multidimensional analog of Gillespie and Hurwitz in [1]. That is, is it necessarily the case that if $s_{m,n}(x)$ is a bounded for all (m,n) and x with continuous elements and P-converges to a continuous function there exists an *RH*-regular matrix transformation that maps $(s_{m,n}(x))$ into a uniformly P-convergent double sequence?

2. Definitions, notations and preliminary results

Definition 2.1 (Pringsheim, 1900). A double sequence $x = [X_{k,l}]$ has *Pringsheim limit* L (denoted by P-lim $x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|X_{k,l} - L| < \epsilon$ whenever $k, l > N$. Such an x is describe more briefly as "P-convergent".

Definition 2.2 (Patterson, 2000). The double sequence y is a *double subsequence* of x provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that if $x_j = x_{n_j,k_j}$, then y is formed by

$$\begin{matrix} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & - . \end{matrix}$$

In [5] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 2.3. The four-dimensional matrix $\mathcal{A}$ is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 2.1 (Hamilton [2], Robison [5]). *The four-dimensional matrix $\mathcal{A}$ is RH-regular if and only if*

RH_1 : $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

RH_2 : $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;

RH_3 : $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*

RH_4 : $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*

RH_5 : $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent;*

RH_6 : *there exist finite positive integers Δ and Γ such that* $\sum_{k,l>\Gamma} |a_{m,n,k,l}| < \Delta$.

3. Main results

Let S represent the double sequence

$$\begin{matrix} s_{1,1}(x), & s_{1,2}(x) & s_{1,3}(x) & \dots \\ s_{2,1}(x), & s_{2,2}(x) & s_{2,3}(x) & \dots \\ s_{3,1}(x), & s_{3,2}(x) & s_{3,3}(x) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

of functions, each of which is continuous in A which is any point set such that for some constant M we have $0 \leq s_{k,l}(x) \leq M$ with $\mathrm{P} - \lim_{k,l} s_{k,l}(x) = 0$. We shall call S an $\mathcal{S}$-double sequence in A.

3.1. Maximal functions. Let x be in A, for each double sequence $(x_{k,l})$ of points of A having x as P-limit and every double sequence (k_m, l_n) of positive integers with $\lim_m k_m = \infty$ and $\lim_n l_n = \infty$ and form

$$\mathrm{P} - \limsup_{m,n} s_{k_m,l_n}(x_{m,n}) = \lambda \tag{3.1}$$

then the least upper bounded of all such numbers λ is the value at x of the maximal function of S in A. We shall denote such functions by $H^2(S; A; x)$.

Theorem 3.1. *If S is an $\mathcal{S}$-double sequence in the compact closed set A, $H^2(S;A;x)$ is an upper semi-continuous function. If B is a closed subset of A then*

$$H^2(S;A;x) \geq H^2(S;B;x).$$

The proof is a direct consequence of the definition for semi-continuous functions and of such it is omitted.

Theorem 3.2. *If S is an $\mathcal{S}$-double sequence in the compact closed set A, $h > 0$ and $H^2(S;A;x) < h$ then*

$$P - \limsup_{m,n} H^2(s_{m,n}, A) \leq h.$$

Proof. Suppose

$$\mathrm{P} - \limsup_{m,n} H^2(s_{m,n}, A) > h.$$

Then for double index subsequence (m_k, n_l)

$$H^2(s_{m_k,n_l}, A) > h.$$

Thus for each (k,l) there is a point $x_{k,l}$ of A such that

$$s_{m_k,n_l}(x_{k,l}) = H^2(s_{m_k,n_l}, A).$$

Therefore

$$s_{m_k,n_l}(x_{k,l}) > h.$$

The double sequence $(x_{k,l})$ has at least one P-limit point δ of A, thus without loss of generality let (k,l) correspond to the double subsequence of $(x_{k,l})$ having limit point δ. Therefore $s_{m_k,n_l}(x_{k,l}) > h$ holds for all (k,l) and

$$\mathrm{P} - \limsup_{k,l} s_{m_k,n_l}(x_{k,l}) \geq h.$$

Also note

$$\mathrm{P} - \limsup_{k,l} s_{m_k,n_l}(x_{k,l})$$

is one of the value of λ of (3.1). Therefore

$$H^2(S;A;\delta) \geq h.$$

Thus we have a contradiction. This completes the proof. □

Theorem 3.3. *If S is an $\mathcal{S}$-double sequence in a compact closed set A, and $h > 0$ then the set A' of points at which $H^2(S;A;x) \geq h$ is closed proper subset of A, nowhere dense in A*

Proof. A' is closed because of the definition of upper semi-continuity of $H^2(S; A; x)$. Note if A' is nowhere dense in A then it is clear that A' is a proper subset of A. Now we need to show that A' is nowhere dense in A we will show that in each open set C which contains a point of A there is an open set which contains a points of A and no point of A'. Suppose this were false for a given open set C containing a point x_0 of A then C itself contains a point of A' that is, there is a point in C for which $H^2 \geq h > \frac{h}{2}$. Thus there is a double sequence index sequence $(n_{1,1})$ and a point $x_{1,1}$ of $A \cap C$ such that

$$s_{n_{1,1}}(x_{1,1}) > \frac{h}{2}$$

and because of continuity of $s_{n_{1,1}}$ there is an open set $C_{1,1} \subseteq C$ and $x_{1,1}$ in $C_{1,1}$ such that $s_{n_{1,1}}(x) > \frac{h}{2}$ in $A \cap C_{1,1}$. The next set is $C_{2,1}$, the order is following that of Definition 2.2. Since $C_{2,1}$ contains a point of A it contains points A' for which $H^2 \geq h > \frac{h}{2}$ there is an index $n_{2,1} > n_{1,1}$ and a point $x_{2,1}$ of $A \cap C_{1,1}$ such that $s_{n_{2,1}}(x_{2,1}) > \frac{h}{2}$ and there is an open set $C_{2,1}$ contained in $C_{1,1}$ and $x_{2,1}$ such that $s_{n_{2,1}}(x > \frac{h}{2})$ throughout $A \cap C_{2,1}$. We contain this process and obtain then following double sequence of set

$$\begin{matrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} \\ C_{2,1} & C_{2,2} & C_{2,3} & - \\ C_{3,1} & C_{3,2} & C_{3,2} & - \\ - & - & - & -. \end{matrix}$$

each of which is contained in the preceding element and whose order is with respect to Definition 2.2 and $s_{n_{k,l}}(x_{k,l}) > \frac{h}{2}$ throughout $A \cap C_{k,l}$. Let $L_{k,l}$ denote the set obtained by adjoining $A \cap C_{k,l}$ with its P-limit points. Thus we obtain a double sequence of closed sets $L_{k,l}$ with $s_{n_{k,l}}(x_{k,l}) \geq \frac{h}{2}$. These set has at least one point in common. Note there is a P-limit point δ such that

$$s_{n_{k,l}}(\delta) \geq \frac{h}{2} \text{ for all } (k, l).$$

If we let (k, l) tends to ∞ in the Pringsheim sense. We are granted $0 \geq \frac{h}{2}$. This grant us a contradiction, since $h > 0$. Therefore A' is nowhere dense in A. □

The proof of the following theorem clearly follows from Theorem 3.1 and of such it is omitted.

Theorem 3.4. *If S is an $\mathcal{S}$-double sequence in a compact closed set A, B is a closed subset of A, and $h > 0$, and if A' and B' denote respectively the set for which $H^2(S; A; x) \geq h$ and $H^2(S; B; x) \geq h$ then $B' \subseteq A'$.*

3.2. $\mathcal{T}$-transformation. In this section we will consider the following type of transformation

$$\sigma_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} s_{k,l}$$

we shall call such transformation $\mathcal{T}$-transformation if it satisfies the following conditions

(1) $a_{m,n,k,l} \geq 0$;

(2) $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;

(3) there exist integers α_i, β_i, ρ_j, and ϕ_j such that

$$\alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \alpha_3 \leq \beta_3 < \cdots \quad \text{and} \quad \rho_1 \leq \phi_1 < \rho_2 \leq \phi_2 < \rho_3 \leq \phi_3 < \cdots$$

with

$$a_{m,n,k,l} = 0 \text{ unless } \alpha_m \leq k \leq \beta_m \;\&\; \rho_n \leq l \leq \phi_n.$$

Condition (3) reduces the transformation to the following

$$\sigma_{m,n}(x) = \sum_{k,l=\alpha_m,\rho_n}^{\beta m,\phi_n} a_{m,n,k,l} s_{k,l}.$$

Note each pairwise row of $(a_{m,n,k,l})$ contains a finite set of nonzero elements and each pairwise column contains at most one such element. The four-dimensional identity is one such example.

Theorem 3.5. *The four-dimensional transformation $\mathcal{T}$ is RH-regular. Also if $\mathcal{T}$-transformation is such the the double sequence is an $\mathcal{S}$-double sequence in a compact closed set A, then*

$$P - \limsup_{m,n} G^2(\sigma_{m,n}; A) \leq P - \limsup_{m,n} G^2(s_{m,n}; A).$$

PROOF. It is clear from condition (1), (2), and (3) that the transformation is RH-regular. Now let us establish that

$$\mathrm{P} - \limsup_{m,n} G^2(\sigma_{m,n}; A) \leq \mathrm{P} - \limsup_{m,n} G^2(s_{m,n}; A).$$

Let

$$\mathrm{P} - \limsup_{m,n} G^2(s_{m,n}; A) = l.$$

For each $\epsilon > 0$ there are an indices K and L such that when $k > K$ and $l > L$ we have $G^2(s_{k,l}; A) < l + \epsilon$. Then $m > M = \alpha_K$ and $n > N = \rho_L$, implies

$$\sigma_{m,n} \leq \sum_{k,l=\alpha_m,\rho_n}^{\beta m,\phi_n} a_{m,n,k,l} G^2(s_{k,l}; A) < (l + \epsilon) \sum_{k,l=\alpha_m,\rho_n}^{\beta m,\phi_n} a_{m,n,k,l} = l + \epsilon$$

throughout A, therefore

$$\mathrm{P} - \limsup_{m,n} G^2(\sigma_{m,n}; A) \leq l + \epsilon.$$

Because $\epsilon > 0$ is arbitrary it follows that

$$\mathrm{P} - \limsup_{m,n} G^2(\sigma_{m,n}; A) \leq l.$$

This completes the proof. □

Note similar to the two-dimensional transformations these four-dimensional transformations are equivalent to four-dimensional triangular transformation. In the four-dimension case we can take B to be the following:

$$b_{r,s,k,l} = \begin{cases} a_{m,n,k,l}, & \text{if } \beta_m \leq r < \beta_{m+1} \ \& \ \alpha_m \leq k \leq \beta_m, \\ & \quad \phi_n \leq s < \phi_{n+1} \ \& \ \rho_n \leq l \leq \phi_n; \\ 0, & \text{if otherwise.} \end{cases}$$

Then the transformation yields the following

$$\mu_{r,s} = \sum_{k,l=1,1}^{r,s} b_{r,s,k,l} s_{k,l}$$

four-dimensional triangular transformation.

3.3. Application of $\mathcal{T}$-transformation to $\mathcal{S}$-double sequences. The following is the first application of $\mathcal{T}$-transformation to $\mathcal{S}$-double sequences.

Theorem 3.6. *Let $h \geq 0$ and also let S be an $\mathcal{S}$-double sequence in the compact closed set A. If for each $q > h$ there is a $\mathcal{T}$-transformation $\sigma^q_{m,n}(x)$ such that*

$$P - \limsup_{m,n} G^2(\sigma^q_{m,n}; A) \leq q,$$

then there is a $\mathcal{T}$-transformation such that

$$P - \limsup_{m,n} G^2(\sigma_{m,n}; A) \leq h.$$

The theorem can be proven using a multidimensional analog of the proof in [1] and of such it is omitted. We now consider the special case with h replace by 0 and q replace by h and obtain the following:

Theorem 3.7. *Let S be an $\mathcal{S}$-double sequence in the compact closed set A. If for each $h > 0$ there is a $\mathcal{T}$-transformation $\sigma^h_{m,n}(x)$ such that*

$$P - \limsup_{m,n} G^2(\sigma^h_{m,n}; A) \leq h,$$

then there is a $\mathcal{T}$-transformation such that $P - \lim_{m,n} \sigma_{m,n} = 0$ uniformly in A.

Likewise the proof is omitted.

Theorem 3.8. *Let $h \geq 0$ and S be an $\mathcal{S}$-double sequence in the compact closed set A, B a closed subset of A. If*

$$P - \limsup_{m,n} G^2(s_{m,n}; B) < h, \tag{3.2}$$

and if for each neighborhood C of B in A there exists a $\mathcal{T}$-transformation such that

$$P - \limsup_{m,n} G^2(\sigma^C_{m,n}; A\bar{C}) < h, \tag{3.3}$$

then there exists a $\mathcal{T}$-transformation such that

$$P - \limsup_{m,n} G^2(\sigma_{m,n}; A) < h. \tag{3.4}$$

Proof. The general goal of this proof is to construct a four-dimensional transformation that grants us the necessary bound. Let us denote a double index (m, n) chosen at random by $(p_{1,1}, q_{1,1})$ and define

$$\sigma_{1,1} = s_{p_{1,1},q_{1,1}}$$

this is the first formula use in the construction of the transformation T. Condition (3.2) ensure us that for sufficiently large m and n, $s_{m,n} < h$. We now choose

$$p_{i_1,1} > p_{1,1} \text{ and } q_{j_1,1} > q_{1,1}; \quad \text{for } i_1, j_1 = 1, 2;\ i_1 = j_1 \neq 1$$

with equality if $i_1 = 1$ or $j_1 = 1$, respectively, such that

$$s_{p_{i_1,1},q_{j_1,1}} < h$$

throughout B. Since all the $s_{p_{i_1,1},q_{j_1,1}}$s are continuous there are neighborhoods $C_{i_1,1,j_1,1}$s of B in A, respectively, such that

$$s_{p_{i_1,1},q_{j_1,1}} < h \text{ throughout } C_{i_1,1,j_1,1}, \text{ respectively.}$$

Let us now form the next three parts of our transformation $T^{C_{i_1,1,j_1,1}}$ by (3.2) for all sufficiently large m and n we have

$$\sigma_{m,n}^{C_{i_1,1,j_1,1}} < h \text{ in } A \cap \bar{C}_{i_1,1,j_1,1}.$$

Then choose $p_{i_1,2}$ and $q_{j_1,2}$ such that

$$\sigma_{p_{i_1,2},q_{j_1,2}}^{C_{i_1,1,j_1,1}} < h \text{ in } A \cap \bar{C}_{i_1,1,j_1,1}$$

such that $\sigma_{p_{i_1,2},q_{j_1,2}}^{C_{i_1,1,j_1,1}}$ contain only elements of S subscripts greater than $(p_{1,1}, q_{1,1})$ order is with respect to Definition 2.2. We can now define three more parts of T as follow:

$$\sigma_{1,2} = \frac{1}{2}\{s_{p_{1,1},q_{2,1}} + \sigma_{p_{1,2},q_{2,2}}^{C_{1,1,2,1}}\},$$

$$\sigma_{2,1} = \frac{1}{2}\{s_{p_{2,1},q_{1,1}} + \sigma_{p_{2,2},q_{1,2}}^{C_{2,1,1,1}}\},$$

and

$$\sigma_{2,2} = \frac{1}{2}\{s_{p_{2,1},q_{2,1}} + \sigma_{p_{2,2},q_{2,2}}^{C_{2,1,2,1}}\}.$$

Now without loss of generality, let ρ_2^1 correspond to the last selected $p_{i_1,1}$ and ρ_2^2 correspond to the last selected $q_{j_1,1}$ and let $p_{i_2,1} > \rho_2^1$, $q_{j_2,1} > \rho_2^2$, $p_{i_2,2}$ and $p_{i_2,2}$

$$(i_2, j_2)' = \{(k,l) : k,l = 1,2,3\} \setminus \{(1,1),(1,2),(2,1),(2,2)\}$$

such that

$$s_{p_{i_2,1},q_{j_2,1}} < h \text{ throughout } B.$$

Since all $s_{p_{i_2,1},q_{j_2,1}}$s are continuous, there are neighborhoods $C_{i_2,1,j_2,1}$s of B in A such that

$$s_{p_{i_2,1},q_{j_2,1}} < h \text{ throughout the respected } C_{i_2,1,j_2,1}.$$

Now we form $T^{C_{i_2,2,j_2,2}}$. Thus by (3.3) for all sufficiently large $p_{i_2,2}$ and $q_{j_2,2}$ we have

$$\sigma_{p_{i_2,2},q_{j_2,2}}^{C_{i_2,1,j_2,1}} < h \text{ in } A \cap \bar{C}_{i_2,1,j_2,1}.$$

In addition

$$\sigma_{p_{i_2,2},q_{j_2,2}}^{C_{i_2,1,j_1,1}} < h \text{ in } B$$

and choose a neighborhood $C_{i_2,2,j_2,2}$ of B in $C_{i_2,1,j_2,1}$ such that

$$\sigma^{C_{i_2,1,j_1,1}}_{p_{i_2,2},q_{j_2,2}} < h \text{ in } C_{i_2,2,j_2,2}.$$

Note it contains only subscripts of S that are greater than $(p_{2,1}, q_{1,1})$. We can now define five more parts of T as follow:

$$\sigma_{3,1} = \frac{1}{2}\{s_{p_{3,1},q_{1,1}} + \sigma^{C_{3,1,1,1}}_{p_{3,2},q_{1,2}}\},$$
$$\sigma_{3,2} = \frac{1}{2}\{s_{p_{3,1},q_{2,1}} + \sigma^{C_{3,1,2,1}}_{p_{3,2},q_{2,2}}\},$$
$$\sigma_{3,3} = \frac{1}{2}\{s_{p_{3,1},q_{3,1}} + \sigma^{C_{3,1,3,1}}_{p_{3,2},q_{3,2}}\},$$
$$\sigma_{2,3} = \frac{1}{2}\{s_{p_{2,1},q_{3,1}} + \sigma^{C_{2,1,3,1}}_{p_{2,2},q_{3,2}}\},$$

and

$$\sigma_{1,3} = \frac{1}{2}\{s_{p_{1,1},q_{3,1}} + \sigma^{C_{1,1,3,1}}_{p_{1,2},q_{3,2}}\}.$$

Next we form $T^{C_{i_3,3,j_3,3}}$ similar to above let us consider the following without loss of generality let ρ^1_3 correspond to the last selected $p_{i_2,1}$ and ρ^2_3 correspond to the last selected $q_{j_2,1}$ and let $p_{i_3,1} > \rho^1_3$ and $q_{j_3,1} > \rho^2_3$, similarly $p_{i_3,2}$, $p_{i_3,2}$, $p_{i_3,3}$, and $p_{i_3,3}$

$$(i_3, j_3)' = \{(k,l) : k,l = 1,2,3,4\} \setminus \{(1,1),(1,2),(2,1),(2,2)\} \cup (i_2, j_2)'$$

such that

$$\sigma^{C_{i_3,2,j_3,2}}_{p_{i_3,3},q_{j_3,3}} < h \text{ in } A \cap \bar{C}_{i_3,2,j_3,2}.$$

Also

$$\sigma^{C_{i_3,2,j_3,2}}_{p_{i_3,3},q_{j_3,3}} < h \text{ in } B$$

then choose neighborhoods $C_{i_3,3,j_3,3}$s of B in $C_{i_3,2,j_3,2}$ such that

$$\sigma^{C_{i_3,2,j_3,2}}_{p_{i_3,3},q_{j_3,3}} < h \text{ in } C_{i_3,3,j_3,3}.$$

Now we form seven more parts of T similar to those of the last group. In particular $\sigma_{4,1}$, $\sigma_{4,2}$, $\sigma_{4,3}$, $\sigma_{4,4}$, $\sigma_{1,4}$, $\sigma_{2,4}$, and $\sigma_{3,4}$. That is $T^{C_{i_3,3,j_3,3}}$. In general let ρ^1 and ρ^2 be the last chosen subscript of the elements of S appearing in $\sigma_{p_{k-1},q_{k-1}}$. We now choose $p_{i_k,k-1}$ and $q_{j_k,k-1}$ greater than ρ^1 and ρ^2, respectively, such that

$$s_{p_{i_k,k},q_{j_k,k}} < h \text{ throughout } B$$

then choose neighborhoods $C_{i_k,k,j_k,k}$s of B in A such that

$$s_{p_{i_k,k}q_{j_k,k}} < h \text{ throughout the respected } C_{i_k,k,j_k,k}\text{s.}$$

Thus we obtain the following indices, neighborhoods, and transformations, respectively:

$$\begin{array}{ccccc}
(p_{i_k,1}, q_{j_k,1}) & (p_{i_k,1}, q_{j_k,2}) & (p_{i_k,1}, q_{j_k,2}) & \cdots & (p_{i_k,1}, q_{j_k,k-1}) \\
(p_{i_k,2}, q_{j_k,1}) & (p_{i_k,2}, q_{j_k,2}) & (p_{i_k,2}, q_{j_k,2}) & \cdots & (p_{i_k,2}, q_{j_k,k-1}) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(p_{i_k,k-1}, q_{j_k,1}) & (p_{i_k,k-1}, q_{j_k,2}) & (p_{i_k,k-1}, q_{j_k,2}) & \cdots & (p_{i_k,k-1}, q_{j_k,k-1}),
\end{array}$$

$$\begin{array}{ccccc}
C_{i_k,1,j_k,1} & C_{i_k,1,j_k,2} & C_{i_k,1,j_k,2} & \cdots & C_{i_k,1,j_k,k-1} \\
C_{i_k,2,j_k,1} & C_{i_k,2,j_k,2} & C_{i_k,2,j_k,2} & \cdots & C_{i_k,2,j_k,k-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
C_{i_k,k-1,j_k,1} & C_{i_k,k-1,j_k,2} & C_{i_k,k-1,j_k,2} & \cdots & C_{i_k,k-1,j_k,k-1},
\end{array}$$

and

$$\begin{array}{ccccc}
T^{C_{i_k,1,j_k,1}} & T^{C_{i_k,1,j_k,2}} & T^{C_{i_k,1,j_k,2}} & \dots & T^{C_{i_k,1,j_k,k-1}} \\
T^{C_{i_k,2,j_k,1}} & T^{C_{i_k,2,j_k,2}} & T^{C_{i_k,2,j_k,2}} & \dots & T^{C_{i_k,2,j_k,k-1}} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
T^{C_{i_k,k-1,j_k,1}} & T^{C_{i_k,k-1,j_k,2}} & T^{C_{i_k,k-1,j_k,2}} & \dots & T^{C_{i_k,k-1,j_k,k-1}}
\end{array}$$

such that

$$A \supset C_{i_k,1,j_k,1} \supset C_{i_k,2,j_k,1} \supset C_{i_k,2,j_k,2} \supset C_{i_k,1,j_k,2} \supset \cdots \supset C_{i_k,k-1,j_k,k-1}$$

the order is that of double subsequences. Observe that

$$\{\begin{cases}
s_{p_{i_k,1},q_{j_k,1}} < h & \text{in } \; C_{i_k,1,j_k,1}, \\
\sigma^{C_{i_k,r,j_k,s}}_{p_{i_k,r+1},q_{j_k}s+1} < h & \text{in } \; A \cap \bar{C}_{i_k,r,j_k,s} \; r,s = 1,2,\dots k-1, \\
\sigma^{C_{i_k,r,j_k,s}}_{p_{i_k,r+1},q_{j_k}s+1} < h & \text{in } \; C_{i_k,r+1,j_k,s+1} \; r,s = 1,2,\dots k-2.
\end{cases}$$

This disjoint partition grant us the following:

$$\begin{aligned}
A = (A \cap \bar{C}_{i_k,1,j_k,1}) + (C_{i_k,1,j_k,1} \cap \bar{C}_{i_k,2,j_k,1}) + \cdots \\
+ (C_{i_k,2,j_k,k-1} \cap \bar{C}_{i_k,1,j_k,k-1}) + C_{i_k,1,j_k,k-1}
\end{aligned}$$

the order is in accordance double subsequence construction in Definition 2.2. In $A \cap \bar{C}_{i_k,1,j_k,1}$ we have $s_{p_{i_k,1},q_{j_k,1}} \leq M$ and

$$\sigma_{p_{i_k,r},q_{j_k,s}}^{C_{i_k,r-1,j_k,s-1}} < h; \quad r > 1,\ s > 1.$$

In $C_{i_k,r,j_k,s} \cap \bar{C}_{i_k,\alpha,j_k,\beta}$; r, s, α, and β are define in accordance with double subsequence definition we have the following

$$s_{p_{i_k,1},q_{j_k,1}} \leq h,\ \sigma_{p_{i_k,r},q_{j_k,s}}^{C_{i_k,\phi-1,j_k,\delta-1}} < h; \quad \phi \neq r\ \delta \neq s\ and\ \sigma_{p_{i_k,r},q_{j_k,s}}^{C_{i_k,r-1,j_k,s-1}} \leq M.$$

In $C_{i_k,1,j_k,k-1}$ we have

$$s_{p_{i_k,1},q_{j_k,1}} \leq h,\ \sigma_{p_{i_k,r},q_{j_k,s}}^{C_{i_k,r-1,j_k,s-1}} < h, \quad \text{and } \sigma_{p_{i_k,r},q_{j_k,s}}^{C_{i_k,r-1,j_k,s-1}} \leq M.$$

Therefore throughout A we are granted

$$\begin{aligned} s_{p_{i_k,1},q_{j_k,1}} &+ \sigma_{p_{i_k,1},q_{j_k,2}}^{C_{i_k,1,j_k,1}} + \cdots + \sigma_{p_{i_k,1},q_{j_k,s}}^{C_{i_k,1,j_k,s-1}} \\ &+ \sigma_{p_{i_k,2},q_{j_k,2}}^{C_{i_k,2,j_k,1}} + \cdots + \sigma_{p_{i_k,2},q_{j_k,s}}^{C_{i_k,2,j_k,s-1}} \\ &+ \quad \vdots \quad + \ \vdots \ \vdots \quad \vdots \\ &+ \sigma_{p_{i_k,r},q_{j_k,2}}^{C_{i_k,r-1,j_k,1}} + \cdots + \sigma_{p_{i_k,r},q_{j_k,s}}^{C_{i_k,r-1,j_k,s-1}} < (r-1)(s-1)h + m. \end{aligned}$$

Thus

$$\sigma_{r,s} = \frac{1}{rs} \left\{ \begin{array}{ccccc} s_{p_{i_k,1},q_{j_k,1}} + \sigma_{p_{i_k,1},q_{j_k,2}}^{C_{i_k,1,j_k,1}} & + \cdots & + & \sigma_{p_{i_k,1},q_{j_k,s}}^{C_{i_k,1,j_k,s-1}} \\ + \ \sigma_{p_{i_k,2},q_{j_k,2}}^{C_{i_k,2,j_k,1}} & + \cdots & + & \sigma_{p_{i_k,2},q_{j_k,s}}^{C_{i_k,2,j_k,s-1}} \\ + \quad \vdots & + \ \vdots & \vdots & \vdots \\ + \ \sigma_{p_{i_k,r},q_{j_k,2}}^{C_{i_k,r-1,j_k,1}} & + \cdots & + & \sigma_{p_{i_k,r},q_{j_k,s}}^{C_{i_k,r-1,j_k,s-1}} \end{array} \right\}.$$

This transformation is an RH-regular $\mathcal{T}$-transformation with

$$\sigma_{r,s} < h + \frac{h+m}{rs}.$$

This grants us the following

$$G^2(\sigma_{r,s}; A) < h + \frac{h+m}{rs}.$$

This yields the result. □

Theorem 3.9. *Let $h \geq 0$ and let S be an $\mathcal{S}$-double sequence in the compact closed set A, B a closed subset of A. If*

$$P - \limsup_{m,n} G^2(s_{m,n}; B) \leq h,$$

and for each neighborhood C of B in A there exists a $\mathcal{T}$-transformation such that

$$P - \limsup_{m,n} G^2(\sigma^C_{m,n}; A\bar{C}) \leq h,$$

then there exists a $\mathcal{T}$-transformation such that

$$P - \limsup_{m,n} G^2(\sigma_{m,n}; A) \leq h.$$

The result follow from the last theorem, with $q > h$ and q satisfies the conditions of h.

3.4. Definition and properties of h-sets and h-order of double sequences. This definition is a multidimensional extension of Gillespie and Hurwitz's in [1].

Definition 3.1. Let S be an $\mathcal{S}$-double sequence in the compact closed set A and let $h > 0$. We define a set of sets $A^{\alpha,\beta}$, where α and β are any Cantor ordinal of the first or second class, by the following scheme of induction:

(1) $A^{0,0} = A^{1,0} = A^{0,1} = A$

(2) if α and β are not P-limiting ordinal, then $A^{\alpha,\beta}$ is the set of points at which $H^2(S; A^{\alpha-1,\beta-1}; x) \geq h$.

(3) if α and β are P-limiting ordinal, then $A^{\alpha,\beta}$ is the greatest common subset of all $A^{\rho,\gamma}$ for $\rho < \alpha$ and $\gamma < \beta$.

The resulting set is called the h-set generated by $(S; A)$ If we use normal ordering such set is call h-sets.

Using the order of Definition 2.2 we are granted multidimensional analog of Gillespie and Hurwitz's of Theorem 10 through Theorem 15 [1] and if we remain true to the ordering in Definition 2.2 the results follow similar to those presented by Gillespie and Hurwitz's in [1] and of such that theorems are stated without proof.

Theorem 3.10. *If B is a closed subset of A and if $B^{\alpha,\beta}$ denote the hsets for $(S; B)$, then $B^{\alpha,\beta} \subset A^{\alpha,\beta}$.*

Theorem 3.11. *For $(S;A)$ each h-set is a closed subset of each preceding h-set; each h-set is a proper subset of each preceding non-empty h-set.*

Theorem 3.12. *For $(S;A)$ there are non-limiting ordinal ρ and γ such that $A^{\rho,\gamma}$ and all following h-sets are empty while all preceding h-sets are non-empty.*

Theorem 3.13. *If α and β are the h-order of $(S;A)$ then*

$$P-\limsup_{m,n} G^2(s_{m,n};A^{\alpha,\beta}) \leq h.$$

Theorem 3.14. *Let $h>0$. If S is an $\mathcal{S}$-double sequence in the compact closed set A then there exists a $\mathcal{T}$-transformation such that*

$$P-\limsup_{m,n} G^2(\sigma_{m,n};A) \leq h.$$

Theorem 3.15. *If S is an $\mathcal{S}$-double sequence in the compact closed set A then there exists a $\mathcal{T}$-transformation such that*

$$P-\limsup_{m,n} \sigma_{m,n}(x) = 0 \text{ uniformly in } A.$$

3.5. Main theorem. We now establish the main theorem.

Theorem 3.16. *Let A be a compact closed set; let the double sequence of function*

$$S = \begin{matrix} s_{1,1}(x), & s_{1,2}(x) & s_{1,3}(x) & \dots \\ s_{2,1}(x), & s_{2,2}(x) & s_{2,3}(x) & \dots \\ s_{3,1}(x), & s_{3,2}(x) & s_{3,3}(x) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

have the following properties:

(1) *for each (m,n) $s_{m,n}(x)$ is continuous in A;*

(2) *for each x in A we have $P-\lim_{m,n} s_{m,n}(x) = s(x)$;*

(3) *$s(x)$ is continuous in A;*

(4) *there exists M such that for all (m,n) and all x in A $|s_{m,n}(x)| \leq M$.*

Then there exists a $\mathcal{T}$-transformation such that

$$P-\lim_{m,n} \sigma_{m,n}(x) = s(x) \text{ uniformly in } A.$$

Proof. For all $x \in A$ condition (1) and (4) grants us $|s(x)| \leq M$, Thus $|s_{m,n}(x) - s(x)| \leq 2M$. Let $s'_{m,n}(x) = |s_{m,n}(x) - s(x)|$ and consider the following double sequence

$$S' = \begin{matrix} s'_{1,1}(x), & s'_{1,2}(x) & s'_{1,3}(x) & \dots \\ s'_{2,1}(x), & s'_{2,2}(x) & s'_{2,3}(x) & \dots \\ s'_{3,1}(x), & s'_{3,2}(x) & s'_{3,3}(x) & \dots \\ \vdots & \vdots & \vdots & \ddots . \end{matrix}$$

It clear that S' is a $\mathcal{S}$-double sequence in A. Theorem 3.15 ensure us that there exists a four-dimensional transformation σ' such that

$$\text{P} - \lim_{m,n} \sigma'_{m,n}(x) = 0 \text{ uniformly in } A.$$

Recall that the coefficients of A are nonnegative and pairwise row sum to 1. Thus we have the following

$$\begin{aligned} |\sigma_{m,n} - s| &= \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} s_{k,l} - s \right| = \left| \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} (s_{k,l} - s) \right| \\ &\leq \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left| s_{k,l} - s \right| = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} s'_{k,l} = \sigma'_{m,n}. \end{aligned}$$

Thus

$$\text{P} - \lim_{m,n} \sigma_{m,n}(x) = s(x) \text{ uniformly in } A. \qquad \square$$

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RICHARD F. PATTERSON
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF NORTH FLORIDA JACKSONVILLE
FLORIDA, 32224
USA

E-mail: rpatters@unf.edu

EKREM SAVAS
DEPARTMENT OF MATHEMATICS
ISTANBUL TICARET UNIVERSITY
USKUDAR- ISTANBUL
TURKEY

E-mail: ekremsavas@yahoo.com

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