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On (m,n)-injectivity and coherence of rings

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Abstract. Let R be a ring. For two positive integers m and n, R is said to be left (m,n)-injective if every left R-homomorphism from an n-generated submodule of ${}_RR^m$ to ${}_RR$ extends to one from ${}_RR^m$ to ${}_RR$. The ring R is called left coherent if each of its finitely generated left ideals is finitely presented. The aim of this article is to investigate (m,n)-injectivity and the coherence of the ring $R[x]/(x^k)$ $(k \geq 1)$. Various sufficient and necessary conditions are obtained for $R[x]/(x^2)$ to be left (m,n)-injective and for $R[x]/(x^k)$ $(k > 2)$ to be left P-injective. Moreover, it is proved that R is left coherent if and only if $R[x]/(x^k)$ is left coherent for every $k \geq 1$ if and only if $R[x]/(x^k)$ is left coherent for some $k \geq 1$.

1. Introduction

Throughout this paper, R is an associative ring with identity. For two positive integers m and n, we write $R^{m\times n}$ for the set of all $m \times n$ matrices over R, and let $R^n = R^{1\times n}$, $R_n = R^{n\times 1}$ and $M_n(R) = R^{n\times n}$. In 2001, (m,n)-injective modules were introduced and discussed in [3]. A left R-module M is called (m,n)-injective if every left R-homomorphism from an n-generated submodule of R^m to M extends to one from R^m to M. The ring R is said to be left (m,n)-injective if ${}_RR$ is (m,n)-injective. Some related notions are recalled here. A ring R is called left FP-injective if R is left (m,n)-injective for all positive integers m and n. If R is left $(1,n)$-injective (resp., left $(1,1)$-injective), then R is called left n-injective (resp., left P-injective). A ring R is called left f-injective if R is left n-injective for every positive integer n. Right versions of these injectivities are defined analogously.

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The ring $R[x]/(x^k)$ $(k \geq 1)$, as an important extension of R, has been discussed in many papers (see [5], [7], [8], [9] et al). In this paper, (m,n)-injectivity and coherence of $R[x]/(x^k)$ are studied. It is well known that $R[x]/(x^2)$ is isomorphic to the trivial extension of R by R, i.e., the ring $R \propto R = \{(a,b) : a,\ b \in R\}$ with addition defined componentwise and multiplication defined by $(a,b)(c,d) = (ac, ad+bc)$. By [6], $R \propto R$ is right self-injective if so is R. In [4], a sufficient but not necessary condition is given for $R \propto R$ to be right (m,n)-injective. In Section 2, we consider the left (m,n)-injectivity of $R \propto R$ and derive an equivalent condition for $R \propto R$ to be left (m,n)-injective. Some known results on (m,n)-injective rings in [4] are obtained as corollaries. The left (m,n)-injectivity of $R[x]/(x^k)$ $(k > 2)$ is investigated in Section 3. For simplicity, we only consider the left P-injectivity and a sufficient and necessary condition for $R[x]/(x^k)$ to be left P-injective is given. A similar argument can be used to obtain an analogous result about the left (m,n)-injectivity of $R[x]/(x^k)$.

Another question we considered is about the coherence of $R[x]/(x^k)$ $(k \geq 1)$. A ring R is said to be left coherent if each of its finitely generated left ideals is finitely presented [1], or equivalently if $l(a)$ is a finitely generated left ideal of R for any $a \in R$ and the intersection of two finitely generated left ideals of R is again finitely generated [10]. A sufficient and necessary condition for $R \propto R$ to be coherent was obtained by Chen and Zhou in [4] where they showed that $R \propto R$ is left coherent if and only if so is R. In Section 4, we generalize the result by showing that R is left coherent if and only if $R[x]/(x^k)$ is left coherent for every $k \geq 1$ if and only if $R[x]/(x^k)$ is left coherent for some $k \geq 1$.

In this paper, if $S \subseteq R^{m\times n}$, we set $l_{R^m}(S) = \{\alpha \in R^m : \alpha A = 0,\ \forall A \in S\}$ and $r_{R_n}(S) = \{\beta \in R_n : A\beta = 0,\ \forall A \in S\}$.

2. (m,n)-injectivity of $R \propto R$

Let R be a ring and m, n be two positive integers. In this section, we investigate the (m,n)-injectivity of the ring $R \propto R$, which is isomorphic to $R[x]/(x^2)$.

Recall that R is left (m,n)-injective [3] if and only if, for any $C \in R^{n\times m}$, every left R-homomorphism from R^nC to R extends to one from R^m to R if and only if $r_{R_n}l_{R^n}(A) = AR_m$ for all $A \in R^{n\times m}$. For convenience, we fix some notations. Set $A = (a_{ij})$, $B = (b_{ij}) \in R^{m\times n}$. Denote $(R^mA : B) = \{\alpha \in R^m : \alpha B \in R^mA\}$, $(AR_n : B) = \{\alpha \in R_n : B\alpha \in AR_n\}$ and $A \propto B = ((a_{ij}, b_{ij})) \in (R \propto R)^{m\times n}$. By calculation, it is clear that $A \propto B = 0$ if and only if $A = 0$, $B = 0$ and $(A \propto B)$ $(C \propto D) = AC \propto (AD + BC)$ for any $A, B \in R^{m\times n}$ and any $C, D \in R^{n\times t}$.

Theorem 2.1. *Let m and n be two positive integers. The following are equivalent for a ring R:*

(1) *$R \propto R$ is a left (m,n)-injective ring;*

(2) *$r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B)) = AR_m + Br_{R_m}(A)$ for any $A, B \in R^{n\times m}$.*

PROOF. Denote $S = R \propto R$.

(1) $\Rightarrow$ (2). First we claim that $Br_{R_m}(A) \subseteq r_{R_n}((R^n A : B))$ for any $A, B \in R^{n\times m}$.

In fact, let $\alpha = B\bar{\alpha}$ with $\bar{\alpha} \in r_{R_m}(A)$. For any $\beta \in (R^n A : B)$, there exists $\gamma \in R^n$ such that $\beta B = \gamma A$. Then $\beta\alpha = \beta B\bar{\alpha} = \gamma A\bar{\alpha} = 0$, i.e., $\alpha \in r_{R_n}$ $((R^n A : B))$. So

$$Br_{R_m}(A) \subseteq r_{R_n}((R^n A : B))$$

and

$$AR_m + Br_{R_m}(A) \subseteq AR_m + r_{R_n}((R^n A : B)) \subseteq r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B)).$$

Next we show that $r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B)) \subseteq AR_m + Br_{R_m}(A)$.

Set $A = (a_{ij}), B = (b_{ij}) \in R^{n\times m}$. Then $A \propto B = ((a_{ij}, b_{ij})) \in S^{n\times m}$. Since S is left (m,n)-injective, $r_{S_n}(l_{S^n}(A \propto B)) = (A \propto B)S_m$. Assume $\alpha \in r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B))$. For any $P \propto Q \in l_{S^n}(A \propto B)$,

$$PA \propto (PB + QA) = (P \propto Q)(A \propto B) = 0.$$

So $PA = 0$ and $PB + QA = 0$, i.e., $P \in l_{R^n}(A) \bigcap (R^n A : B)$. Then $P\alpha = 0$. Thus $(P \propto Q)(0 \propto \alpha) = 0 \propto P\alpha = 0$ and $0 \propto \alpha \in r_{S_n}(l_{S^n}(A \propto B)) = (A \propto B)S_m$. So there exists $C \propto D \in S_m$ such that

$$0 \propto \alpha = (A \propto B)(C \propto D) = AC \propto (AD + BC).$$

Hence $AC = 0$ and $\alpha = AD + BC \in AR_m + Br_{R_m}(A)$. Thus $r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B)) \subseteq AR_m + Br_{R_m}(A)$. Therefore $r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B)) = AR_m + Br_{R_m}(A)$.

(2) $\Rightarrow$ (1). Assume (2). For any $A \in R^{n\times m}$, set $B = 0 \in R^{n\times m}$. Then $(R^n A : B) = R^n$. So the hypothesis implies that

$$r_{R_n} l_{R^n}(A) = AR_m.$$

Now for any $T = \big((a_{ij}, b_{ij})\big) \in S^{n\times m}$, we shall show that $r_{S_n} l_{S^n}(T) = TS_m$.

Denote $A = (a_{ij}), B = (b_{ij})$. Then $A, B \in R^{n\times m}$ and $T = A \propto B$. Let $X \propto Y \in r_{S_n} l_{S^n}(T)$. For any $C \in l_{R^n}(A)$, we have $0 \propto C \in l_{S^n}(T)$ and $0 \propto CX = (0 \propto C)(X \propto Y) = 0$, so $CX = 0$. This implies that $X \in r_{R_n} l_{R^n}(A) = AR_m$. Write $X = AU$ with $U \in R_m$. For any $D \in l_{R^n}(A) \bigcap (R^n A : B)$, we have $DA = 0$ and $DB + HA = 0$ for some $H \in R^n$. Thus $(D \propto H)T = (D \propto H)(A \propto B) = DA \propto (DB + HA) = 0$. It follows that $DX \propto (DY + HX) = (D \propto H)(X \propto Y) = 0$, i.e., $DX = 0$ and $DY + HX = 0$. Consequently,

$$D(Y - BU) = DY - DBU = DY + HAU = DY + HX = 0.$$

This shows that $Y - BU \in r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B)) = AR_m + Br_{R_m}(A)$, so

$$Y = AV + BU + BW$$

for some $V \in R_m$ and $W \in r_{R_m}(A)$. It is easy to see that

$$X \propto Y = AU \propto (AV + BU + BW) = (A \propto B)((U + W) \propto V) \in TS_m.$$

Thus $r_{S_n} l_{S^n}(T) \subseteq TS_m$. Note that the converse inclusion always holds. Therefore $S = R \propto R$ is left (m, n)-injective. □

Corollary 2.2. *If $R \propto R$ is left (m, n)-injective, then $r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B)) = AR_m + r_{R_n}((R^n A : B))$ for any $A, B \in R^{n\times m}$.*

Proof. It is straightforward to verify that

$$AR_m + Br_{R_m}(A) \subseteq AR_m + r_{R_n}((R^n A : B)) \subseteq r_{R_n}(l_{R^n}(A) \bigcap (R^n A : B))$$

for any $A, B \in R^{n\times m}$. Therefore, the result follows immediately from Theorem 2.1. □

Similarly, we can get the following theorem about the right (m, n)-injectivity of $R \propto R$.

Theorem 2.3. *Let R be a ring and m, n be two positive integers. The following are equivalent for R:*

(1) *$R \propto R$ is a right (m, n)-injective ring;*

(2) *$l_{R^n}(r_{R_n}(A) \bigcap (AR_n : B)) = R^m A + l_{R^m}(A)B$ for any $A, B \in R^{m\times n}$.*

Corollary 2.4 ([4, Theorem 1])**.** *Let R be a ring. Suppose that, for any $A, B \in R^{m\times n}$, every right R-homomorphism from $AR_n + Br_{R_n}(A)$ to R extends to one from R_m to R. Then $R \propto R$ is a right (m, n)-injective ring.*

Proof. First note that, if $B=0$, then the hypothesis implies that every right R-homomorphism from AR_n to R extends to one from R_m to R for any $A\in R^{m\times n}$. This shows that R is right (m,n)-injective. As done in the proof of Theorem 2.1, we have $R^mA+l_{R^m}(A)B\subseteq l_{R^n}(r_{R_n}(A)\bigcap(AR_n:B))$. Assume $\alpha\in l_{R^n}(r_{R_n}(A)\bigcap(AR_n:B))$ and define:

$$f: AR_n+Br_{R_n}(A)\to R;\ A\gamma_1+B\gamma_2\mapsto \alpha\gamma_2.$$

If $A\gamma_1+B\gamma_2=0$, then $\gamma_2\in r_{R_n}(A)\bigcap(AR_n:B)$, so $\alpha\gamma_2=0$. Thus f is well-defined. Moreover, it is easy to see that f is a right R-homomorphism. By hypothesis, f can be extended to a right R-homomorphism from R_m to R, i.e., there exists $\xi\in R^m$ such that, for any $A\gamma_1+B\gamma_2\in AR_n+Br_{R_n}(A)$,

$$f(A\gamma_1+B\gamma_2)=\xi(A\gamma_1+B\gamma_2).$$

Thus, for any $\gamma_1\in R_n,\gamma_2\in r_{R_n}(A)$,

$$\xi A\gamma_1=f(A\gamma_1)=0,\ \xi B\gamma_2=f(B\gamma_2)=\alpha\gamma_2.$$

Then $\xi A=\xi AI_n=\xi A(e_1,\ldots,e_n)=0$, where I_n is the identity of $R^{n\times n}$ and e_i is the i-th column of I_n. So $\xi\in l_{R^m}(A)$ and $\alpha-\xi B\in l_{R^n}r_{R_n}(A)=R^mA$. It follows that $\alpha=(\alpha-\xi B)+\xi B\in R^mA+l_{R^m}(A)B$ and $l_{R^n}(r_{R_n}(A)\bigcap(AR_n:B))\subseteq R^mA+l_{R^m}(A)B$. Hence

$$l_{R^n}(r_{R_n}(A)\bigcap(AR_n:B))=R^mA+l_{R^m}(A)B.$$

By Theorem 2.3, the result follows. □

Corollary 2.5 ([4, Theorem 2]). *If $R\propto R$ is right (m,n)-injective, then so is R.*

Proof. Set $B=0$ in Theorem 2.3. We have $l_{R^n}r_{R_n}(A)=R^mA$ for all $A\in R^{m\times n}$. Therefore R is right (m,n)-injective. □

Corollary 2.6. *$R\propto R$ is left P-injective if and only if $r_R(l_R(a)\bigcap(Ra:b))=aR+br_R(a)$ for any $a,b\in R$.*

Corollary 2.7. *$R\propto R$ is left FP-injective if and only if $r_{R_n}(l_{R^n}(A)\bigcap(R^nA:B))=AR_m+Br_{R_m}(A)$ for any positive integers m,n and any $A,B\in R^{n\times m}$.*

Corollary 2.8. *Let n be a fixed positive integer. Then $R\propto R$ is left n-injective if and only if* $r_{R_n}\left(l_{R^n}\left(\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}\right)\bigcap\left(R^n\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}:\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}\right)\right)=\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}R+\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}r_R\left(\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}\right)$ *for any* $\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix},\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix}\in R_n$.

Corollary 2.9. *Let m be a fixed positive integer. Then every m-generated right ideal of $R \propto R$ is a right annihilator if and only if* $r_R(l_R(K) \bigcap (R(a_1,\dots,a_m) : (b_1,\dots,b_m))) = K + (b_1,\dots,b_m) r_{R_m}((a_1,\dots,a_m))$ *for any* $(a_1,\dots,a_m),(b_1,\dots,b_m) \in R^m$, *where* $K = a_1R + \cdots + a_mR$.

3. P-injectivity of $R[x]/(x^k)$

It is well known that $R \propto R$ is isomorphic to $R[x]/(x^2)$. So it is natural to explore the left (m,n)-injectivity of $R[x]/(x^k)$ for an arbitrary positive integer k. For simplicity, we only consider the left P-injectivity of $R[x]/(x^k)$ and acquire an equivalent condition for it. Using a similar argument, an analogous result about the left (m,n)-injectivity of $R[x]/(x^k)$ can be obtained.

We regard $R[x]/(x^k)$ as a subring of $R^{k\times k}$ by identifying the element $a_0 + a_1x + \cdots + a_{k-1}x^{k-1} \in R[x]/(x^k)$ with the matrix

$$\begin{pmatrix} a_0 & a_1 & \dots & a_{k-2} & a_{k-1} \\ & a_0 & a_1 & \dots & a_{k-2} \\ & & \ddots & \ddots & \vdots \\ & & & a_0 & a_1 \\ & & & & a_0 \end{pmatrix} \in R^{k\times k}.$$

Denote by $\psi : R[x]/(x^k) \to R^{k\times k}$ such ring inclusion and

$$S_{(k)} = \{\psi(a_0 + a_1x + \cdots + a_{k-1}x^{k-1}) : a_0, a_1, \dots, a_{k-1} \in R\}.$$

Write $(R^kA : \alpha) = \{r \in R : r\alpha \in R^kA\}$ for any $A \in S_{(k)}$, $\alpha \in R^k$.

Lemma 3.1. *Let R be a ring and n be a fixed positive integer. If, for any $a \in R$, $\alpha = (a_1,\dots,a_n) \in R^n$, $r_R(l_R(a) \bigcap (R^nA : \alpha)) = aR + \alpha r_{R_n}(A)$, where $A = \psi(a + a_1x + \cdots + a_{n-1}x^{n-1}) \in S_{(n)}$, then $r_R(l_R(b) \bigcap (R^mB : \beta)) = bR + \beta r_{R_m}(B)$ for each $1 \le m \le n$ and any $b \in R$, $\beta = (b_1,\dots,b_m) \in R^m$, $B = \psi(b + b_1x + \cdots + b_{m-1}x^{m-1}) \in S_{(m)}$.*

Proof. It suffices to prove the conclusion for $m = n-1$. Suppose $b \in R$, $\beta = (b_1,\dots,b_{n-1}) \in R^{n-1}$ and $B = \psi(b + b_1x + \cdots + b_{n-2}x^{n-2}) \in S_{(n-1)}$. Let $\bar\beta = (b,\beta) = (b, b_1,\dots,b_{n-1}) \in R^n$, $\bar B = \left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right) = \psi(bx + b_1x^2 + \cdots + b_{n-2}x^{n-1}) \in S_{(n)}$. By hypothesis, $r_R((R^n\bar B : \bar\beta)) = \bar\beta r_{R_n}(\bar B)$.

Note that $x \in (R^n\bar{B} : \bar{\beta})$ iff there exists $\bar{\delta} = (\delta, r) \in R^n$, where $r \in R$, $\delta \in R^{n-1}$, such that

$$x(b,\beta) = x\bar{\beta} = \bar{\delta}\bar{B} = (\delta, r)\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = (0, \delta B)$$

iff $x \in l_R(b) \bigcap (R^{n-1}B : \beta)$. This implies that $(R^n\bar{B} : \bar{\beta}) = l_R(b) \bigcap (R^{n-1}B : \beta)$.

Moreover, it is easy to see that $r_{R_n}(\bar{B}) = \left(\begin{smallmatrix} R \\ r_{R_{n-1}}(B) \end{smallmatrix}\right)$. Therefore

$$\begin{aligned} r_R(l_R(b) \bigcap (R^{n-1}B : \beta)) &= r_R((R^n\bar{B} : \bar{\beta})) = \bar{\beta} r_{R_n}(\bar{B}) \\ &= (b, \beta)\left(\begin{smallmatrix} R \\ r_{R_{n-1}}(B) \end{smallmatrix}\right) = bR + \beta r_{R_{n-1}}(B). \end{aligned}$$ □

Lemma 3.2. *Let m be a positive integer. If $S_{(m)}$ is left P-injective and $r_R(l_R(a) \bigcap (R^m A : \alpha)) = aR + \alpha r_{R_m}(A)$ for any $a \in R$, $\alpha = (a_1, \dots, a_m) \in R^m$ and $A = \psi(a + a_1x + \cdots + a_{m-1}x^{m-1}) \in S_{(m)}$, then $S_{(m+1)}$ is left P-injective.*

Proof. Suppose $\bar{A} = \left(\begin{smallmatrix} a & \alpha \\ 0 & A \end{smallmatrix}\right) \in S_{(m+1)}$ with $a \in R$, $\alpha = (a_1, \dots, a_m) \in R^m$ and $A = \psi(a + a_1x + \cdots + a_{m-1}x^{m-1}) \in S_{(m)}$, then $r_{S_{(m)}} l_{S_{(m)}}(A) = AS_{(m)}$ because $S_{(m)}$ is left P-injective. We will show that $r_{S_{(m+1)}} l_{S_{(m+1)}}(\bar{A}) = \bar{A}S_{(m+1)}$.

Assume $\bar{Z} = \left(\begin{smallmatrix} z & \xi \\ 0 & Z \end{smallmatrix}\right) \in r_{S_{(m+1)}} l_{S_{(m+1)}}(\bar{A})$, where $z \in R$, $\xi = (z_1, \dots, z_m) \in R^m$ and $Z = \psi(z + z_1x + \cdots + z_{m-1}x^{m-1}) \in S_{(m)}$. Since $\left(\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right) \in l_{S_{(m+1)}}(\bar{A})$ for any $Y \in l_{S_{(m)}}(A)$,

$$\begin{pmatrix} 0 & YZ \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}) \begin{pmatrix} z & \xi \\ 0 & Z \end{pmatrix} = 0,$$

i.e., $YZ = 0$. Thus $Z \in r_{S_{(m)}} l_{S_{(m)}}(A) = AS_{(m)}$, whence $Z = AH$ for some $H = \psi(h + h_1x + \cdots + h_{m-1}x^{m-1}) \in S_{(m)}$.

For any $t \in l_R(a) \bigcap (R^m A : \alpha)$, we have $ta = 0$ and $t\alpha + \beta A = 0$ for some $\beta = (b_1, \dots, b_m) \in R^m$, i.e., $\left(\begin{smallmatrix} t & \beta \end{smallmatrix}\right)\left(\begin{smallmatrix} a & \alpha \\ 0 & A \end{smallmatrix}\right) = 0$. Let $B = \psi(t + b_1x + \cdots + b_{m-1}x^{m-1}) \in S_{(m)}$. Then $\left(\begin{smallmatrix} t & \beta \\ 0 & B \end{smallmatrix}\right)\bar{A} = \left(\begin{smallmatrix} t & \beta \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} a & \alpha \\ 0 & A \end{smallmatrix}\right) = 0$, i.e., $\left(\begin{smallmatrix} t & \beta \\ 0 & B \end{smallmatrix}\right) \in l_{S_{(m+1)}}(\bar{A})$. It follows that

$$\begin{pmatrix} tz & t\xi + \beta Z \\ 0 & BZ \end{pmatrix} = \begin{pmatrix} t & \beta \\ 0 & B \end{pmatrix}\begin{pmatrix} z & \xi \\ 0 & Z \end{pmatrix} = 0.$$

So $t\xi + \beta Z = 0$. Since $Z = AH$, we have $t(\xi - \alpha H) = t\xi - t\alpha H = t\xi + \beta AH = t\xi + \beta Z = 0$. Then $t\left(z_m - \alpha\left(\begin{smallmatrix} h_{m-1} \\ \vdots \\ h_1 \\ h \end{smallmatrix}\right)\right) = 0$. Thus $z_m - \alpha\left(\begin{smallmatrix} h_{m-1} \\ \vdots \\ h_1 \\ h \end{smallmatrix}\right) \in$

$r_R(l_R(a)\bigcap(R^mA:\alpha)) = aR + \alpha r_{R_m}(A)$. Write

$$z_m - \alpha\begin{pmatrix} h_{m-1} \\ \vdots \\ h_1 \\ h \end{pmatrix} = ar + \alpha\begin{pmatrix} g_{m-1} \\ \vdots \\ g_1 \\ g \end{pmatrix}$$

with $r \in R$, $\begin{pmatrix} g_{m-1} \\ \vdots \\ g_1 \\ g \end{pmatrix} \in r_{R_m}(A)$. Then $z_m = ar + \alpha\begin{pmatrix} h_{m-1}+g_{m-1} \\ \vdots \\ h_1+g_1 \\ h+g \end{pmatrix}$. Set $G = \psi(g + g_1x + \cdots + g_{m-1}x^{m-1})$. Since $A\begin{pmatrix} g_{m-1} \\ \vdots \\ g_1 \\ g \end{pmatrix} = 0$, $AG = 0$. So $Z = AH = A(H+G)$. Then $z = a(h+g)$ and

$$(z_1,\ldots,z_{m-1}) = (a,a_1,\ldots,a_{m-1})\begin{pmatrix} h_1+g_1 & h_2+g_2 & h_3+g_3 & \ldots & h_{m-1}+g_{m-1} \\ h+g & h_1+g_1 & h_2+g_2 & \ldots & h_{m-2}+g_{m-2} \\ & h+g & h_1+g_1 & \ldots & h_{m-3}+g_{m-3} \\ & & \ddots & \ddots & \vdots \\ & & & h+g & h_1+g_1 \\ & & & & h+g \end{pmatrix}$$

$$= a(h_1+g_1,\ldots,h_{m-1}+g_{m-1})$$

$$+ (a_1,\ldots,a_{m-1})\begin{pmatrix} h+g & h_1+g_1 & h_2+g_2 & \ldots & h_{m-2}+g_{m-2} \\ & h+g & h_1+g_1 & \ldots & h_{m-3}+g_{m-3} \\ & & \ddots & \ddots & \vdots \\ & & & h+g & h_1+g_1 \\ & & & & h+g \end{pmatrix}$$

$$= a(h_1+g_1,\ldots,h_{m-1}+g_{m-1})$$

$$+ (a_1,\ldots,a_{m-1},a_m)\begin{pmatrix} h+g & h_1+g_1 & h_2+g_2 & \ldots & h_{m-2}+g_{m-2} \\ & h+g & h_1+g_1 & \ldots & h_{m-3}+g_{m-3} \\ & & \ddots & \ddots & \vdots \\ & & & h+g & h_1+g_1 \\ & & & & h+g \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

$$= a(h_1+g_1,\ldots,h_{m-1}+g_{m-1})$$

$$+ \alpha\begin{pmatrix} h+g & h_1+g_1 & h_2+g_2 & \ldots & h_{m-2}+g_{m-2} \\ & h+g & h_1+g_1 & \ldots & h_{m-3}+g_{m-3} \\ & & \ddots & \ddots & \vdots \\ & & & h+g & h_1+g_1 \\ & & & & h+g \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Since $z_m = ar + \alpha\begin{pmatrix} h_{m-1}+g_{m-1} \\ \vdots \\ h_1+g_1 \\ h+g \end{pmatrix}$,

$$\xi = (z_1,\ldots,z_{m-1},z_m) = a(h_1+g_1,\ldots,h_{m-1}+g_{m-1},r)$$

$$+\,\alpha\begin{pmatrix} h+g & h_1+g_1 & h_2+g_2 & \dots & h_{m-2}+g_{m-2} & h_{m-1}+g_{m-1} \\ & h+g & h_1+g_1 & \dots & h_{m-3}+g_{m-3} & h_{m-2}+g_{m-2} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & h+g & h_1+g_1 & h_2+g_2 \\ & & & & h+g & h_1+g_1 \\ 0 & 0 & 0 & \dots & 0 & h+g \end{pmatrix} = a\eta+\alpha(H+G),$$

where $\eta = (h_1+g_1, \dots, h_{m-1}+g_{m-1}, r) \in R^m$. From this, we can see that $\left(\begin{smallmatrix} h+g & \eta \\ 0 & H+G \end{smallmatrix}\right) \in S_{(m+1)}$ and

$$\begin{aligned} \bar{Z} = \begin{pmatrix} z & \xi \\ 0 & Z \end{pmatrix} &= \begin{pmatrix} a(h+g) & a\eta+\alpha(H+G) \\ 0 & A(H+G) \end{pmatrix} \\ &= \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix}\begin{pmatrix} h+g & \eta \\ 0 & H+G \end{pmatrix} \in \bar{A}S_{(m+1)}. \end{aligned}$$

Hence $r_{S_{(m+1)}} l_{S_{(m+1)}}(\bar{A}) = \bar{A}S_{(m+1)}$, and this shows that $S_{(m+1)}$ is left P-injective. $\square$

Theorem 3.3. *Let n be a positive integer. The following are equivalent for a ring R:*

(1) *$R[x]/(x^n)$ is a left P-injective ring;*

(2) *$S_{(n)}$ is a left P-injective ring;*

(3) *$r_R(l_R(a) \bigcap (R^{n-1}A : \alpha)) = aR + \alpha r_{R_{n-1}}(A)$ for any $a \in R$, $\alpha = (a_1, \dots, a_{n-1}) \in R^{n-1}$ and $A = \psi(a + a_1x + \dots + a_{n-2}x^{n-2}) \in S_{(n-1)}$.*

Proof. We only need to show (2) $\Leftrightarrow$ (3).

(2) $\Rightarrow$ (3). Suppose $a \in R$, $\alpha = (a_1, \dots, a_{n-1}) \in R^{n-1}$. Set $A = \psi(a + a_1x + \dots + a_{n-2}x^{n-2}) \in S_{(n-1)}$ and $\bar{A} = \left(\begin{smallmatrix} a & \alpha \\ 0 & A \end{smallmatrix}\right) \in S_{(n)}$. Then $r_{S_{(n)}} l_{S_{(n)}}(\bar{A}) = \bar{A}S_{(n)}$ by hypothesis.

Let $t = \alpha\mu$ with $\mu \in r_{R_{n-1}}(A)$. For any $r \in (R^{n-1}A : \alpha)$, $r\alpha + \gamma A = 0$ for some $\gamma \in R^{n-1}$ and hence

$$rt = r\alpha\mu = r\alpha\mu + \gamma A\mu = (r\alpha + \gamma A)\mu = 0.$$

This shows $t \in r_R((R^{n-1}A : \alpha))$ and $\alpha r_{R_{n-1}}(A) \subseteq r_R((R^{n-1}A : \alpha))$. So

$$aR + \alpha r_{R_{n-1}}(A) \subseteq aR + r_R((R^{n-1}A : \alpha)) \subseteq r_R(l_R(a) \bigcap (R^{n-1}A : \alpha)).$$

Conversely, assume $z \in r_R(l_R(a) \bigcap (R^{n-1}A : \alpha))$. For any $\bar{B} = \left(\begin{smallmatrix} b & \beta \\ 0 & B \end{smallmatrix}\right) \in l_{S_{(n)}}(\bar{A})$,

$$\begin{pmatrix} ba & b\alpha + \beta A \\ 0 & BA \end{pmatrix} = \begin{pmatrix} b & \beta \\ 0 & B \end{pmatrix}\begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} = \bar{B}\bar{A} = 0.$$

So $ba = 0$ and $b\alpha + \beta A = 0$, i.e., $b \in l_R(a) \bigcap (R^{n-1}A : \alpha)$. Thus $bz = 0$. Let $\xi = (0, \dots, 0, z) \in R^{n-1}$. Then $b\xi = 0$ and

$$\begin{pmatrix} b & \beta \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b\xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $\left(\begin{smallmatrix} 0 & \xi \\ 0 & 0 \end{smallmatrix}\right) \in r_{S_{(n)}} l_{S_{(n)}}(\bar{A}) = \bar{A} S_{(n)}$. Write

$$\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & \alpha \\ 0 & A \end{pmatrix} \begin{pmatrix} d & \eta \\ 0 & D \end{pmatrix}$$

with $d \in R$, $\eta = (y_1, \dots, y_{n-1}) \in R^{n-1}$ and $\left(\begin{smallmatrix} d & \eta \\ 0 & D \end{smallmatrix}\right) \in S_{(n)}$. Then $AD = 0$ and $\xi = a\eta + \alpha D$. Let $\lambda = \begin{pmatrix} y_{n-2} \\ \vdots \\ y_1 \\ d \end{pmatrix}$ be the last column of D. Then $A\lambda = 0$ and $z = ay_{n-1} + \alpha\lambda \in aR + \alpha r_{R_{n-1}}(A)$. This implies that $r_R(l_R(a) \bigcap (R^{n-1}A : \alpha)) \subseteq aR + \alpha r_{R_{n-1}}(A)$. Hence $r_R(l_R(a) \bigcap (R^{n-1}A : \alpha)) = aR + \alpha r_{R_{n-1}}(A)$.

(3) $\Rightarrow$ (2). Assume (3). By Lemma 3.1, we get that $r_R(l_R(a) \bigcap (R^m A : \alpha)) = aR + \alpha r_{R_m}(A)$ for each $1 \le m \le n-1$ and any $a \in R$, $\alpha = (a_1, \dots, a_m) \in R^m$, $A = \psi(a + a_1 x + \dots + a_{m-1}x^{m-1}) \in S_{(m)}$. In particular, $r_R(l_R(a) \bigcap (Ra : b)) = aR + br_R(a)$ for any $a, b \in R$. So $S_{(2)} = R \propto R$ is left P-injective by Corollary 2.6. Hence, by Lemma 3.2, $S_{(3)}$ is left P-injective. Proceeding in this manner, we can get that $S_{(m)}$ is left P-injective for all $2 \le m \le n$. In particular, $S_{(n)}$ is left P-injective, and the proof is completed. □

Corollary 3.4. *If $R[x]/(x^n)$ is a left P-injective ring, then $R[x]/(x^m)$ is left P-injective for all $1 \le m \le n$.*

Proof. By Theorem 3.3 and Lemma 3.1. □

4. Coherence of $R[x]/(x^n)$

Let n be a positive integer. In this section, we explore the interplay between the coherence of a ring R and the coherence of $R[x]/(x^n)$ $(n \ge 1)$. We denote $S_{(n)} = \{\psi(a_0 + a_1 x + \dots + a_{n-1}x^{n-1}) \in R^{n\times n} : a_0, a_1, \dots, a_{n-1} \in R\}$ as in Section 3 and write $(R_n a : \alpha) = \{H \in S_{(n)} : H\alpha \in R_n a\}$ for any $a \in R$, $\alpha \in R_n$.

It was proved in [4] that a ring R is left coherent if and only if $R \propto R$ is left coherent. This is a special case of the main result of this section. We first give the following lemma which appears in the proof of [4, Theorem 12].

Lemma 4.1. *If R is a left coherent ring, then $(Ra : b)$ is a finitely generated left ideal of R for any $a, b \in R$.*

Lemma 4.2. *Let R be a ring and n be a positive integer. If $S_{(n)}$ is left coherent, then $(R_n a : \alpha)$ is a finitely generated left ideal of $S_{(n)}$ for any $a \in R$, $\alpha \in R_n$.*

Proof. Suppose $a \in R$, $\alpha = \begin{pmatrix} a_n \\ \vdots \\ a_1 \end{pmatrix} \in R_n$. Let $A = \psi(a)$, $B = \psi(a_1 + a_2 x + \cdots + a_n x^{n-1})$. Then $A, B \in S_{(n)}$. Denote $(S_{(n)}A : B) = \{H \in S_{(n)} : HB \in S_{(n)}A\}$. Note that $H \in (R_n a : \alpha)$ iff $H\alpha = \gamma a$ for some $\gamma = \begin{pmatrix} r_n \\ \vdots \\ r_1 \end{pmatrix} \in R_n$ iff $HB = GA$, where $G = \psi(r_1 + r_2 x + \cdots + r_n x^{n-1}) \in S_{(n)}$ iff $H \in (S_{(n)}A : B)$. This implies that $(R_n a : \alpha) = (S_{(n)}A : B)$. Since $S_{(n)}$ is left coherent, $(S_{(n)}A : B)$ is a finitely generated left ideal of $S_{(n)}$ by Lemma 4.1. So $(R_n a : \alpha)$ is a finitely generated left ideal of $S_{(n)}$. □

Theorem 4.3. *The following are equivalent for a ring R:*

(1) *R is left coherent;*

(2) *$R[x]/(x^n)$ is left coherent for all $n \geq 1$;*

(3) *$R[x]/(x^n)$ is left coherent for some $n \geq 1$.*

Proof. Since $R[x]/(x^n) \cong S_{(n)}$, we proceed the proof for $S_{(n)}$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Assume (3). We first show that R is left P-coherent, i.e., $l_R(a)$ is a finitely generated left ideal of R for any $a \in R$.

Set $A = \psi(ax^{n-1}) \in S_{(n)}$. Note that $l_{S_{(n)}}(A) = \{\psi(b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}) : b_0 \in l_R(a),\ b_1, \ldots, b_{n-1} \in R\}$. Since $S_{(n)}$ is left coherent, $l_{S_{(n)}}(A)$ is a finitely generated left ideal of $S_{(n)}$. Write

$$\begin{aligned} l_{S_{(n)}}(A) = S_{(n)}\psi(a_1 + a_{11}x + \cdots + a_{1(n-1)}x^{n-1}) + \ldots \\ + S_{(n)}\psi(a_m + a_{m1}x + \cdots + a_{m(n-1)}x^{n-1}) \end{aligned}$$

with all $a_i,\ a_{ij} \in R$. It follows that

$$l_R(a) = Ra_1 + \cdots + Ra_m,$$

so R is left P-coherent.

Now since $R[x]/(x^n)$ is left coherent, $M_k(R[x]/(x^n))$ is left coherent for each $k \geq 1$. So $(M_k(R))[x]/(x^n) \cong M_k(R[x]/(x^n))$ is left coherent. Thus, as above,

$M_k(R)$ is left P-coherent for each $k \geq 1$, and so R is left coherent by [2, Proposition 2.7].

(1) $\Rightarrow$ (2). We prove the conclusion by induction on n. Since $S_{(1)} \cong R$, $S_{(n)}$ is left coherent for $n = 1$.

Assume $S_{(n)}$ is left coherent for some $n \geq 1$, we show that $S_{(n+1)}$ is left coherent. First we verify that $S_{(n+1)}$ is left P-coherent, i.e., for any $\bar{A} \in S_{(n+1)}$, $l_{S_{(n+1)}}(\bar{A})$ is a finitely generated left ideal of $S_{(n+1)}$.

Write $\bar{A} = \left(\begin{smallmatrix} A & \alpha \\ 0 & a \end{smallmatrix}\right)$ with $a \in R, \alpha = \begin{pmatrix} a_n \\ \vdots \\ a_1 \end{pmatrix} \in R_n, A = \psi(a + a_1 x + \cdots + a_{n-1}x^{n-1}) \in S_{(n)}$. Since $S_{(n)}$ is left coherent, $l_{S_{(n)}}(A)$ is a finitely generated left ideal of $S_{(n)}$ and $l_R(a)$ is a finitely generated left ideal of R. By Lemma 4.2, $(R_n a : \alpha)$ is a finitely generated left ideal of $S_{(n)}$. So $l_{S_{(n)}}(A) \bigcap (R_n a : \alpha)$ is finitely generated. Write

$$l_{S_{(n)}}(A) \bigcap (R_n a : \alpha) = S_{(n)}G_1 + \cdots + S_{(n)}G_m, \quad l_R(a) = Rt_1 + \cdots + Rt_m,$$

where all $G_i = \psi(g_i + g_{i1}x + \cdots + g_{i(n-1)}x^{n-1}) \in S_{(n)}$, $t_i \in R$. Then $G_i A = 0$, $G_i\alpha + \eta_i a = 0$ for some $\eta_i = \begin{pmatrix} d_{in} \\ \vdots \\ d_{i1} \end{pmatrix} \in R_n$, and $t_i a = 0$. Then $\psi(t_i x^n) \in l_{S_{(n+1)}}(\bar{A})$ for all $1 \leq i \leq m$. Let $\theta_i = \begin{pmatrix} d_{in} \\ g_{i(n-1)} \\ \vdots \\ g_{i1} \end{pmatrix}$. Then $\left(\begin{smallmatrix} G_i & \theta_i \\ 0 & g_i \end{smallmatrix}\right) \in S_{(n+1)}$. Since $G_i A = 0$ and $G_i\alpha + \eta_i a = 0$, we get $G_i\alpha + \theta_i a = 0$. This implies that $\left(\begin{smallmatrix} G_i & \theta_i \\ 0 & g_i \end{smallmatrix}\right) \in l_{S_{(n+1)}}(\bar{A})$, $\forall 1 \leq i \leq m$.

Assume $\left(\begin{smallmatrix} B & \beta \\ 0 & b \end{smallmatrix}\right) \in l_{S_{(n+1)}}(\bar{A})$, where $b \in R$, $\beta = \begin{pmatrix} b_n \\ \vdots \\ b_1 \end{pmatrix} \in R_n$, $B = \psi(b + b_1 x + \cdots + b_{n-1}x^{n-1}) \in S_{(n)}$. Then

$$\begin{pmatrix} BA & B\alpha + \beta a \\ 0 & ba \end{pmatrix} = \begin{pmatrix} B & \beta \\ 0 & b \end{pmatrix} \begin{pmatrix} A & \alpha \\ 0 & a \end{pmatrix} = 0.$$

So $BA = 0$, and $B\alpha + \beta a = 0$, thus $B \in l_{S_{(n)}}(A) \bigcap (R_n a : \alpha) = S_{(n)}G_1 + \cdots + S_{(n)}G_m$. Write

$$B = Z_1 G_1 + \cdots + Z_m G_m$$

with all $Z_i = \psi(z_i + z_{i1}x + \cdots + z_{i(n-1)}x^{n-1}) \in S_{(n)}$. Then

$$(\beta - (Z_1\theta_1 + \cdots + Z_m\theta_m))a = \beta a + Z_1 G_1 \alpha + \cdots + Z_m G_m \alpha = B\alpha + \beta a = 0.$$

Hence

$$(b_n - (\xi_1\theta_1 + \cdots + \xi_m\theta_m))a = 0,$$

where ξ_i is the first row of Z_i. It follows that $b_n - (\xi_1\theta_1 + \cdots + \xi_m\theta_m) \in l_R(a) = Rt_1 + \cdots + Rt_m$. Set

$$b_n - (\xi_1\theta_1 + \cdots + \xi_m\theta_m) = r_1t_1 + \cdots + r_mt_m$$

with all $r_i \in R$. Then $b_n = \xi_1\theta_1 + \cdots + \xi_m\theta_m + r_1t_1 + \cdots + r_mt_m$. Let $\lambda_i = \begin{pmatrix} 0 \\ z_{i(n-1)} \\ \vdots \\ z_{i1} \end{pmatrix}$. Then $\left(\begin{smallmatrix} Z_i & \lambda_i \\ 0 & z_i \end{smallmatrix}\right) \in S_{(n+1)}$. Since $B = \sum_{i=1}^m Z_iG_i$ and $b_n = \sum_{i=1}^m \xi_i\theta_i + \sum_{i=1}^m r_it_i$,

$$\beta = \sum_{i=1}^m (Z_i\theta_i + \lambda_i g_i) + \begin{pmatrix} \sum_{i=1}^m r_it_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} B & \beta \\ 0 & b \end{pmatrix} &= \begin{pmatrix} \sum_{i=1}^m Z_iG_i & \sum_{i=1}^m (Z_i\theta_i + \lambda_i g_i) \\ 0 & \sum_{i=1}^m z_ig_i \end{pmatrix} + \begin{pmatrix} 0 & \ldots & 0 & \sum_{i=1}^m r_it_i \\ 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \end{pmatrix} \\ &= \sum_{i=1}^m \begin{pmatrix} Z_i & \lambda_i \\ 0 & z_i \end{pmatrix} \begin{pmatrix} G_i & \theta_i \\ 0 & g_i \end{pmatrix} + \sum_{i=1}^m \psi(r_i)\psi(t_ix^n). \end{aligned}$$

This means that $l_{S_{(n+1)}}(\bar{A}) = \sum_{i=1}^m S_{(n+1)} \left(\begin{smallmatrix} G_i & \theta_i \\ 0 & g_i \end{smallmatrix}\right) + \sum_{i=1}^m S_{(n+1)}\psi(t_ix^n)$ is finitely generated. Thus we have proved that $S_{(n)}$ being left coherent implies that $S_{(n+1)}$ is left P-coherent, i.e., $R[x]/(x^n)$ being left coherent implies that $R[x]/(x^{n+1})$ is left P-coherent.

Since $R[x]/(x^n)$ is left coherent, $(M_k(R))[x]/(x^n) \cong M_k(R[x]/(x^n))$ is left coherent for each $k \geq 1$. As above, $M_k(R[x]/(x^{n+1})) \cong (M_k(R))[x]/(x^{n+1})$ is left P-coherent for each $k \geq 1$. So $R[x]/(x^{n+1})$ is left coherent by [2, Proposition 2.7]. Thus, by induction, R being left coherent implies that $R[x]/(x^n)$ is left coherent for each $n \geq 1$ and the result follows. $\square$

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