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On the counting function of Stanley sequences

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Abstract. For a finite sequence $A = \{a_1 < a_2 < \cdots < a_t\}$ of nonnegative integers which contains no 3-term arithmetic progression, the Stanley sequence S generated by A is defined as follows: for $k \geq t$, a_{k+1} is the least integer $a > a_k$ such that $\{a_1, a_2, \dots, a_k, a\}$ contains no 3-term arithmetic progression. Recently, Moy proved that $\liminf S(x)/\sqrt{x} \geq \sqrt{2}$, which solves a problem posed by Erdős et al., where $S(x)$ is the counting function of S. In this note we show that $\limsup S(x)/\sqrt{x} \geq 1.77$.

1. Introduction

Let $\mathbb{N}_0$ denote the set of nonnegative integers. For a finite set $A = \{a_1 < a_2 < \cdots < a_t\} \subset \mathbb{N}_0$ which contains no 3-term arithmetic progression, we denote by $S = \{a_1, a_2, \dots\}$ the sequence defined by the following recursion: if $k \geq t$ and $a_1, \dots, a_k$ have been defined, let a_{k+1} be the smallest integer $a > a_k$ such that $\{a_1, \dots, a_k\} \cup \{a\}$ contains no 3-term arithmetic progression. This sequence is called *the Stanley sequence* generated by A. Stanley sequence were considered, for instance, in [1]–[5].

Recently Moy [3] proved that for any $\varepsilon > 0$ and $x \geq x_0(\varepsilon, A)$,

$$S(x) \geq (\sqrt{2} - \varepsilon)\sqrt{x}.$$

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This solved a problem posed by Erdős *et al.* [1]. That means

$$\liminf_{x\to\infty} \frac{S(x)}{\sqrt{x}} \ge \sqrt{2}.$$

In this note we study

$$\limsup_{x\to\infty} \frac{S(x)}{\sqrt{x}}.$$

We have the following results:

Theorem 1. *For a given finite set $A \subset \mathbb{N}_0$ containing no 3-term arithmetic progression, let S be the Stanley sequence generated by A and $S(x)$ be its counting function. Then*

$$\limsup_{x\to\infty} \frac{S(x)}{\sqrt{x}} \ge 1/\sqrt{\tau} > 1.77,$$

where τ is the maximum value of

$$\frac{t}{2\sqrt{2}} \log\left(\sqrt{2t^2-1}+\sqrt{2t^2}\right) + \frac{1}{2}t^2\left(1-\sqrt{2t^2-1}\right)$$

on $[1/\sqrt{2}, 1]$.

We also pose the following problems.

Problem 1. *Is there any finite set $A \subset \mathbb{N}_0$ containing no* 3*-term arithmetic progression such that*

$$\limsup_{x\to\infty} \frac{S(x)}{\sqrt{x}} < +\infty?$$

Problem 2. *Is there any finite set $A \subset \mathbb{N}_0$ containing no* 3*-term arithmetic progression such that*

$$\liminf_{x\to\infty} \frac{S(x)}{\sqrt{x}} < +\infty?$$

2. Proof of the theorem

As in [3], define

$$H(S,n) = |\{(a_i,a_j) : i<j,\ n = 2a_j - a_i\}|.$$

Lemma 1 ([3, Lemma 2.3])**.** *We have*

$$\sum_{0\le n\le x} H(S,n) \ge x - S(x) - \max A.$$

Proof of Theorem 1. Let

$$\alpha = \liminf_{x\to\infty} \frac{S(x)}{\sqrt{x}}, \quad \beta = \limsup_{x\to\infty} \frac{S(x)}{\sqrt{x}}, \quad t = \frac{\alpha}{\beta}.$$

We assume that $\beta < 2$ as otherwise the assertion is immediate, and notice that then, in view of Moy's result, $t > \sqrt{2}/2$, whence $0 < 2t^2 - 1 \le 1$ which will be used later.

Fix $\varepsilon \in (0,1)$ and find a positive integer x_0 such that for any $x \ge x_0$ we have

$$(\alpha - \varepsilon)\sqrt{x} < S(x) < (\beta + \varepsilon)\sqrt{x}.$$

For the rest of the proof we assume that x runs over a strictly increasing sequence of positive integers such that

$$S(x) = (\alpha + o(1))\sqrt{x}.$$

The crucial observation (originating from Moy's paper) is that every sufficiently large integer is either in S or the largest term of a three-term arithmetic progression having its two smallest terms in S. Hence, the number of such progressions with the largest term not exceeding x is at least $x + o(x)$. Noticing that the smallest term of such a progression s and its second smallest term t satisfy $2t - s \le x$, we conclude that

$$\begin{aligned} x + o(x) \le \sum_{\substack{s,t\in S\\ s<t\le(x+s)/2}} 1 &= \sum_{\substack{s\in S\\ s\le x}} \left(S\left(\frac{x+s}{2}\right) - S(s)\right) \\ &= \sum_{\substack{s\in S\\ s\le x}} S\left(\frac{x+s}{2}\right) - \frac{1}{2}S(x)(S(x)-1) \\ &= \sum_{\substack{s\in S\\ s\le x}} S\left(\frac{x+s}{2}\right) - \frac{1}{2}\alpha^2 x + o(x). \end{aligned} \tag{1}$$

To estimate the sum in the right-hand side we use partial summation:

$$\begin{aligned} \sum_{\substack{s\in S\\ s\le x}} S\left(\frac{x+s}{2}\right) &= \sum_{s\le x} S\left(\frac{x+s}{2}\right)(S(s) - S(s-1)) \\ &= S(x)^2 - \sum_{s\le x-1} \left(S\left(\frac{x+s+1}{2}\right) - S\left(\frac{x+s}{2}\right)\right) S(s) \end{aligned}$$

$$
\begin{aligned}
&\le S(x)^2-(\alpha-\varepsilon)\sum_{x_0\le s\le x}\left(S\left(\frac{x+s+1}{2}\right)-S\left(\frac{x+s}{2}\right)\right)\sqrt{s}\\
&\le S(x)^2-(\alpha-\varepsilon)S(x)\sqrt{x}+(\alpha-\varepsilon)S((x+x_0)/2)\sqrt{x_0-1}\\
&\quad+(\alpha-\varepsilon)\sum_{x_0\le s\le x}S\left(\frac{x+s}{2}\right)(\sqrt{s}-\sqrt{s-1})\\
&=(\alpha-\varepsilon)\sum_{x_0\le s\le x}S\left(\frac{x+s}{2}\right)(\sqrt{s}-\sqrt{s-1})+O(\varepsilon x). \qquad (2)
\end{aligned}
$$

Now we split the last sum into two parts $s>\gamma x$ and $x_0\le s\le\gamma x$, where γ will be chosen later to obtain the optimal splitting point. The part of the last sum corresponding to $s>\gamma x$ can be estimated by

$$
\sum_{\gamma x<s\le x}S(x)\big(\sqrt{s}-\sqrt{s-1}\,\big)=S(x)\big(\sqrt{x}-\sqrt{[\gamma x]}\,\big)=\alpha\big(1-\sqrt{\gamma}\,\big)x+o(x). \qquad (3)
$$

For the remaining part of the sum, we have

$$
\begin{aligned}
&\sum_{x_0\le s\le\gamma x}S\left(\frac{x+s}{2}\right)\big(\sqrt{s}-\sqrt{s-1}\,\big)\le\frac{\beta+\varepsilon}{\sqrt{2}}\sum_{x_0\le s\le\gamma x}\sqrt{x+s}\left(\frac{1}{2\sqrt{s}}+O\left(\frac{1}{s}\right)\right)\\
&\quad\le\frac{\beta+\varepsilon}{2\sqrt{2}}\sum_{1\le s\le\gamma x}\sqrt{1+\frac{x}{s}}+o(x)\le\frac{\beta+\varepsilon}{2\sqrt{2}}\int_1^{\gamma x}\sqrt{1+\frac{x}{t}}dt+o(x)\\
&\quad\le\frac{\beta+\varepsilon}{2\sqrt{2}}x\int_0^{\gamma}\sqrt{1+\frac{1}{t}}dt+o(x)\\
&\quad\le\frac{\beta+\varepsilon}{2\sqrt{2}}\left(\sqrt{\gamma(\gamma+1)}+\log\big(\sqrt{\gamma}+\sqrt{\gamma+1}\,\big)\right)x+o(x). \qquad (4)
\end{aligned}
$$

From (1)–(4) we get

$$
(\alpha-\varepsilon)\left(\alpha\big(1-\sqrt{\gamma}\,\big)+\frac{\beta+\varepsilon}{2\sqrt{2}}\left(\sqrt{\gamma(\gamma+1)}+\log\big(\sqrt{\gamma}+\sqrt{\gamma+1}\,\big)\right)\right)-\frac{1}{2}\alpha^2+O(\varepsilon)\ge 1,
$$

and furthermore, since ε can be chosen arbitrarily small,

$$
\left(\frac{1}{2}-\sqrt{\gamma}\right)\alpha^2+\frac{\alpha\beta}{2\sqrt{2}}\left(\sqrt{\gamma(\gamma+1)}+\log\big(\sqrt{\gamma}+\sqrt{\gamma+1}\,\big)\right)\ge 1.
$$

Dividing through by β^2, we derive that

$$
\beta^{-2}\le\left(\frac{1}{2}-\sqrt{\gamma}\right)t^2+\frac{t}{2\sqrt{2}}\left(\sqrt{\gamma(\gamma+1)}+\log\big(\sqrt{\gamma}+\sqrt{\gamma+1}\,\big)\right).
$$

For a fixed $t \in (\sqrt{2}/2, 1]$, the right-hand side takes its minimal value at $\gamma = 2t^2 - 1$. So we choose $\gamma = 2t^2 - 1$. Then

$$\beta^{-2} \leq \frac{t}{2\sqrt{2}} \log\left(\sqrt{2t^2-1} + \sqrt{2t^2}\right) + \frac{1}{2}t^2\left(1 - \sqrt{2t^2-1}\right).$$

This establishes the estimate $\beta \geq 1/\sqrt{\tau}$, and numerical investigation shows that $\tau < 0.318214$, implying $1/\sqrt{\tau} > 1.77$.

This completes the proof of Theorem 1. □

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