Publ. Math. Debrecen
82/1 (2013), 107–123
DOI: 10.5486/PMD.2013.5341

On the mod p^2 determination of $\sum_{k=1}^{p-1} H_k/(k\cdot 2^k)$: another proof of a conjecture by Sun

By ROMEO MEŠTROVIĆ (Kotor)

Abstract. For a positive integer n let $H_n=\sum_{k=1}^{n} 1/k$ be the nth harmonic number. Z. W. Sun conjectured that for any prime $p\geq 5$,

$$\sum_{k=1}^{p-1}\frac{H_k}{k\cdot 2^k}\equiv\frac{7}{24}pB_{p-3}\pmod{p^2}.$$

This conjecture is recently confirmed by Z. W. Sun and L. L. Zhao. In this note we give another proof of the above congruence by establishing congruences for all the sums of the form $\sum_{k=1}^{p-1} 2^{\pm k}H_k^r/k^s \pmod{p^{4-r-s}}$ with $(r,s)\in\{(1,1),(1,2),(2,1)\}$.

1. The main results

Given positive integers n and m, the *harmonic numbers of order* m are those rational numbers $H_{n,m}$ defined as

$$H_{n,m}=\sum_{k=1}^{n}\frac{1}{k^m}.$$

For simplicity, we will denote by

$$H_n:=H_{n,1}=\sum_{k=1}^{n}\frac{1}{k}$$

the nth *harmonic number* (in addition, we define $H_0=0$).

Mathematics Subject Classification: Primary: 11B75; Secondary: 11A07, 11B68, 05A19, 05A10.
Key words and phrases: harmonic numbers of order m, congruence, Bernoulli numbers.

Recently, Z. W. Sun [9] obtained basic congruences modulo a prime $p \geq 5$ for several sums of terms involving harmonic numbers. In particular, Sun established $\sum_{k=1}^{p-1} H_k^r \pmod{p^{4-r}}$ for $r = 1, 2, 3$. Further generalizations of these congruences have been recently obtained by R. Tauraso in [13]. More recently, Z. W. Sun [10] initiated and studied congruences involving both harmonic and Lucas sequences (especially, including Fibonacci numbers or Lucas numbers).

Recall that Bernoulli numbers $B_0, B_1, B_2, \ldots$ are given by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad (n = 1, 2, 3, \ldots).$$

In this note we establish six congruences involving harmonic numbers contained in the following result.

Theorem 1.1. *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{2^k H_k}{k} \equiv -q_p(2)^2 + \frac{2}{3} p q_p(2)^3 + \frac{p}{12} B_{p-3} \pmod{p^2}, \tag{1}$$

$$\sum_{k=1}^{p-1} \frac{2^k H_k}{k^2} \equiv -\frac{1}{3} q_p(2)^3 + \frac{23}{24} B_{p-3} \pmod{p}, \tag{2}$$

$$\sum_{k=1}^{p-1} \frac{H_k}{k^2 \cdot 2^k} \equiv \frac{5}{8} B_{p-3} \pmod{p}, \tag{3}$$

$$\sum_{k=1}^{p-1} \frac{2^k H_k^2}{k} \equiv -\frac{1}{3} q_p(2)^3 + \frac{11}{24} B_{p-3} \pmod{p}, \tag{4}$$

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k \cdot 2^k} \equiv \frac{7}{8} B_{p-3} \pmod{p} \tag{5}$$

and

$$\sum_{k=1}^{p-1} \frac{2^k H_{k,2}}{k} \equiv -\frac{1}{3} q_p(2)^3 - \frac{25}{24} B_{p-3} \pmod{p}. \tag{6}$$

As an application, we obtain a result obtained quite recently by Z. W. Sun and L. L. Zhao in [11].

Corollary 1.2 ([11, Theorem 1.1])**.** *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{H_k}{k \cdot 2^k} \equiv \frac{7}{24} p B_{p-3} \pmod{p^2} \tag{7}$$

and

$$\sum_{k=1}^{p-1} \frac{H_{k,2}}{k \cdot 2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p}. \tag{8}$$

Remark 1.1. The congruence (7) is conjectured by Z. W. Sun in [9, Conjecture 1.1] and quite recently proved by Z. W. Sun and L. L. Zhao in [11]. We point out that Lemma 2.3 from [11] presents the main auxiliary result in the proof of (7) and its proof is based on a polynomial congruence recently obtained by L. L. Zhao and Z. W. Sun in [14, Theorem 1.2]. Moreover, in this proof the authors also use the congruence $\sum_{k=1}^{p-1} H_k/k^2 \equiv B_{p-3} \pmod{p}$ obtained by Sun and Tauraso in [12, the congruence (5.4)].

Notice also that the first congruence in [9, Conjecture 1.1] is also proved by the author of this note in [6, Theorem 1.1 (1.3)].

Reducing the modulus in (1) we have

Corollary 1.3. *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{2^k H_k}{k} \equiv -q_p(2)^2 \pmod{p}. \tag{9}$$

This paper is organized as follows. In the next section, using numerous classical and recent combinatorial congruences, we prove the congruence (1). Applying (1) and some auxiliary results, in Section 3 we establish the congruences (2) and (3). Section 4 is devoted to the proof of the congruences (4), (5) and (6) based on the previous congruences and an identity for harmonic numbers. As an application, in Section 5 we prove Corollary 1.2 which contains two congruences recently obtained by Z. W. Sun and L. L. Zhao in [11].

2. Proof of the congruence (1)

The following result is well known and elementary.

Lemma 2.1 (see e.g., [9, Lemma 2.1(2.2)])**.** *If $p \geq 3$ is a prime, then*

$$\binom{p-1}{k} \equiv (-1)^k - (-1)^k pH_k + (-1)^k \frac{p^2}{2}(H_k^2 - H_{k,2}) \pmod{p^3} \tag{10}$$

for each $k = 1, 2, \dots, p-1$. In particular, we have

$$\binom{p-1}{k} \equiv (-1)^k - (-1)^k pH_k \pmod{p^2}. \tag{11}$$

Lemma 2.2. *If $p>3$ is a prime, then*

$$H_{p-1}:=\sum_{k=1}^{p-1}\frac{1}{k}\equiv -\frac{p^2}{3}B_{p-3}\pmod{p^3},\tag{12}$$

$$H_{p-1,2}:=\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv \frac{2p}{3}B_{p-3}\pmod{p^2},\tag{13}$$

$$H_{p-1,3}:=\sum_{k=1}^{p-1}\frac{1}{k^3}\equiv 0\pmod{p^2},\tag{14}$$

$$\begin{aligned}H_{(p-1)/2}:=\sum_{k=1}^{(p-1)/2}\frac{1}{k}&\equiv -2q_2(p)+pq_2(p)^2\\ &-\frac{2p^2}{3}q_2(p)^3-\frac{7p^2}{12}B_{p-3}\pmod{p^3},\end{aligned}\tag{15}$$

$$H_{(p-1)/2,2}:=\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv \frac{7p}{3}B_{p-3}\pmod{p^2}\tag{16}$$

and

$$H_{(p-1)/2,3}:=\sum_{k=1}^{(p-1)/2}\frac{1}{k^3}\equiv -2B_{p-3}\pmod{p}.\tag{17}$$

Proof. The congruence (12) is proved in [5]; see also [7, Theorem 5.1(a)], while (13) is a particular case of [7, Corollary 5.1]. The well known congruence (14) is a particular case of [2, Theorem 3 (b)] and (15) is in fact the congruence (c) in [7, Theorem 5.2]. Further, the congruences (16) and (17) are the congruences (a) with $k=2$ and (b) with $k=3$ in [7, Corollary 5.2], respectively. □

We will also need the following six congruences recently established by Z. H. Sun [8] and Dilcher and Skula [3].

Lemma 2.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv -2q_p(2)-\frac{7p^2}{12}B_{p-3}\pmod{p^3},\tag{18}$$

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2}\equiv -q_p(2)^2+p\left(\frac{2}{3}q_p(2)^3+\frac{7}{6}B_{p-3}\right)\pmod{p^2},\tag{19}$$

$$\sum_{k=1}^{p-1} \frac{1}{k\cdot 2^k} \equiv q_p(2) - \frac{p}{2} q_p(2)^2 \pmod{p^2}, \tag{20}$$

$$\sum_{k=1}^{p-1} \frac{1}{k^2\cdot 2^k} \equiv -\frac{1}{2} q_p(2)^2 \pmod{p}, \tag{21}$$

$$\sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3} q_p(2)^3 - \frac{7}{24} B_{p-3} \pmod{p} \tag{22}$$

and

$$\sum_{k=1}^{p-1} \frac{1}{k^3\cdot 2^k} \equiv \frac{1}{6} q_p(2)^3 + \frac{7}{48} B_{p-3} \pmod{p}. \tag{23}$$

Proof. The congruences (18)–(21) are in fact the congruences (i)–(iv) in [8, Theorem 4.1]. By the congruence (5) in [3, Theorem 1],

$$\sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3} q_p(2)^3 - \frac{7}{48} \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \pmod{p},$$

whence inserting (17), we immediately obtain (22). Finally, (23) follows immediately from (22) by applying the substitution trick $k \mapsto p-k$ and the fact that $2^p \equiv 2 \pmod{p}$ by Fermat little theorem. □

Lemma 2.4. *Let n be an arbitrary positive integer. Then*

$$\sum_{1\le k\le i\le n} \frac{2^k-1}{ki} = \sum_{j=1}^{n} \frac{1}{j^2}\binom{n}{j}. \tag{24}$$

Proof. Using the well known identities $\sum_{k=j}^{i}\binom{k-1}{j-1} = \binom{i}{j}$ with $j \le i$ (see e.g., [10, Lemma 2.1]), $\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{k-1}{j-1}$ with $j \le k$, and the fact that $\binom{k}{j} = 0$ when $k < j$, we have

$$\begin{aligned}
\sum_{1\le k\le i\le n} \frac{2^k-1}{ki} &= \sum_{1\le k\le i\le n} \frac{(1+1)^k-1}{ki} = \sum_{1\le k\le i\le n} \frac{1}{i}\sum_{j=1}^{k}\frac{1}{k}\binom{k}{j} \\
&= \sum_{1\le k\le i\le n} \frac{1}{i}\sum_{j=1}^{n}\frac{1}{j}\binom{k-1}{j-1} = \sum_{j=1}^{n}\frac{1}{j}\sum_{1\le k\le i\le n}\frac{1}{i}\binom{k-1}{j-1} \\
&= \sum_{j=1}^{n}\frac{1}{j}\sum_{j\le k\le i\le n}\frac{1}{i}\binom{k-1}{j-1} = \sum_{j=1}^{n}\frac{1}{j}\sum_{i=j}^{n}\frac{1}{i}\sum_{k=j}^{i}\binom{k-1}{j-1}
\end{aligned}$$

$$= \sum_{j=1}^{n} \frac{1}{j} \sum_{i=j}^{n} \frac{1}{i} \binom{i}{j} = \sum_{j=1}^{n} \frac{1}{j} \sum_{i=j}^{n} \frac{1}{j} \binom{i-1}{j-1}$$
$$= \sum_{j=1}^{n} \frac{1}{j^2} \sum_{i=j}^{n} \binom{i-1}{j-1} = \sum_{j=1}^{n} \frac{1}{j^2} \binom{n}{j},$$

as desired. □

Lemma 2.5. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \binom{p-1}{k} \equiv \frac{3p}{4} B_{p-3} \pmod{p^2}. \tag{25}$$

Proof. By the congruence (11) of Lemma 2.1, we have

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \binom{p-1}{k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} (1 - pH_k) \pmod{p^2}$$
$$= \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} - p \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k^2} \pmod{p^2}. \tag{26}$$

Using (13) and (16) of Lemma 2.2, we have

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} = 2 \sum_{\substack{1 \le j \le p-1 \\ 2 \mid j}} \frac{1}{j^2} - \sum_{k=1}^{p-1} \frac{1}{k^2}$$
$$= \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k^2} - \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{p}{2} B_{p-3} \pmod{p^2}. \tag{27}$$

Similarly, by (14) and (17) of Lemma 2.2 we find that

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3} \equiv -\frac{1}{2} B_{p-3} \pmod{p}. \tag{28}$$

Since $p \mid H_{p-1}$, it follows that for each $k = 1, 2, \ldots, p-1$,

$$H_k = H_{p-1} - \sum_{i=1}^{p-k-1} \frac{1}{p-i} \equiv \sum_{i=1}^{p-k-1} \frac{1}{i} = H_{p-k-1} \pmod{p}. \tag{29}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^kH_k}{k^2}&=\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\left(H_{k-1}+\frac{1}{k}\right)\\&=\sum_{k=1}^{p-1}\frac{(-1)^kH_{k-1}}{k^2}+\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}=\sum_{k=1}^{p-1}\frac{(-1)^{p-k}H_{p-k-1}}{(p-k)^2}+\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\\&\equiv-\sum_{k=1}^{p-1}\frac{(-1)^kH_k}{k^2}+\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\pmod p\end{aligned}$$

from which taking (28) we have

$$\sum_{k=1}^{p-1}\frac{(-1)^kH_k}{k^2}\equiv\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3}\equiv-\frac{1}{4}B_{p-3}\pmod{p}. \tag{30}$$

Finally, substituting (27) and (30) into (26) we obtain (25). □

Proof of the congruence (1). The identity (24) of Lemma 2.4 with $n=p-1$ may be written as

$$\sum_{1\le k<i\le p-1}\frac{2^k}{ki}+\sum_{k=1}^{p-1}\frac{2^k}{k^2}-\sum_{1\le k\le i\le p-1}\frac{1}{ki}=\sum_{k=1}^{p-1}\frac{1}{k^2}\binom{p-1}{k}. \tag{31}$$

Further, from (12) of Lemma 2.2 we see that $H_{p-1}\equiv 0\,(\bmod\ p^2)$ (the well known Wolstenholme's theorem [1] or [4]), and thus for each $k=0,1,2,\ldots,p-2$,

$$\sum_{i=k+1}^{p-1}\frac{1}{i}\equiv-\sum_{i=1}^{k}\frac{1}{i}=-H_k\pmod{p^2}.$$

Therefore,

$$\sum_{1\le k<i\le p-1}\frac{2^k}{ik}=\sum_{k=1}^{p-1}\frac{2^k}{k}\sum_{i=k+1}^{p-1}\frac{1}{i}\equiv-\sum_{k=1}^{p-1}\frac{2^kH_k}{k}\pmod{p^2}. \tag{32}$$

Further, from the shuffle relation

$$2\sum_{1\le k\le i\le p-1}\frac{1}{ki}=\left(\sum_{k=1}^{p-1}\frac{1}{k}\right)^2+\sum_{k=1}^{p-1}\frac{1}{k^2}=H_{p-1}^2+H_{p-1,2}$$

by setting the Wolstenholme's congruence $H_{p-1}\equiv 0\,(\bmod\,p^2)$ and the congruence (13) of Lemma 2.2, we obtain

$$\sum_{1\le k\le i\le p-1}\frac{1}{ki}\equiv\frac{p}{3}B_{p-3}\pmod{p^2}. \tag{33}$$

Finally, substituting (25) of Lemma 2.5, (19) of Lemma 2.3, (32) and (33) into the equality (31), we get the desired congruence (1). □

3. Proof of the congruences (2) and (3)

Lemma 3.1. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n-1}\frac{(-2)^k}{k}\binom{n}{k}=\begin{cases}-2H_n+H_{(n-1)/2}+\dfrac{2^n}{n} & \text{if } n \text{ is odd}\\ -2H_n+H_{n/2}-\dfrac{2^n}{n} & \text{if } n \text{ is even.}\end{cases}\tag{34}$$

Proof. Suppose that n is an arbitrary positive integer. We follow proof of Lemma 4.1 in [8]. By the binomial formula, for each $t>0$ and $x\in\mathbb{R}$, we have

$$\frac{(1-xt)^n-1}{t}=\sum_{k=1}^{n}\frac{\binom{n}{k}(-xt)^k}{t}\,dt=\sum_{k=1}^{n}\binom{n}{k}(-x)^k t^{k-1}.\tag{35}$$

Since $\int_0^1 t^{k-1}\,dt=1/k$, setting $y=1-xt$ (35) gives

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-x)^k}{k}\binom{n}{k}&=\int_0^1\sum_{k=1}^{n}\binom{n}{k}(-x)^k t^{k-1}\,dt=\int_0^1\frac{(1-xt)^n-1}{t}\,dt\\ &=-\frac{1}{x}\int_1^{1-x}\frac{x(y^n-1)}{1-y}\,dy=\int_1^{1-x}\sum_{k=1}^{n}y^{k-1}\,dy\\ &=\sum_{k=1}^{n}\frac{(1-x)^k-1}{k}.\end{aligned}\tag{36}$$

Taking $x=2$ into (36), we obtain

$$\sum_{k=1}^{n-1}\frac{(-2)^k}{k}\binom{n}{k}+\frac{(-2)^n}{n}=-2\sum_{\substack{1\le k\le n\\ k\ \text{odd}}}\frac{1}{k}=\begin{cases}-2H_n+H_{(n-1)/2} & \text{if } n \text{ is odd}\\ -2H_n+H_{n/2} & \text{if } n \text{ is even}\end{cases}$$

which yields the identity (34). □

Lemma 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{(-2)^k}{k}\binom{p}{k}\equiv pq_p(2)^2-\frac{2}{3}p^2q_p(2)^3+\frac{1}{12}p^2B_{p-3}\pmod{p^3}\tag{37}$$

and

$$\sum_{k=1}^{p-1}\frac{(-2)^k}{k}\binom{p-1}{k}\equiv -2q_p(2)+pq_p(2)^2-\frac{2}{3}p^2q_p(2)^3+\frac{1}{12}p^2B_{p-3}\pmod{p^3}.\tag{38}$$

Proof. Setting $n=p$ in the first equality of (34) of Lemma 3.1 and using the congruences (12) and (15) from Lemma 2.2 reduced modulo p^2, we obtain (37). Similarly, taking $n=p-1$ into the second equality of (34) from Lemma 3.1 and substituting the congruences (12) and (15) from Lemma 2.2 into this, we obtain the congruence (38) and the proof is completed. $\square$

Proof of the congruences (2) and (3). First notice that

$$\sum_{k=1}^{p-1}\frac{2^kH_k^2}{k}=\sum_{k=1}^{p-1}\frac{2^k\left(H_{k-1}+\frac{1}{k}\right)^2}{k}=\sum_{k=1}^{p-1}\frac{2^kH_{k-1}^2}{k}+2\sum_{k=1}^{p-1}\frac{2^kH_{k-1}}{k^2}+\sum_{k=1}^{p-1}\frac{2^k}{k^3}. \tag{39}$$

Further, using (11) of Lemma 2.1 and the identity $\binom{p-1}{k-1}=\frac{k}{p}\binom{p}{k}$, we find that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{2^kpH_{k-1}}{k^2}&\equiv\sum_{k=1}^{p-1}\frac{2^k}{k^2}\left(1-(-1)^{k-1}\binom{p-1}{k-1}\right)\pmod{p^2}\\&=\sum_{k=1}^{p-1}\frac{2^k}{k^2}+\sum_{k=1}^{p-1}\frac{(-2)^k}{k^2}\binom{p-1}{k-1}\\&=\sum_{k=1}^{p-1}\frac{2^k}{k^2}+\frac{1}{p}\sum_{k=1}^{p-1}\frac{(-2)^k}{k}\binom{p}{k}\pmod{p^2}.\end{aligned} \tag{40}$$

Substituting the congruences (19) from Lemma 2.3 and (37) from Lemma 3.2 into (40), after dividing by p we obtain

$$\sum_{k=1}^{p-1}\frac{2^kH_{k-1}}{k^2}\equiv\frac{5}{4}B_{p-3}\pmod{p}, \tag{41}$$

whence taking $H_{k-1}=H_k-1/k$ and inserting (22) from Lemma 2.3, we immediately obtain (2).

Since by (29) $H_{p-k-1}\equiv H_k\,(\mathrm{mod}\,p)$ for each $k=1,2,\dots,p-1$ and $2^p\equiv 2\,(\mathrm{mod}\,p)$, we have

$$\sum_{k=1}^{p-1}\frac{2^kH_{k-1}}{k^2}=\sum_{k=1}^{p-1}\frac{2^{p-k}H_{p-k-1}}{(p-k)^2}\equiv\sum_{k=1}^{p-1}\frac{2^{1-k}H_k}{k^2}=2\sum_{k=1}^{p-1}\frac{H_k}{k^2\cdot 2^k}\pmod{p}. \tag{42}$$

Comparing (41) and (42) yields (3). $\square$

4. Proof of the congruences (4), (5) and (6)

Lemma 4.1. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n-1}\frac{(-2)^{k-1}}{k}\binom{n}{k-1}=\begin{cases}\dfrac{2^{n-1}(1-n)}{n+1} & \text{if } n \text{ is odd}\\ \dfrac{(n-1)2^{n-1}+1}{n+1} & \text{if } n \text{ is even.}\end{cases}\tag{43}$$

Proof. Multiplying by $-1/2$ the identity (34) of Lemma 3.1, it becomes

$$\sum_{k=1}^{n-1}\frac{(-2)^{k-1}}{k}\binom{n}{k}=\begin{cases}H_{n-1}-\dfrac{1}{2}H_{(n-1)/2}-\dfrac{2^{n-1}-1}{n} & \text{if } n \text{ is odd}\\ H_n-\dfrac{1}{2}H_{n/2}+\dfrac{2^{n-1}}{n} & \text{if } n \text{ is even.}\end{cases}\tag{44}$$

Now the identities $\binom{n}{k-1}=\binom{n+1}{k}-\binom{n}{k}$ and (44) for any odd positive integer n give

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{(-2)^{k-1}}{k}\binom{n}{k-1}&=\sum_{k=1}^{n-1}\frac{(-2)^{k-1}}{k}\binom{n+1}{k}-\sum_{k=1}^{n-1}\frac{(-2)^{k-1}}{k}\binom{n}{k}\\&=\sum_{k=1}^{n}\frac{(-2)^{k-1}}{k}\binom{n+1}{k}-\frac{(-2)^{n-1}(n+1)}{n}-\sum_{k=1}^{n-1}\frac{(-2)^{k-1}}{k}\binom{n}{k}\\&=\left(H_{n+1}-\frac{1}{2}H_{(n+1)/2}+\frac{2^n}{n+1}\right)-\frac{2^{n-1}(n+1)}{n}\\&\quad-\left(H_{n-1}-\frac{1}{2}H_{(n-1)/2}-\frac{2^{n-1}-1}{n}\right)\\&=(H_{n+1}-H_{n-1})-\frac{1}{2}(H_{(n+1)/2}-H_{(n-1)/2})+\frac{2^n}{n+1}-\frac{n2^{n-1}+1}{n}\\&=\left(\frac{1}{n}+\frac{1}{n+1}\right)-\frac{1}{n+1}+\frac{2^n}{n+1}-\frac{n2^{n-1}+1}{n}\\&=\frac{2^{n-1}(1-n)}{n+1},\end{aligned}\tag{45}$$

which is in fact the first equality in (43). Similarly, using the identities $\binom{n}{k-1}=\binom{n+1}{k}-\binom{n}{k}$ and (44) for even positive integer n, we obtain the second equality in (43). □

Lemma 4.2. *Let n be an arbitrary positive integer. Then*

$$(-1)^n\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}2^kH_k=(2^n-2)H_{n-1}+H_{[n/2]}+\frac{2^n-2}{n}\tag{46}$$

where $[x]$ denotes the integer part of x.

Proof. We proceed by induction on n. An immediate computation shows that (46) is satisfied for $n=1$ and $n=2$. For every $n=1,2,\dots$ put

$$S_n=\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}2^kH_k.$$

Then using the identity $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$ we have

$$\begin{aligned}
S_{n+1}&=\sum_{k=1}^{n}(-1)^{k-1}\binom{n+1}{k}2^kH_k\\
&=\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n+1}{k}2^kH_k+(-1)^{n-1}(n+1)2^nH_n\\
&=\sum_{k=1}^{n-1}(-1)^{k}\binom{n}{k}2^kH_k+\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k-1}2^kH_k+(-1)^{n-1}(n+1)2^nH_n\\
&=S_n+2\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k-1}2^{k-1}\left(H_{k-1}+\frac{1}{k}\right)+(-1)^{n-1}(n+1)2^nH_n\\
&=S_n+2\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k-1}2^{k-1}H_{k-1}+2\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k}\binom{n}{k-1}2^{k-1}\\
&\quad+(-1)^{n-1}(n+1)2^nH_n\\
&=S_n-2\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}2^kH_k-2(-1)^{n-1}n2^{n-1}H_{n-1}\\
&\quad+2\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k}\binom{n}{k-1}2^{k-1}+(-1)^{n-1}(n+1)2^nH_n\\
&=S_n-2S_n+2\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k}\binom{n}{k-1}2^{k-1}\\
&\quad-2(-1)^{n-1}n2^{n-1}\left(H_n-\frac{1}{n}\right)+(-1)^{n-1}(n+1)2^nH_n\\
&=-S_n+2\sum_{k=1}^{n-1}\frac{(-2)^{k-1}}{k}\binom{n}{k-1}+(-1)^{n-1}2^n(H_n+1).
\end{aligned}\tag{47}$$

Notice that both equalities (43) from Lemma 4.1 for any positive integer n can be written as

$$\sum_{k=1}^{n-1}\frac{(-2)^{k-1}}{k}\binom{n}{k-1}=(-1)^n\frac{(n-1)2^{n-1}}{n+1}+(1+(-1)^n)\frac{1}{2(n+1)}.\tag{48}$$

Next, substituting (48) into (47) multiplied by $(-1)^{n+1}$, we find that

$$(-1)^{n+1}S_{n+1}=(-1)^nS_n+\frac{2^{n+1}-1-(-1)^n}{n+1}+2^nH_n. \tag{49}$$

By the induction hypothesis, we have

$$(-1)^nS_n=(-1)^n\sum_{k=1}^{n-1}(-1)^{k-1}\binom{n}{k}2^kH_k=(2^n-2)H_{n-1}+H_{[n/2]}+\frac{2^n-2}{n}$$

which substituting into (49) after routine calculations gives

$$(-1)^{n+1}S_{n+1}=(2^{n+1}-2)H_n+H_{[(n+1)/2]}+\frac{2^{n+1}-2}{n+1}.$$

This concludes the induction proof. $\square$

Proof of the congruences (4), (5), and (6). Using the identities $\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}$, $H_k=H_{k-1}+1/k$ $(1\le k\le n)$ and the congruence (11) from Lemma 2.1, the left hand side of (46) in Lemma 4.2 for $n=p$ is

$$\begin{aligned}\sum_{k=1}^{p-1}(-1)^{k-1}\binom{p}{k}2^kH_k&=\sum_{k=1}^{p-1}\frac{2^kp}{k}(-1)^{k-1}\binom{p-1}{k-1}\left(H_{k-1}+\frac{1}{k}\right)\\&\equiv\sum_{k=1}^{p-1}\frac{2^kp}{k}(1-pH_{k-1})\left(H_{k-1}+\frac{1}{k}\right)\pmod{p^3}=p\sum_{k=1}^{p-1}\frac{2^kH_{k-1}}{k}\\&\quad+p\sum_{k=1}^{p-1}\frac{2^k}{k^2}-p^2\sum_{k=1}^{p-1}\frac{2^kH_{k-1}^2}{k}-p^2\sum_{k=1}^{p-1}\frac{2^kH_{k-1}}{k^2}\pmod{p^3}.\end{aligned} \tag{50}$$

Further, note that by (1) of Theorem 1.1, (19) of Lemma 2.3 and the identity $H_{k-1}=H_k-1/k$,

$$\sum_{k=1}^{p-1}\frac{2^kH_{k-1}}{k}=\sum_{k=1}^{p-1}\frac{2^kH_k}{k}-\sum_{k=1}^{p-1}\frac{2^k}{k^2}\equiv-\frac{13p}{12}B_{p-3}\pmod{p^2}. \tag{51}$$

Taking (51), (41) and (19) of Lemma 2.3 into (50), we find that

$$\begin{aligned}\sum_{k=1}^{p-1}(-1)^{k-1}\binom{p}{k}2^kH_k&\equiv-pq_p(2)^2+\frac{2p^2}{3}q_p(2)^3-\frac{7p^2}{6}B_{p-3}\\&\quad-p^2\sum_{k=1}^{p-1}\frac{2^kH_{k-1}^2}{k}\pmod{p^3}.\end{aligned} \tag{52}$$

On the other hand, by (46) of Lemma 4.2 with $n=p$,

$$\sum_{k=1}^{p-1}(-1)^{k-1}\binom{p}{k}2^kH_k=(2-2^p)H_{p-1}-H_{(p-1)/2}-2q_p(2). \tag{53}$$

Furthermore, since by Wolstenholme's theorem and Fermat little theorem, $p^3\mid(2-2^p)H_{p-1}$, taking this and (15) of Lemma 2.2 into (53) we get

$$\sum_{k=1}^{p-1}(-1)^{k-1}\binom{p}{k}2^kH_k\equiv -pq_p(2)^2+\frac{2}{3}p^2q_p(2)^3+\frac{7p^2}{12}B_{p-3}\pmod{p^3}. \tag{54}$$

Now substituting (54) into (52) and after dividing this by p^2, we obtain

$$\sum_{k=1}^{p-1}\frac{2^kH_{k-1}^2}{k}\equiv -\frac{7}{4}B_{p-3}\pmod{p}, \tag{55}$$

whence taking $H_{k-1}=H_k-1/k$, and next applying (41) and (22) of Lemma 2.3, we immediately get (4).

In order to prove the congruence (5), notice that by (29) $H_{p-k}\equiv H_{k-1}\pmod{p}$ for each $k=1,2,\ldots,p-1$. Hence, using this, the congruence (55), Fermat little theorem, and applying (4), (2) and (22), we find that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k^2}{k\cdot 2^k}&=\sum_{k=1}^{p-1}\frac{H_{p-k}^2}{(p-k)\cdot 2^{p-k}}\equiv\sum_{k=1}^{p-1}\frac{H_{k-1}^2}{(-k)\cdot 2^{1-k}}\pmod{p}\\&=-\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^kH_{k-1}^2}{k}=-\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k\left(H_k-\frac{1}{k}\right)^2}{k}\\&=-\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^kH_k^2}{k}+\sum_{k=1}^{p-1}\frac{2^kH_k}{k^2}-\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k^3}\equiv\frac{7}{8}B_{p-3}\pmod{p}.\end{aligned}$$

This is in fact (5).

For establishing the congruence (6), observe that by (10) of Lemma 2.1,

$$p^2H_{k,2}\equiv 2-2pH_k+p^2H_k^2-2(-1)^k\binom{p-1}{k}\pmod{p^3},$$

whence we have

$$\begin{aligned}p^2\sum_{k=1}^{p-1}\frac{2^kH_{k,2}}{k}&\equiv 2\sum_{k=1}^{p-1}\frac{2^k}{k}-2p\sum_{k=1}^{p-1}\frac{2^kH_k}{k}+p^2\sum_{k=1}^{p-1}\frac{2^kH_k^2}{k}\\&\quad-2\sum_{k=1}^{p-1}\frac{(-2)^k}{k}\binom{p-1}{k}\pmod{p^3}.\end{aligned} \tag{56}$$

Finally, substituting the congruences (18) of Lemma 2.3, (1), (4) of Theorem 1.1 and (38) of Lemma 3.2 into (56), and dividing the obtained congruence by p^2, we obtain (6). This completes the proof. $\square$

5. Proof of Corollary 1.2

Lemma 5.1. *If $p > 3$ is a prime, then*

$$\sum_{1\le k\le i\le p-1} \frac{2^k}{ik^2} \equiv -\frac{5}{4}B_{p-3} \pmod{p}, \tag{57}$$

$$\sum_{1\le k\le i\le p-1} \frac{2^k}{i^2k} \equiv \frac{3}{4}B_{p-3} \pmod{p}, \tag{58}$$

$$\sum_{1\le k\le i\le p-1} \frac{2^k}{ik} \equiv \frac{13}{12}pB_{p-3} \pmod{p^2}. \tag{59}$$

Proof. Since $H_{p-1} \equiv 0(\bmod p^2)$ (the well known Wolstenholme's theorem), for each $k = 1, 2, \dots p-1$,

$$\sum_{i=k}^{p-1} \frac{1}{i} \equiv -\sum_{i=1}^{k-1} \frac{1}{i} = -H_{k-1} \pmod{p^2}. \tag{60}$$

Applying (60), (11) of Lemma 2.1 and taking the identity $\binom{p-1}{k-1} = \frac{k}{p}\binom{p}{k}$, we find that

$$\begin{aligned} p\sum_{1\le k\le i\le p-1} \frac{2^k}{ik^2} &= \sum_{k=1}^{p-1}\frac{2^k}{k^2}\sum_{i=k}^{p-1}\frac{p}{i} \equiv -\sum_{k=1}^{p-1}\frac{2^k}{k^2}pH_{k-1} \pmod{p^2} \\ &\equiv -\sum_{k=1}^{p-1}\frac{2^k}{k^2}\left(1-(-1)^{k-1}\binom{p-1}{k-1}\right) \\ &= -\sum_{k=1}^{p-1}\frac{2^k}{k^2} - \sum_{k=1}^{p-1}\frac{(-2)^k}{k^2}\binom{p-1}{k-1} \\ &= -\frac{1}{p}\sum_{k=1}^{p-1}\frac{2^k}{k^2} - \sum_{k=1}^{p-1}\frac{(-2)^k}{k}\binom{p}{k} \pmod{p^2}. \end{aligned} \tag{61}$$

Finally, taking the congruence (19) of Lemma 2.3 and (37) of Lemma 3.2 into the right hand side of (61), we immediately obtain (57).

Further, from (13) of Lemma 2.2 we see that $H_{p-1,2}\equiv 0\,(\bmod\, p)$ and therefore, for each $k=1,2,\dots,p-1$,

$$\sum_{i=k}^{p-1}\frac{1}{i^2}\equiv -\sum_{i=1}^{k-1}\frac{1}{i^2}=-H_{k-1,2}\pmod p.$$

Applying this, $H_{k-1,2}=H_{k,2}-1/k^2$, (4) of Theorem 1.1 and (22) of Lemma 2.3, we obtain

$$\begin{aligned}\sum_{1\le k\le i\le p-1}\frac{2^k}{i^2k}&=\sum_{k=1}^{p-1}\frac{2^k}{k}\sum_{i=k}^{p-1}\frac{1}{i^2}\equiv -\sum_{k=1}^{p-1}\frac{2^kH_{k-1,2}}{k}\pmod p\\&=-\sum_{k=1}^{p-1}\frac{2^kH_{k,2}}{k}+\sum_{k=1}^{p-1}\frac{2^k}{k^3}\equiv\frac{3}{4}B_{p-3}\pmod p\end{aligned}$$

which is in fact (58).

Finally, by (60) we have

$$\begin{aligned}\sum_{1\le k\le i\le p-1}\frac{2^k}{ik}&=\sum_{k=1}^{p-1}\frac{2^k}{k}\sum_{i=k}^{p-1}\frac{1}{i}\equiv -\sum_{k=1}^{p-1}\frac{2^k}{k}H_{k-1}\pmod{p^2}\\&=-\sum_{k=1}^{p-1}\frac{2^kH_k}{k}+\sum_{k=1}^{p-1}\frac{2^k}{k^2}\pmod{p^2},\end{aligned}$$

whence substituting the congruences (2) of Theorem 1.1 and (19) of Lemma 2.3, we obtain (59). $\square$

Proof of Corollary 1.2. Since $1/(p-k)\equiv -(p+k)/k^2\,(\bmod\, p^2)$, we find that

$$\begin{aligned}2^p\sum_{k=1}^{p-1}\frac{H_k}{k\cdot 2^k}&=2^p\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\sum_{i=1}^{k}\frac{1}{i}=2^p\sum_{1\le i\le k\le p-1}\frac{1}{ik\cdot 2^k}\\&=2^p\sum_{1\le p-i\le p-k\le p-1}\frac{1}{(p-i)(p-k)2^{p-k}}\equiv\sum_{1\le k\le i\le p-1}\frac{(p+i)(p+k)2^k}{i^2k^2}\pmod{p^2}\\&\equiv\sum_{1\le k\le i\le p-1}\frac{(pi+pk+ik)2^k}{i^2k^2}\pmod{p^2}\\&=p\left(\sum_{1\le k\le i\le p-1}\frac{2^k}{ik^2}+\sum_{1\le k\le i\le p-1}\frac{2^k}{i^2k}\right)+\sum_{1\le k\le i\le p-1}\frac{2^k}{ik}\pmod{p^2}.\end{aligned}\tag{62}$$

The substitution of congruences (57)–(59) of Lemma 5.1 into (62) immediately yields

$$2^p\sum_{k=1}^{p-1}\frac{H_k}{k\cdot 2^k}\equiv\frac{7}{12}pB_{p-3}\pmod{p^2},$$

which multiplying by 2^{-p} and using the fact that by Fermat little theorem $2^{-p}\equiv 2^{-1}\,(\mathrm{mod}\,p)$, immediately implies (7).

It remains to prove (8). By (10) of Lemma 2.1,

$$p^2H_{k,2}\equiv 2-2pH_k+p^2H_k^2-2(-1)^k\binom{p-1}{k}\pmod{p^3},$$

whence we have

$$\begin{aligned}p^2\sum_{k=1}^{p-1}\frac{H_{k,2}}{k\cdot 2^k}\equiv{}&2\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}-2p\sum_{k=1}^{p-1}\frac{H_k}{k\cdot 2^k}+p^2\sum_{k=1}^{p-1}\frac{H_k^2}{k\cdot 2^k}\\&-2\sum_{k=1}^{p-1}\frac{(-1)^k}{k\cdot 2^k}\binom{p-1}{k}\pmod{p^3}.\end{aligned}\tag{63}$$

Taking $n=p-1$ and $x=1/2$ into (36) from the proof of Lemma 3.1, we obtain

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k\cdot 2^k}\binom{p-1}{k}=\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}-H_{p-1}.\tag{64}$$

Substituting (64) into (63) yields

$$p^2\sum_{k=1}^{p-1}\frac{H_{k,2}}{k\cdot 2^k}\equiv -2p\sum_{k=1}^{p-1}\frac{H_k}{k\cdot 2^k}+p^2\sum_{k=1}^{p-1}\frac{H_k^2}{k\cdot 2^k}+2H_{p-1}\pmod{p^3}.\tag{65}$$

Finally, substituting the congruences (7) of Corollary 1.2, (5) of Theorem 1.1 and (12) of Lemma 2.2 into (65), and dividing this by p^2, we obtain the congruence (8), and the proof is completed. □

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ROMEO MEŠTROVIĆ
DEPARTMENT OF MATHEMATICS
MARITIME FACULTY
UNIVERSITY OF MONTENEGRO
DOBROTA 36
85330 KOTOR
MONTENEGRO

E-mail: romeo@ac.me
URL: `https://sites.google.com/site/romeomestrovic/`

(Received September 26, 2011; revised October 10, 2011)