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# A finiteness condition for verbal conjugacy classes in a group

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**Abstract.** Given a group $G$ and a word $w$, we denote by $G_w$ the set of all $w$-values in $G$ and by $w(G)$ the corresponding verbal subgroup. The main result of the paper is the following theorem. Let $k$ be a positive integer and let $w$ be either the word $\gamma_k$ or the word $\delta_k$. Suppose that $G$ is a group in which $\langle x^{G_w}\rangle$ is Chernikov for all $x \in G$. Then $\langle x^{w(G)}\rangle$ is Chernikov for all $x \in G$ as well.

## 1. Introduction

Let $w$ be a word in $n$ variables, and let $G$ be a group. The verbal subgroup $w(G)$ of $G$ determined by $w$ is the subgroup generated by the set $G_w$ consisting of all values $w(g_1, \dots, g_n)$, where $g_1, \dots, g_n$ are elements of $G$. A word $w$ is said to be concise if whenever $G_w$ is finite for a group $G$, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise, but it was later proved that this problem has a negative solution in its general form (see [4], p. 439). On the other hand, many relevant words are known to be concise. For instance, Turner-Smith [7] showed that the lower central words $\gamma_k$ and the derived words $\delta_k$ are concise; here the words $\gamma_k$ and $\delta_k$ are defined by the positions $\gamma_1 = \delta_0 = x$, $\gamma_{k+1} = [\gamma_k, \gamma_1]$ and $\delta_{k+1} = [\delta_k, \delta_k]$. The corresponding verbal subgroups for these words are the familiar $k$th term of the lower central series of $G$ denoted by $\gamma_k(G)$ and the $k$th derived group of $G$ denoted by $G^{(k)}$.

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There are several natural ways to look at Hall's question from a different angle. The circle of problems arising in this context can be characterized as follows.

Given a word $w$ and a group $G$, assume that certain restrictions are imposed on the set $G_w$. How does this influence the properties of the verbal subgroup $w(G)$?

If $X$ and $Y$ are non-empty subsets of a group $G$, we will write $X^Y$ to denote the set $\{y^{-1}xy \mid x \in X,\ y \in Y\}$. In [2] groups $G$ with the property that $x^{G_w}$ is finite for all $x \in G$ were called $FC(w)$-groups. Recall that $FC$-groups are precisely groups with finite conjugacy classes. The main result of [2] tells us that if $w$ is a concise word, then a group $G$ is an $FC(w)$-group if and only if $x^{w(G)}$ is finite for all $x \in G$. In particular, it follows that if $w$ is a concise word and $G$ is an $FC(w)$-group, then the verbal subgroup $w(G)$ is $FC$. Later it was shown in [1] that there exists a function $f = f(m, w)$ such that if, under the hypothesis of the above theorem, $x^{G_w}$ has at most $m$ elements for all $x \in G$, then $x^{w(G)}$ has at most $f$ elements for all $x \in G$. In view of these results we would like to consider the following question.

Given a concise word $w$ and a group $G$, assume that for all $x \in G$ the subgroup $\langle x^{G_w} \rangle$ satisfies a certain finiteness condition. Is it true that a similar condition is also satisfied by $\langle x^{w(G)} \rangle$ for all $x \in G$?

Here and throughout the paper $\langle M \rangle$ denotes the subgroup generated by the set $M$. The main result of the present paper is as follows.

**Theorem 1.1.** *Let $k$ be a positive integer and let $w$ be either the word $\gamma_k$ or the word $\delta_k$. Suppose that $G$ is a group in which $\langle x^{G_w} \rangle$ is Chernikov for all $x \in G$. Then $\langle x^{w(G)} \rangle$ is Chernikov for all $x \in G$ as well.*

Recall that a group $G$ is Chernikov if it has a subgroup of finite index that is a direct product of finitely many groups of type $C_{p^\infty}$ for various primes $p$ (quasicyclic $p$-groups). By a deep result obtained independently by Shunkov [6] and Kegel and Wehrfritz [3] Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup. The minimal subgroup of finite index of a Chernikov group $G$ is called the radicable part of $G$. In general a group $G$ is called radicable if the equation $x^n = a$ has a solution in $G$ for every positive integer $n$ and every $a \in G$. It is well-known that a periodic abelian radicable group is a direct product of quasicyclic $p$-subgroups.

A proof of Theorem 1.1 in the case where $w = \gamma_k$ can be obtained from the case $w = \delta_k$ by simply replacing everywhere in the proof the term "$\delta_k$-commutators" by "$\gamma_k$-commutators". That is why we do not provide an explicit proof for the case $w = \gamma_k$ concentrating instead on proving Theorem 1.1 in the case $w = \delta_k$.

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## 2. Preliminary lemmas

We start the section with the following well-known lemma (see for example [5, Lemma 3.13]).

**Lemma 2.1.** *Suppose that $R$ is a radicable abelian normal subgroup of the group $G$ and suppose that $H$ is a subgroup of $G$ such that $[R, \underbrace{H, \dots, H}_{r}] = 1$ for some natural number $r$. If $H/H'$ is periodic, then $[R, H] = 1$.*

From this we can easily deduce the following useful corollaries.

**Corollary 2.2.** *In a periodic nilpotent group $G$ every radicable abelian subgroup $Q$ is central.*

Proof. Arguing by induction on the nilpotency class of $G$ we assume that the image of $Q$ in $G/Z(G)$ is central. Therefore $Q$ is contained in a normal abelian subgroup of $G$. In particular $\langle Q^G \rangle$ is a normal abelian radicable subgroup and the result is now immediate from Lemma 2.1. □

Let $G$ be a group acted on by a group $A$. As usual, $[G, A]$ denotes the subgroup generated by all elements of the form $x^{-1}x^a$, where $x \in G$, $a \in A$. It is well-known that $[G, A]$ is a normal subgroup of $G$. If $B$ is a normal subset of $A$ such that $A = \langle B \rangle$, then $[G, A] = \langle [G, b];\ b \in B \rangle$.

**Corollary 2.3.** *Let $A$ be a periodic group acting on a periodic radicable abelian group $G$. Then $[G, A, A] = [G, A]$.*

Proof. To show this, we can assume that $[G, A, A] = 1$. In this case Lemma 2.1 yields at once that $[G, A] = 1$. □

**Lemma 2.4.** *Let $A$ be a finite group acting on a periodic radicable abelian group $G$. Then $[G, A]$ is radicable.*

Proof. Since $[G, A] = \prod_{a\in A}[G, a]$, it is sufficient to show that $[G, a]$ is radicable for every $a \in A$. Let $x \in [G, a]$ and let $n$ be a positive integer. Then there exist $g \in G$ such that $x = [g, a]$ and $g_1 \in G$ such that $g_1^n = g$. Since $G$ is abelian, we have $[g_1, a]^n = [g_1^n, a]$. Hence for every $x \in [G, a]$ and every positive integer $n$, there exists an element $[g_1, a] \in [G, a]$ such that $x = [g_1, a]^n$; that is, $[G, a]$ is radicable, as required. □

**Lemma 2.5.** *Let $A$ be a radicable Chernikov group acting on a Chernikov group $B$. Then $[B, A, A] = 1$.*

Proof. Denote by $B_0$ the radicable part of $B$. By [5, Theorem 3.29.2], $A/C_A(B_0)$ is finite. Since $A$ is radicable, it follows that $A$ has no subgroups of finite index and so $[B_0, A] = 1$. On the other hand, $B/B_0$ is finite and therefore $A/C_A(B/B_0)$ is also finite. Again, since $A$ $A$ has no subgroups of finite index, it follows that $[B, A] \leq B_0$. Hence $[B, A, A] \leq [B_0, A] = 1$. □

**Lemma 2.6.** *Let $G$ be a group and $y$ an element of $G$. Suppose that $x_1, \dots, x_k \in G$ are $\delta_k$-commutators for $k \geq 0$. Then $[y, x_1, \dots, x_k]$ is a $\delta_k$-commutator as well.*

Proof. Note that $x_1, \dots, x_k$ can be viewed as $\delta_i$-commutators for each $i \leq k$. It is clear that $[y, x_1]$ is a $\delta_1$-commutator. Arguing by induction on $k$ assume that $k \geq 1$ and $[y, x_1, \dots, x_{k-1}]$ is a $\delta_{k-1}$-commutator. Then $[y, x_1, \dots, x_k] = [[y, x_1, \dots, x_{k-1}], x_k]$ is a $\delta_k$-commutator. □

Throughout the paper, whenever $G$ is a Chernikov group we denote by $G_0$ the radicable part of $G$ and by $G^*$ the subgroup $[G_0, G]$.

**Lemma 2.7.** *Let $G$ be a Chernikov group for which there exists a positive integer $m$ such that $G$ can be generated by elements of order dividing $m$. If $G^* = 1$, then $G$ is finite.*

Proof. Since $G^* = 1$, it follows that $G_0$ is central. The Schur Theorem [5, Theorem 4.12] yields that $G'$ is finite. Since $G$ can be generated by elements of order dividing $m$, we conclude that $G$ has finite exponent. In particular $G$ has no subgroups of type $C_{p^\infty}$. Thus, $G$ must be finite. □

**Lemma 2.8.** *Let $G$ be a group such that $\langle x^G \rangle$ is Chernikov for every $x \in G$. Then all abelian radicable subgroups of $G$ generate an abelian radicable subgroup.*

Proof. Let $T$ be the subgroup of $G$ generated by all abelian radicable subgroups. Let $A$ be an arbitrary abelian radicable subgroup in $G$ and choose $x \in G$. Then Lemma 2.1 together with Lemma 2.5 shows that the product $\langle x^G \rangle_0 A$ is abelian. Thus, all subgroups of the form $\langle x^G \rangle_0$ lie in the center of $T$. Therefore $G/Z(T)$ is a periodic $FC$-group. Since $T$ has no subgroups of finite index, it centralizes every finite normal subgroup and we conclude that the image of $T$ in $G/Z(T)$ is central. Therefore $T$ is nilpotent of class at most two. Corollary 2.2 now enables us to deduce that $T$ is abelian, as required. □

We will also require the following lemma.

**Lemma 2.9.** *Let $X$ be a normal set in a locally finite group $G$. Let $a \in G$ and assume that the set $a^X$ is finite. Then the set $a^{\langle X \rangle}$ is likewise finite.*

Proof. Let $x_1, \dots, x_n$ be elements of $X$ with the property that $a^X = \{a^{x_1}, \dots, a^{x_n}\}$ and let $Y = \langle x_1, \dots, x_n \rangle$. Since $Y$ is finite, the class $a^Y$ is finite as well. Let $N = \langle X \rangle$. We will show that $a^N = a^Y$. Choose $y \in N$. Then $y$ can be written as a product $y = y_1 \dots y_m$, where $y_i \in X$. It is sufficient to show that $a^y \in a^Y$. If $m = 1$, then $y \in X$ and so $a^y \in \{a^{x_1}, \dots, a^{x_n}\} \subseteq a^Y$. Thus, assume that $m \geq 2$ and use induction on $m$. Suppose that $a^{y_1} = a^{x_1}$. Set $z_i = x_1 y_i {x_1}^{-1}$ for $i = 2, \dots, m$. Since $X$ is a normal set of $G$, $z_i \in X$. Write

$$a^y = a^{x_1 y_2 \cdots y_m} = a^{x_1 y_2 \cdots y_m {x_1}^{-1} x_1} = a^{z_2 \cdots z_m x_1}.$$

By induction $a^{z_2 \cdots z_m} \in a^Y$. Since $x_1 \in Y$, it follows that $a^y \in a^Y$. This completes the proof. □

## 3. Proof of Theorem 1.1

Assume the hypothesis of Theorem 1.1 with $w = \delta_k$ and let $X$ denote the set of all $\delta_k$-commutators in $G$. By the hypothesis $\langle a^X \rangle$ is Chernikov for all $a \in G$. Set $H = G^{(k)}$. We wish to show that $\langle a^H \rangle$ is Chernikov for all $a \in G$. First we will deal with the particular case where $a \in X$. Thus, choose $a \in X$ and let $D = \langle a^X \rangle$. As usual, the normal closure of a subset $S \subseteq G$ is the minimal normal subgroup of $G$ containing $S$.

**Lemma 3.1.** *With the above notation, the normal closure of $D^*$ in $G$ is an abelian radicable subgroup.*

Proof. By Corollary 2.3 $D^* = [D_0, \underbrace{D, \dots, D}_{k}]$. Since $a \in X$, every element of $a^X$ is also a $\delta_k$-commutator. It follows that $D$ is generated by the normal set $X \cap D$. Therefore the subgroup $D^*$ is generated by subgroups of the form $[D_0, b_1, \dots, b_k]$, where $b_1, \dots, b_k \in X \cap D$.

Let us show that for every choice of $b_1, \dots, b_k \in X \cap D$ the subgroup $[D_0, b_1, \dots, b_k]$ is contained in a normal abelian radicable subgroup of $G$. Thus, fix $b_1, \dots, b_k \in X \cap D$ and put $K = [D_0, b_1, \dots, b_k]$. Since $D_0$ is abelian, it is clear that for every $d_1, d_2 \in D_0$ we have

$$[d_1, b_1, \dots, b_k][d_2, b_1, \dots, b_k] = [d_1 d_2, b_1, \dots, b_k].$$

Now, Lemma 2.6 shows that every element of $K$ is a $\delta_k$-commutator and Lemma 2.4 yields that $K$ is radicable. Since $\langle g^K \rangle$ is Chernikov for every $g \in G$, it follows from Lemma 2.5 that $[g^K, K, K] = 1$. In particular $[g, K, K] = 1$ and we conclude that $K$ commutes with $K^g$ for every $g \in G$. Therefore $\langle K^G \rangle$ is abelian. Since $\langle K^G \rangle$ is generated by radicable subgroups, it follows that $\langle K^G \rangle$ is radicable.

Now choose other elements $b'_1, \dots, b'_k \in X \cap D$ and set $K_1 = [D_0, b'_1, \dots, b'_k]$. Repeating the above argument we conclude that $\langle K_1^G \rangle$ is abelian and radicable. Thus, the product $\langle K^G \rangle \langle K_1^G \rangle$ is nilpotent of class at most two and Corollary 2.2 tells us that $\langle K^G \rangle \langle K_1^G \rangle$ is abelian. Thus, all subgroups of the form $\langle [D_0, x_1, \dots, x_k]^G \rangle$, where $x_1, \dots, x_k \in X \cap D$, commute and the lemma follows. □

Set $R = \langle \langle y^X \rangle^*;\ y \in X \rangle$. This notation will be kept throughout the rest of the paper.

**Corollary 3.2.** *The subgroup $R$ is abelian and radicable.*

Proof. Choose $y_1, y_2 \in X$. Let $R_1$ be the normal closure of $\langle {y_1}^X \rangle^*$ and $R_2$ that of $\langle {y_2}^X \rangle^*$. By Lemma 3.1 both $R_1$ and $R_2$ are abelian radicable subgroups. We conclude that the product $R_1 R_2$ is nilpotent of class at most two and Corollary 2.2 shows that $R_1 R_2$ is abelian. The result follows. □

In the next lemma we use terminology and some results from the paper [2]. For the reader's convenience we will briefly explain it. Let $w$ be a word, $G$ a group and $H$ a subgroup of $w(G)$. We say that $H$ has finite $w$-index if the elements of $G_w$ lie in finitely many right cosets of $H$ in $w(G)$. A group $G$ is an $FC(w)$-group if and only if the subgroup $C_{w(G)}(x)$ has finite $w$-index for every element $x$ of $G$.

**Lemma 3.3.** *The group $G$ is locally finite.*

Proof. First of all we notice that $G$ is torsion since $\langle y^X \rangle$ is torsion for every $y \in G$. Since $R$ is abelian (Corollary 3.2), it is sufficient to prove the local finiteness of $G$ under the assumption that $R = 1$. Choose $y \in X$. By the above assumption $\langle y^X \rangle^* = 1$. Since $\langle y^X \rangle$ is generated by conjugates of $y$, it follows from Lemma 2.7 that $\langle y^X \rangle$ is finite. This implies that $C_G(y) \cap H$ has finite $\delta_k$-index.

This happens for every choice of $y \in X$. Since every element of $H$ is a product of finitely many $\delta_k$-commutators, [2, Lemma 2.1] shows that $H$ is an $FC(\delta_k)$-group. In particular, the main result of [2] tells us that the $k$th derived group of $H$ is an $FC$-group. Now the local finiteness of $G$ is obvious. □

**Lemma 3.4.** *The subgroup* $\langle a^H \rangle$ *is Chernikov.*

Proof. Recall that $a \in X$ and $D = \langle a^X \rangle$. Set $E = \langle a^H \rangle$. We wish to show that $E$ is Chernikov. Let us show first that $ER/R$ is finite. It suffices to show this under the additional assumption that $R = 1$. In this case $D^* = 1$ and so $D$ is finite by Lemma 2.7. In particular $a^X$ is finite and since $G$ is locally finite, we use Lemma 2.9 to conclude that $E$ is finite. Thus, indeed $ER/R$ is finite. Set $R_1 = E \cap R$. Choose elements $e_1, \dots, e_s \in X$ such that every conjugate of $a$ in $H$ belong to a coset $e_i R_1$ for some $i = 1, \dots, s$. We have $E = \langle R_1, e_1, \dots, e_s \rangle$. Since the set $e_1, \dots, e_s$ generates $E$ and is normal in $H$ modulo $R_1$, it follows that

$$[R_1, E] = [R_1, e_1] \dots [R_1, e_s].$$

By Lemma 2.4 $[R_1, e_i] = [R_1, e_i, e_i]$ and Lemma 2.6 shows that $[R_1, e_i] \subseteq X$. We conclude that $[R_1, e_i] = [R_1, e_i, e_i] \le [X, e_i] \le \langle {e_i}^X \rangle$ and so the subgroups $[R_1, e_i]$ are Chernikov for every $i = 1, \dots, s$. Therefore $[R_1, E]$ is Chernikov and we can pass to the quotient $H\langle a \rangle / [R_1, E]$. Without loss of generality we assume that $[R_1, E] = 1$. In this case, $R_1 \le Z(E)$ and $E/Z(E)$ is finitely generated. Lemma 3.3 shows that $E/Z(E)$ is finite. The Schur Theorem now tells us that the derived group $E'$ is finite. We see that $D$ is a Chernikov group generated by elements of the same order and its derived group $D'$ is finite. It follows that $D$ is finite. Now Lemma 2.9 enables us to deduce that $E$ is finite. This completes the proof. □

**Corollary 3.5.** *If* $g \in H$, *then* $\langle g^H \rangle$ *is Chernikov.*

Proof. This follows directly from Lemma 3.4 and the fact that every element of $H$ is a product of finitely many elements from $X$. □

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Combining Corollary 3.5 with Lemma 2.8 we deduce that all abelian radicable subgroups of $H$ generate an abelian radicable subgroup. This will be denoted by $T$.

To complete the proof of Theorem 1.1 we need to show that $\langle b^H \rangle$ is Chernikov for every $b \in G$. Thus, let $b \in G$. Set $B = \langle b^X \rangle$ and $C = \langle b^H \rangle$. By the hypothesis, $B$ is Chernikov. Since $T$ contains all the abelian radicable subgroups of $H$, the image of $B$ in $H\langle b\rangle/T$ is finite. Therefore Lemma 2.9 shows that also the image of $C$ is finite. Let us define

$$S = \langle [T, b_1, \dots, b_k] \mid b_i \in X\rangle.$$

For every choice $b_1, \dots, b_k \in X$ the subgroup $[T, b_1, \dots, b_k]$ is a radicable subgroup (Lemma 2.4) contained in $X$ (Lemma 2.6). Thus, $S$ is a normal radicable subgroup of $G$. Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be the list of the radicable subgroups contained in $S \cap X$. Then $S = \langle S_\lambda \mid \lambda \in \Lambda\rangle$. Since $S_\lambda \subseteq X$, we have $[S_\lambda, b] \leq [X, b] \leq B$ for every $\lambda$ and so we deduce that $[S, b] \leq B$. In particular, $[S, x]$ is Chernikov for every $x \in G$. Set $T_1 = C \cap T$. Now choose in $C$ finitely many conjugates of $b$, say $c_1, \dots, c_n$, such that $C = \langle T_1, c_1, \dots, c_n\rangle$ and the set $c_1T_1, \dots, c_nT_1$ is normal in $C/T_1$. Then $[S, C] = [S, c_1] \dots [S, c_n]$. Since every subgroup $[S, c_i]$ is Chernikov, so is $[S, C]$. Moreover the subgroup $[S, C]$ is normal in $H\langle b\rangle$ and so we can consider the quotient $H\langle b\rangle/[S, C]$. Thus, we assume that $[S, C] = 1$.

Suppose temporarily that $S = 1$. Then $T$ is contained in the $k$th term of the upper central series of $H$ and Lemma 2.1 shows that actually $T \leq Z(H)$. In this case $B_0$, the radicable part of $B$, is normal in $H\langle b\rangle$ and so we can consider the quotient $H\langle b\rangle/B_0$. The image of $B$ in the quotient is finite. By Lemma 2.9 the image of $C$ must be finite as well. This proves the theorem in the particular case where $S = 1$.

Let us now drop the assumption that $S = 1$. The above argument shows that the image of $C$ in $H\langle b\rangle/S$ is Chernikov. Taking into account that $[S, C] = 1$ we deduce that $C/Z(C)$ is Chernikov. Polovickii's Theorem [5, p. 129] now tells us that $C'$, the derived group of $C$, is Chernikov and we can pass to the quotient $H\langle b\rangle/C'$. Thus, we assume that $C' = 1$. Now $C$ is an abelian group generated by elements of the same order (namely, of order equal to that of $b$). We deduce that $C$ has finite exponent. Taking into account that $B \leq C$ we observe that $B$ is a Chernikov group of finite exponent. Hence, $B$ is finite. But then by Lemma 2.9, $C$ must be finite as well. The proof is now complete. □

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