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Homogeneity is superstable

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Abstract. Let X be a set, (Y, d) a metric space, G a semigroup with unit. We assume that G acts on X and Y, respectively. Given a mapping $g : G \times X \to \mathbb{R}_+$ we consider the following condition for mappings f from X into Y:

$$d(f(\alpha x), \alpha f(x)) \leq g(\alpha, x) \quad \text{for} \quad \alpha \in G, x \in X\,.$$

We prove that under suitable assumptions on g and the acting of G on Y the mapping f is homogeneous, i.e.

$$f(\alpha x) = \alpha f(x) \quad \text{for} \quad \alpha \in G, x \in X\,.$$

A topological version of the problem for mappings from a vector space into a topological vector space is considered, too.

0. Let E_1 be a real vector space, E_2 a real normed space and let $\varepsilon \in [0, \infty)$. In [3] JÓZEF TABOR considered the problem of stability of linear mappings from E_1 into E_2. It has been proved there that every mapping $f : E_1 \to E_2$ satisfying the condition

$$\|f(\alpha x) - \alpha f(x)| \leq \varepsilon \quad \text{for} \quad \alpha \in \mathbb{R},\ x \in E_1 \tag{1}$$

is homogeneous.

This result raised the following question asked by K. BARON during the seminar of R. GER (Katowice, October 1992):

Is it true that every solution of the inequality

$$\|f(\alpha x) - \alpha f(x)\| \leq \varepsilon|\alpha| \quad \text{for} \quad \alpha \in \mathbb{R},\ x \in E_1 \tag{2}$$

is homogeneous?

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It turns out that conditions (1) and (2) are equivalent. For the proof it is sufficient to replace in (1) and (2) α by $\frac{1}{\alpha}$ and x by αx for $\alpha \neq 0$ and to note that (1) implies $f(0) = 0$. But the problem can be generalized. Then it becomes interesting though we known the answer to the original question of K. Baron.

Both conditions (1) and (2) mean that the expression $f(\alpha x) - \alpha f(x)$ is suitably bounded. Therefore some kinds of boundedness will be involved in their generalizations. We shall use the notion of a metric space in the first generalization and the notion of a topological vector space in the second one.

1. We begin our considerations with the most general result.

Lemma. *Let X be a set, Y a Hausdorff topological space and let $g_1 : X \to X$, $g_2 : Y \to Y$, $f : X \to Y$ be any mappings. If g_2 is continuous then the following conditions are equivalent:*

(i) $g_2(f(x)) = f(g_1(x))$ *for* $x \in X$,

(ii) *there exists a sequence of mappings $f_n : X \to Y$ such that*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \textit{for } \ x \in X \tag{3}$$

and

$$\lim_{n\to\infty} g_2(f_n(x)) = f(g_1(x)) \quad \textit{for } \ x \in X\,. \tag{4}$$

Proof. To obtain (ii) from (i) we need only to put $f_n := f$. Now suppose that (ii) holds. Making use of (4), (3) and continuity of g_2 we obtain

$$f(g_1(x)) = \lim_{n\to\infty} g_2(f_n(x)) = g_2(f(x)) \quad \text{for } \ x \in X\,.$$

As one expects, the continuity of g_2 is an essential assumption in the Lemma.

Example. Take $X = Y = [0,1]$, d – the usual metric,

$$g_1(x) = x \quad \text{for } x \in [0,1]\,, \quad g_2(y) = \begin{cases} 1 & \text{for } y = 0 \\ y^2 & \text{for } y \in (0,1] \end{cases},$$

$$f(x) = \begin{cases} 0 & \text{for } x \in [0,1] \\ 1 & \text{for } x = 1 \end{cases}, \quad f_n(x) = \begin{cases} \frac{1}{n} & \text{for } x \in [0,1) \\ 1 & \text{for } x = 1 \end{cases}.$$

Then (3) and (4) hold but (i) does not.

Before we prove the next theorems we need to recall some definitions and to establish some notations.

Let X be a set and G a semigroup with the unit 1. We say that G acts on X if we are given a mapping $F : G \times X \to X$ such that

$$F(\beta, F(\alpha, x)) = F(\beta\alpha, x) \quad \text{for} \ \ \alpha, \beta \in G, \ x \in X \,,$$
$$F(1, x) = x \quad \text{for} \ \ x \in X \,.$$

We write αx instead of $F(\alpha, x)$.

By $\mathbb{R}_+$ we denote the set of non-negative real numbers, K stands for the real or complex field and by 0^0 we mean 1.

Theorem 1. *Let X be a set, (Y, d) a metric space, G a semigroup with unit acting on X and Y, respectively. We assume that for each $\alpha \in G$ the mapping $Y \ni y \to \alpha y$ is continuous. Let $g : G \times X \to \mathbb{R}_+$ be a given mapping and let $f : X \to Y$ satisfy the following condition:*

$$d(f(\alpha x), \alpha f(x)) \leq g(\alpha, x) \quad \textit{for} \ \ \alpha \in G, \ x \in X \,. \tag{5}$$

If there exists a sequence α_n of invertible elements of G such that

$$\lim_{n\to\infty} g(\alpha\alpha_n, (\alpha_n)^{-1}x) = 0 \quad \textit{for} \ \ \alpha \in G, \ x \in X \,, \tag{6}$$

then

$$f(\alpha x) = \alpha f(x) \quad \textit{for} \ \ \alpha \in G, \ x \in X \,.$$

Proof. Inserting into (5) $\alpha\alpha_n$ and $(\alpha_n)^{-1}x$ in place of α and x, respectively, we obtain

$$d(f(\alpha x), \alpha(\alpha_n f((\alpha_n)^{-1}x)) \leq g(\alpha\alpha_n, (\alpha_n)^{-1}x) \quad \text{for} \ \ \alpha \in G, x \in X, n \in \mathbb{N} \,,$$

whence by (6) we have

$$\lim_{n\to\infty} \alpha(\alpha_n f((\alpha_n)^{-1}x)) = f(\alpha x) \quad \text{for} \ \ \alpha \in G, x \in X \,. \tag{7}$$

Taking $\alpha = 1$ we obtain from (7)

$$\lim_{n\to\infty} \alpha_n f((\alpha_n)^{-1}x) = f(x) \quad \text{for} \ \ x \in X \,. \tag{8}$$

Fix an $\alpha \in G$. We put

$$f_n(x) := \alpha_n f(\alpha_n))^{-1}x) \quad \text{for } x \in X \,,$$
$$g_1(x) := \alpha x \quad \text{for } x \in X \,,$$
$$g_2(y) := \alpha y \quad \text{for } y \in Y \,.$$

Then (8) means that f_n satsifies (3) and by (7) it satisfies (4). By the Lemma we obtain

$$f(g_1(x)) = g_2(f(x)) \quad \text{for} \quad x \in X,$$

i.e.

$$f(\alpha x) = \alpha f(x) \quad \text{for} \quad x \in X.$$

This completes the proof.

Corollary 1. *Let X be a normed space, $L(X)$ the semigroup of continuous linear operators on X with the composition as a binary operation and let $p_1, p_2 \in \mathbb{R}_+$, $p_1 \neq p_2$. Let $k : X \to \mathbb{R}_+$ be a given mapping such that*

$$k(Ax) \leq \|A\|^{p_2} k(x) \quad \text{for} \quad A \in L(X),\ x \in X. \tag{9}$$

If a mapping $f : X \to X$ satisfies the condition

$$\|f(Ax) - Af(x)\| \leq \|A\|^{p_1} k(x) \quad \text{for} \quad A \in L(X),\ x \in X, \tag{10}$$

then there exists an $\alpha \in K$ such that

$$f(x) = \alpha x \quad \text{for} \quad x \in X.$$

Proof. We put

$$g(A, x) := \|A\|^{p_1} k(x) \quad \text{for} \quad A \in L(X),\ x \in X,$$
$$A_n = \alpha_n I,$$

where I denotes the identity mapping, $\alpha_n = \frac{1}{n}$ if $p_1 > p_2$ and $\alpha_n = n$ if $p_1 < p_2$. Then (10) means that condition (5) is satisfied. By (9) we have for $x \in X$

$$g(AA_n, (A_n)^{-1}x) = \|AA_n\|^{p_1} k((A_n)^{-1}x) \leq$$
$$\leq \|A\|^{p_1} \|A_n\|^{p_1} \|(A_n)^{-1}\|^{p_2} k(x) = \|A\|^{p_1} |\alpha_n|^{p_1 - p_2} k(x) \xrightarrow[n \to \infty]{} 0,$$

which means that condition (6) holds. By Theorem 1 we obtain

$$f(Ax) = Af(x) \quad \text{for} \quad A \in L(X),\ x \in X. \tag{11}$$

We are going to prove that for each $x \in X$ there exists an $\alpha \in K$ such that $f(x) = \alpha x$. Inserting into (11) $x = 0$, $A = 2I$ we obtain that $f(0) = 0 = \alpha 0$ for $\alpha \in K$. Now suppose for the proof by contradiction that there exists an $x \in X$, $x \neq 0$ such that $f(x) \neq \alpha x$ for each $\alpha \in K$. Then x and $f(x)$ are linearly independent and therefore there exists an $A \in L(X)$ such that $Af(x) = 0$ and $Ax = x$. Now applying (11) we obtain that

$f(x) = 0$, which gives a contradiction. We have just proved that for each $x \in X$ there exists an $\alpha \in K$ such that $f(x) = \alpha x$. We need to prove yet that α does not depend on x. Let $x_1, x_2 \in X,\ x_1 \neq 0,\ x_2 \neq 0,\ x_1 \neq x_2$ and let $f(x_1) = \alpha_1 x_1, f(x_2) = \alpha_2 x_2$. Take an $A \in L(X)$ such that $Ax_1 = x_2$. Then by (11) we obtain

$$\alpha_1 x_2 = \alpha_1 A(x_1) = A(\alpha_1 x_1) = Af(x_1) = f(Ax_1) = f(x_2) = \alpha_2 x_2 \,.$$

Hence $\alpha_1 = \alpha_2$, which completes the proof.

Corollary 2. *Let K be a real or complex field and let $p, p_1, p_2 \in \mathbb{R}_+$, $p \neq p_2$. Let X be a vector space over K, Y a normed space over K and let $k : X \to \mathbb{R}_+$ be a given mapping such that*

$$k(\alpha x) \leq |\alpha|^{p_2} k(x) \quad \textit{for} \ \ \alpha \in K,\ x \in X \,. \tag{12}$$

If a mapping $f : X \to Y$ satisfies the condition

$$\|f(\alpha x) - |\alpha|^p f(x)\| \leq |\alpha|^{p_1} k(x) \quad \textit{for} \ \ \alpha \in K,\ x \in X \,, \tag{13}$$

then

$$f(\alpha x) = |\alpha|^p f(x) \quad \textit{for} \ \ \alpha \in K,\ x \in X \,. \tag{14}$$

Proof. An acting of K on X is the multiplication by scalars whereas an acting of K on Y is defined by the formula

$$\alpha * y := |\alpha|^p y \quad \text{for} \ \ \alpha \in K,\ y \in Y \,.$$

We put

$$g(\alpha, x) := |\alpha|^{p_1} k(x) \quad \text{for} \ \ \alpha \in K,\ x \in X \,.$$

Then (13) means that condition (5) is fulfilled. We take $\alpha_n = \frac{1}{n}$ if $p_1 > p_2$ and $\alpha_n = n$ if $p_1 < p_2$. Then by (12) we obtain for $\alpha \in K,\ x \in X$:

$$g(\alpha\alpha_n, (\alpha_n)^{-1} x) \leq |\alpha|^{p_1} |\alpha_n|^{p_1 - p_2} k(x) \xrightarrow[n \to \infty]{} 0 \,,$$

which proves that condition (7) is satisfied. By Theorem 1 we obtain

$$f(\alpha x) = \alpha * f(x) = |\alpha|^p f(x) \quad \text{for} \ \ \alpha \in K,\ x \in X \,.$$

It is obvious that in Corollary 2 X and Y may be replaced by their subsets X_1 and Y_1 such that $KX_1 \subset X_1$ and $K * Y_1 \subset Y_1$, respectively. We also note that the function f of the form (14) satisfies condition (13) for any function k. Hence assuming (12) we get equivalency of (13) and (14). For $p = 1$ condition (14) means the absolute homogeneity of f. Therefore by applying condition (13) we can weaken the definitions of a norm and a seminorm.

Corollary 3. *Let K, p_1, p_2, X, Y, k be as in Corollary 2. If a mapping $f : X \to Y$ satisfies the condition*

$$\|f(\alpha x) - \alpha f(x)\| \le |\alpha|^{p_1} k(x) \quad \text{for } \alpha \in K,\ x \in X\,,$$

then

$$f(\alpha x) = \alpha f(x) \quad \text{for } \alpha \in K,\ x \in X\,.$$

Proof. The proof runs similarly as that of Corollary 2.

Similarly as before, X and Y in Corollary 3 may be replaced by their subsets X_1 and Y_1 closed with respect to multiplication by scalars. Putting in Corollary 3 $p_1 = 1$, $p_2 = 0$, $k(x) = \varepsilon$ we obtain that each mapping $f : X \to Y$ satisfying (2) is homogeneous. This gives a positive answer to the question of K. Baron. Similarly, setting in Corollary 2 $p = 1$, $p_1 = 1$, $p_2 = 0$, $k(x) = \varepsilon$ we obtain an analogous result concerning absolute homogeneity. Inequality (2) may be interpreted as some kind of approximate homogeneity. Similarly the inequality

$$\|f(\alpha x) - |\alpha| f(x)\| \le \varepsilon|\alpha| \quad \text{for } \ \alpha \in \mathbb{R},\ x \in X$$

may be treated as an absolute approximate homogeneity. Then, according to the terminology introduced by R. Ger [1], our results mean that the equation of homogeneity as well as the absolute homogeneity are superstable.

2. Inequality (2) can be considered not only in a normed space but in a topological vector space as well. For this purpose we have to rewrite condition (2) in a different form. Denote

$$V := \{x \in X : \|x\| \le \varepsilon\}\,.$$

Then (2) can be written as

$$f(\alpha x) - \alpha f(x) \in \alpha V \quad \text{for } \ \alpha \in \mathbb{R},\ x \in X\,.$$

Now it is clear that the condition

$$f(\alpha x) - \alpha f(x) \in g(\alpha, x) V \quad \text{for } \ \alpha \in \mathbb{R},\ x \in X\,,$$

where $V \subset X$ and g maps $\mathbb{R} \times X$ into $\mathbb{R}$, generalizes condition (2).

We recall that a subset V of a topological vector space over K is called bounded if for each neighrbourhood U of zero there exists an $r \in K \setminus \{0\}$ such that $rV \subset U$ (cf. [2]).

Theorem 2. *Let X be a vector space over K, Y a topological vector space over K and let X_1 and Y_1 be subsets of X and Y, respectively, such that $KX_1 \subset X_1$ and $KY_1 \subset Y_1$. Let $V \subset Y$ be a bounded set and $g : K \times Y_1 \to K$ a mapping such that there exists a sequence α_n of non-zero elements of K such that*

$$\lim_{n\to\infty} g(\alpha\alpha_n, (\alpha_n)^{-1}x) = 0 \quad \text{for } \alpha \in K,\ x \in X_1 . \tag{15}$$

If a mapping $f : X_1 \to Y_1$ satisfies the condition

$$f(\alpha x) - \alpha f(x) \in g(\alpha, x)V \quad \text{for } \alpha \in K,\ x \in X_1 , \tag{16}$$

then

$$f(\alpha x) = \alpha f(x) \quad \text{for } \alpha \in K,\ x \in X_1 . \tag{17}$$

Proof. We have by (16)

$$\frac{f(\alpha_n x)}{\alpha_n} \in f(x) + \frac{g(\alpha_n, x)}{\alpha_n} V \quad \text{for } x \in X_1,\ n \in \mathbb{N} .$$

Thus

$$\frac{\alpha}{\alpha_n} f(\alpha_n x) \in \alpha f(x) + \frac{\alpha}{\alpha_n} g(\alpha_n, x)V \quad \text{for } \alpha \in K,\ x \in X_1,\ n \in \mathbb{N} .$$

This condition and (16) yield

$$\frac{\alpha}{\alpha_n} f(\alpha_n x) - f(\alpha x) \in \frac{\alpha}{\alpha_n} g(\alpha_n, x)V - g(\alpha, x)V \qquad \text{for } \alpha \in K,\ x \in X_1,\ n \in \mathbb{N} .$$

Now replacing α and x by $\alpha\alpha_n$ and $(\alpha_n)^{-1}x$ respectively, we obtain

$$\alpha f(x) - f(\alpha x) \in \alpha g(\alpha\alpha_n, (\alpha_n)^{-1}x)V - g(\alpha\alpha_n, (\alpha_n)^{-1}x)V \qquad \text{for } \alpha \in K,\ x \in X_1,\ n \in \mathbb{N} .$$

Since V is bounded, this condition and (15) imply (17).

Modifying slightly the proof of Theorem 2 we obtain the following topological analogue of Corollary 2:

Theorem 3. *Let $X, Y, X_1, Y_1, V, g, \alpha_n$ be as in Theorem 2 and let $p \in \mathbb{R}_+$. If a mapping $f : X_1 \to Y_1$ satisfies the condition*

$$f(\alpha x) - |\alpha|^p f(x) \in g(\alpha, x)V \quad \text{for } \alpha \in K,\ x \in X_1 ,$$

then

$$f(\alpha x) = |\alpha|^p f(x) \quad \text{for } \alpha \in K, x \in X_1 .$$

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