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Finsler connection properties generated by the two-vector angle developed on the indicatrix-inhomogeneous level

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Abstract. The Finsler spaces in which the tangent Riemannian spaces are conformally flat prove to be characterized by the condition that the indicatrix is a space of constant curvature. In such spaces the Finslerian two-vector angle can be obtained from the respective two-vector angle of the associated Riemannian space. This observation entails the problem to obtain the angle-preserving connection on general indicatrix-inhomogeneous level, that is, when the indicatrix curvature value $\mathcal{C}_{\text{Ind.}}$ is permitted to be an arbitrary smooth function of the indicatrix position point x. The problem has been completely solved by means of the proposed method to determine the coefficients of nonlinear connection from the separable equation of preservation of the normalized angle. The obtained connection is metrical with the deflection part which is proportional to the gradient of the function $H(x)$ entering the equality $\mathcal{C}_{\text{Ind.}} \equiv H^2$, and is uniquely determined up to the torsion tensor of the associated Riemannian space. Also, the involved deformation of space is covariant-constant. Important tensorial information is obtainable by the help of the coincidence-limit method applied to geodesics of the indicatrix space. When the transitivity of covariant derivative is used, from the commutators of covariant derivatives the associated curvature tensor can be found. The developed theory is applied to the Finsleroid space.

Motivation and introduction

A Finsler space is given by the pair (M, F), where M is a differentiable manifold and $F = F(x, y)$ is a Finsler metric function introduced on the tangent bundle TM of M. The function $F = F(x, y)$ depends on the points $x \in M$ and on the tangent vectors $y \in T_xM$, where $T_xM \subset TM$ is the tangent space supported

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by the x. The notion of connection in Finsler space can be studied departing from various convenient sets of axioms (see [1]–[9] and references therein).

The embedded position of the indicatrix $\mathcal{I}_x \subset T_xM$ in the tangent Riemannian space $\mathcal{R}_{\{x\}} = \{T_xM, g_{\{x\}}(y)\}$ (where $g_{\{x\}}(y)$ denotes the Finslerian metric tensor with x considered fixed and y used as being the variable) induces the Riemannian metric on the indicatrix through the well-known method and in this sense makes the indicatrix a Riemannian space. Therefore, the geodesics can be introduced on the indicatrix by applying the conventional Riemannian methods.

In any (sufficiently smooth) Finsler space the two-vector angle $\alpha_{\{x\}}(y_1, y_2)$ can locally be determined with the help of the indicatrix geodesic arc, which motivates the important question whether the Finsler geometry can be profoundly settled down by developing and applying the connection which preserves the angle.

In general, the angle $\alpha_{\{x\}}(y_1, y_2)$ is complicated and cannot be determined in an explicit tensorial form, except for rare Finsler metric functions. The lucky example is given by the Finsler space $\mathcal{F}^N$ which is characterized by the condition that at each p. x the indicatrix $\mathcal{I}_x \subset T_xM$ is a space of constant curvature. Considering arbitrary dimension $N \geq 3$, it is possible to prove that the tangent Riemannian space $\mathcal{R}_{\{x\}}$ is conformally flat if and only if the indicatrix $\mathcal{I}_x$ is a space of constant curvature (see Proposition 2.1). Therefore, the Finsler space $\mathcal{F}^N$ can alternatively be characterized by the condition that the tangent Riemannian spaces $\mathcal{R}_{\{x\}}$ are of the conformally flat nature.

Our consideration will be local in both the base manifold and the tangent space. The indices $i, j, \ldots$ will refer to local admissible coordinates $\{x^i\}$ on the base manifold M. From any given Finsler metric function $F = F(x, y)$ we can construct the covariant tangent vector $\hat{y} = \{y_i\}$ and the Finslerian metric tensor $\{g_{ij}\}$ in the conventional way: $y_i = (1/2)\partial F^2/\partial y^i$ and $g_{ij} = \partial y_i/\partial y^j$. The contravariant tensor $\{g^{ij}\}$ is defined by the reciprocity conditions $g_{ij}g^{jk} = \delta_i^k$, where δ stands for the Kronecker symbol. We shall also use the tensor $C_{ijk} = (1/2)\partial g_{ij}/\partial y^k$. By l we shall denote the unit vectors, namely, $l = y/F(x, y)$, such that $F(x, l) = 1$. In addition to the Finsler metric tensor g_{mn}, it is convenient to use the tensor $h_{mn} = g_{mn} - l_m l_n$ having the property $h_{mn}y^n = 0$. We shall raise and lower the indices $i, j, \ldots$ of tensorial objects by means of the tensors g_{ij} and g^{jk}, for example, $C^i{}_{jk} = g^{in}C_{njk}$.

Because of the conformal flatness of the spaces $\mathcal{R}_{\{x\}}$, the Finsler space $\mathcal{F}^N$ produces on the same base manifold M the *associated Riemannian space*, to be denoted by $\mathcal{R}^N = (M, S)$, where $S = \sqrt{a_{mn}(x)y^m y^n}$ is the Riemannian metric function constructed from a positive-definite Riemannian metric tensor $a_{mn}(x)$.

The respective transformation $\bar{y} = \bar{y}(x, y)$ which makes the Finsler space

$\mathcal{F}^N$ a Riemannian space is positively homogeneous with respect to the variable y. We denote the degree of homogeneity by $H(x)$. The remarkable equality $\mathcal{C}_{\text{Ind.}} \equiv H^2$ is fulfilled (which was established in [10-12]), where $\mathcal{C}_{\text{Ind.}} = \mathcal{C}_{\text{Ind.}}(x)$ denotes the value of the curvature of the indicatrix $\mathcal{I}_x \subset T_xM$. The relevant conformal multiplier p^2 is constructed from the Finsler metric function F according to $p = (F(x,y))^{1-H(x)}/H(x)$. The metric tensor a_{ij} relates to the Finslerian metric tensor g_{ij} of the space $\mathcal{F}^N$ according to the formulas (2.10)–(2.15) which determine the transformation. We can induce the angle $\alpha^{\text{Riem}}_{\{x\}}$ conventionally defined in the Riemannian space $\mathcal{R}^N$ into the Finsler space $\mathcal{F}^N$, which yields simply $\alpha_{\{x\}}(y_1, y_2) = \big(1/H(x)\big)\alpha^{\text{Riem}}_{\{x\}}(\bar{y}_1, \bar{y}_2)$.

To explicate the coefficients $N^m{}_n$ of nonlinear connection from the Finsler angle $\alpha = \alpha_{\{x\}}(y_1, y_2)$, we should successfully propose the preservation equation. The nearest possibility is to formulate the equation $d_i\alpha = 0$ in accordance with the formulas (1.10) and (1.13), applying the separable operator d_i indicated in (1.9).

This possibility has been realized in the preceding work [10], [11]. Namely, in that work the separable preservation equation $d_i\alpha = 0$ has been solved in the Finsler space $\mathcal{F}^N$ under the assumption that $\mathcal{C}_{\text{Ind.}} = \text{const}$, which implies $H = \text{const}$. The coefficients $N^m{}_n$, and also the connection, have been obtained.

In the present paper, we overcome the restriction $\mathcal{C}_{\text{Ind.}} = \text{const}$, permitting the indicatrix curvature value $\mathcal{C}_{\text{Ind.}}$ to depend on the points $x \in M$ which support the indicatrix. We call the space $\mathcal{F}^N$ *indicatrix-homogeneous,* if the value is a constant, whence $H = \text{const}$. If the dependence $\mathcal{C}_{\text{Ind.}} = \mathcal{C}_{\text{Ind.}}(x)$ does hold, we say that the space $\mathcal{F}^N$ is *indicatrix-inhomogeneous,* in which case $H_i \neq 0$, where $H_i = \partial H/\partial x^i$. The representations obtained in the previous work [10], [11] are the $(H_i \to 0)$-limits of their generalized counterparts developed in the present study.

It appears that in general the angle preservation equation formulated in the separable way does not permit any solution for the coefficients $N^m{}_n$.

This conclusion can be drawn from the implications which are derivable by the help of the coincidence-limit method [13] which extracts the tensorial information from behavior of Riemannian geodesics. To this end we should use the distance function $E = E(x, y_1, y_2)$ in the indicatrix space with $E = (1/2)\alpha^2$, keeping in mind that the angle α measures the length of the indicatrix geodesic arc (in accordance with (1.1)) and, therefore, establishes the geodesic distance in the indicatrix treated as a Riemannian space. Evaluating various partial derivatives of the function E with respect to y_1 and y_2 and finding the coincidence limits when $y_2 \to y_1$, we can obtain a valuable information on the derivatives of the Finsler metric tensor of Finsler space. Performing the required evaluations on the level of the second-order partial derivatives $\partial^2/\partial y_1^m \partial y_2^n$, and, then, applying the operation $y_2 \to y_1$ to the resultant expressions, it is possible to arrive at the following general conclusion: *In any Finsler space the assumption $d_i\alpha = 0$ of the separable type entails the equality $\mathcal{D}_i h_{mn} = (2/F) h_{mn} d_i F$.*

If we additionally postulate $d_i F = 0$, we obtain $\mathcal{D}_i h_{mn} = 0$ and, therefore, the metricity $\mathcal{D}_i g_{mn} = 0$ which is formulated with the covariant derivative $\mathcal{D}$ arisen from the deflectionless connection. We can apply the derivative $\partial^2/\partial y^l \partial y^h$ to the equality $\mathcal{D}_i g_{mn} = 0$, which leads after simple evaluations to the equality $\mathcal{D}_i S_n{}^k{}_{jm} = 0$. Here, the $S_n{}^k{}_{jm}$ is the tensor which describes the curvature of indicatrix (the tensor can be found in Section 5.8 in [1]).

Clearly, the condition $\mathcal{D}_i S_n{}^k{}_{jm} = 0$ is fulfilled in but rare cases of Finsler space. They include the indicatrix-homogeneous case of the space $\mathcal{F}^N$. The indicatrix-inhomogeneous space $\mathcal{F}^N$ is not complied with the condition.

Therefore, accounting for the dependence $H = H(x)$ in the angle-generated connection coefficients of the Finsler space $\mathcal{F}^N$ is neither a straightforward task nor a trivial problem.

These important (and rather unexpected?) implications enforce us to look for more capable ideas to formulate the preservation of angle. The attractive idea is to substitute the normalized angle $\alpha_{\{x\}}^{\{H(x)\}}(y_1, y_2) = H(x)\alpha_{\{x\}}(y_1, y_2)$ (see (1.26)) with the initial angle $\alpha_{\{x\}}(y_1, y_2)$ in the separable preservation law, according to (1.27). The law obtained is of the recurrent-type (1.28), namely $d_i\alpha + (1/H)H_i\alpha = 0$. It appears that this preservation complies with the indicatrix-inhomogeneous Finsler space $\mathcal{F}^N$. The reason thereto is the following assertion obtainable by the help of the coincidence-limit method: *In any Finsler space the assumption $d_i\alpha + (1/H)H_i\alpha = 0$ entails the equality $\mathcal{D}_i h_{mn} = (2/F)h_{mn} d_i F - (2/H)H_i h_{mn}$.* When $d_i F = 0$, the equality reduces to $\mathcal{D}_i g_{mn} = -(2/H)H_i h_{mn}$, which in turn entails the extension of the previous vanishing $\mathcal{D}_i S_n{}^k{}_{jm} = 0$ such that the right-hand part of this extension (written in (1.32)) is just the expression which is obtained when the characteristic representation $S_n{}^k{}_{jm} = C(x)(h_{nm}h_j^k - h_{nj}h_m^k)$ of the tensor $S_n{}^k{}_{jm}$ of the space $\mathcal{F}^N$ under study is inserted under the action of the covariant derivative $\mathcal{D}_i$.

Thus, we are entitled to use the recurrent-type equation $d_i\alpha + (1/H)H_i\alpha = 0$. Solving the equation with respect to the coefficients $N^m{}_n(x, y)$ results in the explicit representation (2.31). The $N^m{}_n(x, y)$ obtainable in this way can naturally be interpreted as the *coefficients of the non-linear connection produced by the angle in the space $\mathcal{F}^N$ studied on the general indicatrix-inhomogeneous level.*

With the knowledge of the coefficients $N^m{}_i(x,y)$, we can evaluate the derivative coefficients $N^m{}_{ij} = \partial N^m{}_i/\partial y^j$ and express the Finslerian connection coefficients $T^m{}_{ij}$ through the Riemannian connection coefficients $L^m{}_{ij} = L^m{}_{ij}(x)$ and the function $H = H(x)$ (by the help of the formulas (1.34) and (2.32)).

In this way, the *metrical non-linear Finsler connection* $\mathcal{F}N = \{N^m{}_i, T^m{}_{ij}\}$ is induced in the indicatrix-inhomogeneous space $\mathcal{F}^N$ from the metrical linear connection $\mathcal{R}L = \{L^m{}_j, L^m{}_{ij}\}$ evidenced in the Riemannian space $\mathcal{R}^N$, where $L^m{}_j = -L^m{}_{ji}y^i$. The involved function $H = H(x)$ may depend on x in arbitrary smooth way.

The *deflection tensor* $\Delta^k{}_{im} = -N^k{}_{im} - T^k{}_{im}$ is non-vanishing as far as $H_i \neq 0$, for $\Delta^k{}_{im} = (1/H)H_i h^k_m$. So, in distinction from the connection developed in the indicatrix-homogeneous case, in the indicatrix-inhomogeneous space the connection $\mathcal{F}N$ is no more deflectionless. Nevertheless, the connection is metrical and the equality $N^m{}_j = -T^m{}_{ji}y^i$ holds. The connection coefficients $T^m{}_{ji}$ are not symmetric with respect to the subscripts j, i.

The connection $\mathcal{F}N$ gives rise to the covariant derivative $\mathcal{T}$ whose remarkable properties are listed in (1.38)–(1.41).

We say that the transformation $y = y(x, \bar{y})$ performs the *deformation* $\mathbf{C}$ of the space $\mathcal{F}^N$. The formulas (2.16)–(2.20) describe the basic properties of the deformation $\mathbf{C}$.

The indicatrix-inhomogeneous space $\mathcal{F}^N$ under study arises from the Riemannian space $\mathcal{R}^N$ as a result of such a deformation: $\mathcal{F}^N = \mathbf{C}\cdot\mathcal{R}^N$. The same interpretation refers also to the connections, namely $\mathcal{F}N = \mathbf{C}\cdot\mathcal{R}L$. The deformation is $\mathcal{T}$-covariant constant: $\mathcal{T}\cdot\mathbf{C} = 0$. Also, the covariant derivative $\mathcal{T}$ is the manifestation of the *transitivity* of the connection under this transformation, in short, $\mathcal{T} = \mathcal{C}\cdot\nabla$, where ∇ is the covariant derivative applicable in the Riemannian space $\mathcal{R}^N$ (these properties of the introduced deformation have been established in Section II.4 of [12]).

In the Riemannian geometry we have merely $H = 1$. In the Finsler space $\mathcal{F}^N$, the $H(x)$ plays the role of the input function which changes the indicatrix curvature value.

The present study of the $\mathcal{F}^N$-space was essentially influenced by the recent publications [3] and [6].

Developing the attractive idea to measure the angle by means of the area, Tamássy proved the theorem in [6] which states that a diffeomorphism between two Finsler spaces is an isometry iff it keeps the angle. The obvious importance of the theorem motivates the desire to go farther and find a particular Finsler space in which the connection is obtainable from the angle in terms of clear and handy representations. The Finsler space $\mathcal{F}^N$ suits well the purpose.

In [3], Kozma and Tamássy attracted the attention to the circumstance that in the Riemannian geometry we can use naturally the metrical and linear connection on the tangent bundle of the variables x, y. Like to the constructions developed in the preceding work [10,11] dealt with the indicatrix-homogeneous case, in the present indicatrix-inhomogeneous study of the space $\mathcal{F}^N$ the export of this connection by the help of the deformation $\mathbf{C}$ generates the required Finsler connection.

The deformation $\mathcal{F}^N = \mathbf{C} \cdot \mathcal{R}^N$ is a *conformal isometry*, namely the space $\breve{\mathcal{F}}^N = \{M, \breve{g}_{ij}\}$ with $\breve{g}_{ij} = g_{ij}/p^2$ is isometric to the Riemannian space $\mathcal{R}^N = (M, a_{ij})$, where p^2 is the conformal multiplier displayed in (2.10) and (2.15). Indeed, from the formulas (2.17) and (2.18) it follows that $\breve{g}_{mn} = a_{ij}\bar{y}^i_m\bar{y}^j_n$.

In Section 1, the key representations necessary for using the indicatrix-inhomogeneous $\mathcal{F}^N$-space have been exposed.

In Section 2 we extend Proposition 2.1 of the preceding work [10], [11] in the following essential aspect. In [10], [11], the assumption was made that the respective conformal multiplier is of the power dependence on the Finsler metric function. Instead, we shall show by means of an attentive analysis that the power dependence is a direct consequence of the property that the tangent Riemannian spaces $\mathcal{R}_{\{x\}}$ are of the conformally flat nature (see Propositions 2.1 and 2.2). We also describe the basic properties of the deformation $\mathbf{C}$ and explain how the angle representation $\alpha_{\{x\}}(y_1, y_2) = 1/H(x)\alpha^{\text{Riem}}_{\{x\}}(\bar{y}_1, \bar{y}_2)$ can be established. After that, we solve the preservation equation of the normalized angle with respect to the coefficients $N^m{}_n$. The outcome is given by the formula (2.31) which indicates the representation of the coefficients $N^m{}_n$ which is valid for an arbitrary Finsler space of the type $\mathcal{F}^N$. The representation involves the vector field U^i which determines the key transformation $y = \mathbf{C}(x, \bar{y})$. Given a particular Finsler space of the type $\mathcal{F}^N$, the formula (2.31) yields the coefficients $N^m{}_n$ in a completely explicit way when the field U^i is known explicitly.

The Finsleroid case to which Section 3 is devoted provides us with such an example, for the required field U^i is explicitly given by means of the representation (3.6) (which was earlier found in Section 6 of [7]). Therefore, we can straightforwardly apply the developed theory of the $\mathcal{F}^N$-space to the metric function of the Finsleroid type. The expansion (1.42) for the respective Finsleroid coefficients $N^m{}_n$ has been evaluated. The explicit representation of the entailed derivative coefficients $N^k{}_{im}$ is also indicated. The respective validity of the representations (1.29) and (1.30) of the tensors $\mathcal{D}_i h_{mn}$ and $N^k{}_{imn}$ on the indicatrix-inhomogeneous level of study of the Finsleroid space has been verified by direct evaluations presented in detail in [12]. Thus we have got prepared the connection $\mathcal{F}N$ in the Finsleroid space at our disposal with an arbitrary input smooth function $\mathcal{C}_{\text{Ind.}}(x)$.

In the last Section 4 the significance of the connection problem solved in the present paper has been emphasized. Also, the converse method of derivation of the connection in the space $\mathcal{F}^N$ under study, namely the method which is directly based on the deformation of the Riemannian connection, has been proposed. The method is qualitative rather than evaluative.

Below we are interested in spaces of the dimension $N \ge 3$. The two-dimensional case has been studied in the work [8], [9].

The present work was preceded by the *arXiv*-publication [12] in which various required methods of evaluations have been developed.

1. Basic representations

Let U_x be a simply connected and geodesically complete region on the indicatrix $\mathcal{I}_x$ supported by a point $x \in M$. Any point pair $u_1, u_2 \in U_x$ can be joined by the respective arc $\mathcal{A}_{\{x\}}(l_1, l_2) \subset \mathcal{I}_x$ of the Riemannian geodesic line drawn on U_x. By identifying the length of the arc with the angle notion we arrive at the **geodesic-arc angle** $\alpha_{\{x\}}(y_1, y_2)$, where $y_1, y_2 \in T_xM$ are the two vectors whose direction rays $0y_1$ and $0y_2$ intersect the indicatrix at the points u_1 and u_2. We obtain

$$\alpha_{\{x\}}(y_1, y_2) = \|\mathcal{A}_{\{x\}}(l_1, l_2)\|. \tag{1.1}$$

The coefficients $N^k{}_i = N^k{}_i(x, y)$ are required to construct the operator

$$d_i = \frac{\partial}{\partial x^i} + N^k{}_i \frac{\partial}{\partial y^k}. \tag{1.2}$$

These coefficients are assumed naturally to be positively homogeneous of degree 1 with respect to the vector argument y.

The derivative coefficients

$$N^k{}_{nm} = \frac{\partial N^k{}_n}{\partial y^m}, \qquad N^k{}_{nmj} = \frac{\partial N^k{}_{nm}}{\partial y^j} \tag{1.3}$$

fulfill the identities $N^k{}_{nm} y^m = N^k{}_n$, $N^k{}_{nmj} y^m = N^k{}_{nmj} y^j = 0$, and $N^k{}_{nmj} = N^k{}_{njm}$. The coefficients are used to construct the covariant derivatives

$$\mathcal{D}_k F = d_k F, \qquad \mathcal{D}_k l^m = d_k l^m - N^m{}_{kn} l^n, \qquad \mathcal{D}_k l_m = d_k l_m + N^h{}_{km} l_h, \tag{1.4}$$

and

$$\mathcal{D}_k g_{mn} = d_k g_{mn} + N^h{}_{km} g_{hn} + N^h{}_{kn} g_{mh}. \tag{1.5}$$

The identities

$$\frac{\partial \mathcal{D}_k F}{\partial y^m} = \mathcal{D}_k l_m, \qquad \frac{\partial \mathcal{D}_k l_m}{\partial y^n} = \mathcal{D}_k g_{mn} + l_h N^h{}_{kmn} \tag{1.6}$$

are obviously valid, together with

$$\frac{\partial \mathcal{D}_i g_{mn}}{\partial y^j} = 2\mathcal{D}_i C_{mnj} + N^t{}_{imj} g_{tn} + N^t{}_{inj} g_{mt}, \tag{1.7}$$

where

$$\mathcal{D}_i C_{mnj} = d_i C_{mnj} + N^t{}_{ij} C_{mnt} + N^t{}_{im} C_{tnj} + N^t{}_{in} C_{mtj}. \tag{1.8}$$

The covariant derivative of the tensor $h_{mn} = g_{mn} - l_m l_n$ will be constructed in the manner similar to (1.5), namely $\mathcal{D}_k h_{mn} = d_k h_{mn} + N^h{}_{km} h_{hn} + N^h{}_{kn} h_{mh}$.

To deal with the two-vector angle $\alpha = \alpha_{\{x\}}(y_1, y_2)$, we merely extend the operator d_i in the *separable way*, namely

$$d_i = \frac{\partial}{\partial x^i} + N^k{}_i(x, y_1)\frac{\partial}{\partial y_1^k} + N^k{}_i(x, y_2)\frac{\partial}{\partial y_2^k}, \qquad y_1, y_2 \in T_x M, \tag{1.9}$$

and introduce the covariant derivative $\mathcal{D}_i \alpha$ according to

$$\mathcal{D}_i \alpha = d_i \alpha. \tag{1.10}$$

In the associated Riemannian space $\mathcal{R}^N$ we have the separable operator

$$d_i^{\mathrm{Riem}} = \frac{\partial}{\partial x^i} + L^k{}_i(x, y_1)\frac{\partial}{\partial y_1^k} + L^k{}_i(x, y_2)\frac{\partial}{\partial y_2^k}, \qquad y_1, y_2 \in T_x M, \tag{1.11}$$

with the linear coefficients $L^k{}_i(x, y_1) = -L^k{}_{ij}(x) y_1^j$ and $L^k{}_i(x, y_2) = -L^k{}_{ij}(x) y_2^j$ in which

$$L^m{}_{ij} = a^m{}_{ij} + S^m{}_{ij}, \tag{1.12}$$

where $a^m{}_{ij} = a^m{}_{ij}(x)$ stands for the Christoffel symbols constructed from the Riemannian metric tensor $a_{ij}(x)$ of the space $\mathcal{R}^N$, and $S^m{}_{ij} = S^m{}_{ij}(x)$ is the *torsion tensor* of the Riemannian connection of the space $\mathcal{R}^N$, such that $S^m{}_{ij} = -S^m{}_{ji}$. When applied to the Riemannian two-vector angle $\alpha^{\mathrm{Riem}}_{\{x\}}(y_1, y_2) = \arccos(a_{mn}(x) y_1^m y_2^n / S_1 S_2)$, where $S_1 = \sqrt{a_{mn}(x) y_1^m y_1^n}$ and $S_2 = \sqrt{a_{mn}(x) y_2^m y_2^n}$, the operator reveals the fundamental property of angle preservation

$$d_i^{\mathrm{Riem}} \alpha^{\mathrm{Riem}}_{\{x\}}(y_1, y_2) = 0, \qquad y_1, y_2 \in T_x M.$$

By analogy, one may assume that the Finsler coefficients $N^k{}_i$ fulfill the *separable angle-preservation equation*

$$\mathcal{D}_i\alpha = 0 \tag{1.13}$$

to try developing the theory in which the properties

$$\mathcal{D}_kF = 0, \qquad \mathcal{D}_kl_m = 0, \qquad \mathcal{D}_kl^m = 0, \tag{1.14}$$

together with the metricity

$$\mathcal{D}_kg_{mn} = 0 \tag{1.15}$$

hold fine. This metricity, taken in conjunction with the identities indicated in (1.6), just entails that

$$l_hN^h{}_{kmn} = 0. \tag{1.16}$$

The following valuable implication can be deduced from angle by applying the coincidence-limit method: *In any Finsler space the assumption $d_i\alpha = 0$ of the separable type entails the equality*

$$\mathcal{D}_ih_{mn} = \frac{2}{F}h_{mn}d_iF \tag{1.17}$$

(take below the formula (1.29), keeping $H = \text{const}$). If we additionally postulate $d_iF = 0$, we obtain $\mathcal{D}_ih_{mn} = 0$ and, therefore, $\mathcal{D}_ig_{mn} = 0$.

Thus, the separable angle-preservation equation entails the following remarkable implication:

$$\text{Preservation of angle and length} \Longrightarrow \text{metricity}, \tag{1.18}$$

that is, the two conditions $d_i\alpha = 0$ and $d_iF = 0$ entail $\mathcal{D}_ig_{mn} = 0$.

When $\mathcal{D}_ig_{mn} = 0$, from the identity (1.7) it follows that

$$2\mathcal{D}_iC_{mnj} + N^t{}_{imj}g_{tn} + N^t{}_{inj}g_{mt} = 0, \tag{1.19}$$

which in turn entails that, because the tensor C_{mnj} is totally symmetric, the tensor $N_{nimj} = N^t{}_{imj}g_{tn}$ must be totally symmetric with respect to the subscripts n, m, j: $N_{nimj} = N_{minj} = N_{jimn} = N_{nijm}$, whence

$$N^k{}_{imn} = -\mathcal{D}_iC^k{}_{mn}, \tag{1.20}$$

where $\mathcal{D}_iC^k{}_{mn} = d_iC^k{}_{mn} - N^k{}_{it}C^t{}_{mn} + N^t{}_{im}C^k{}_{tn} + N^t{}_{in}C^k{}_{mt}$.

Thus, *in any Finsler space the two conditions* $d_i\alpha = 0$ *and* $d_iF = 0$ *entail the representation* (1.20) *for the coefficients* ${N^k}_{imn}$.

The formula (1.20) tells us also that (1.16) can be regarded as a direct implication of the identity $y^kC_{knj} = 0$ shown by the tensor C_{knj}.

Because of the identity

$$\frac{\partial {C^k}_{mn}}{\partial y^j} - \frac{\partial {C^k}_{jn}}{\partial y^m} = -2\left({C^h}_{nm}{C^k}_{hj} - {C^h}_{nj}{C^k}_{hm}\right), \tag{1.21}$$

the components

$$\widetilde{R}_n{}^m{}_{ij} = \frac{\partial {C^m}_{ni}}{\partial y^j} - \frac{\partial {C^m}_{nj}}{\partial y^i} + {C^h}_{ni}{C^m}_{hj} - {C^h}_{nj}{C^m}_{hi} \tag{1.22}$$

of the curvature tensor $\widehat{R}_{\{x\}} = \{\widehat{R}_n{}^m{}_{ij}(x,y)\}$ arisen in the tangent Riemannian space $\mathcal{R}_{\{x\}}$ reduce to

$$\widehat{R}_n{}^m{}_{ij} = \frac{1}{F^2}S_n{}^m{}_{ij} \quad \text{with} \quad S_n{}^m{}_{ij} = \left({C^h}_{nj}{C^m}_{hi} - {C^h}_{ni}{C^m}_{hj}\right)F^2. \tag{1.23}$$

By differentiating the coefficients (1.20) with respect to y^j and making the interchange of the indices m, j, it is easy to conclude after a short evaluation that owing to the vanishing $\partial {N^k}_{imn}/\partial y^j - \partial {N^k}_{ijn}/\partial y^m = 0$ and the above identity (1.21), the representation (1.20) entails the vanishing

$$\mathcal{D}_iS_n{}^k{}_{jm} = 0, \tag{1.24}$$

where $\mathcal{D}_iS_n{}^k{}_{jm} = d_iS_n{}^k{}_{jm} - {N^k}_{ih}S_n{}^h{}_{jm} + {N^t}_{in}S_t{}^k{}_{jm} + {N^t}_{ij}S_n{}^k{}_{tm} + {N^t}_{im}S_n{}^k{}_{jt}$.

Thus the following assertion is valid.

Proposition 1.1. *In an arbitrary Finsler space of any dimension* $N \geq 3$, *the possibility of determination of the coefficients* ${N^m}_n$ *from the separable equation* $d_i\alpha = 0$ *supplemented by the condition* $d_iF = 0$ *implies* $\mathcal{D}_iS_n{}^k{}_{jm} = 0$.

In the indicatrix-homogeneous case of the space $\mathcal{F}^N$ under study, we have the representation $S_{nmij} = \text{const}(h_{nj}h_{mi} - h_{ni}h_{mj})$, which complies with the necessary condition $\mathcal{D}_iS_n{}^k{}_{jm} = 0$ because of the property $\mathcal{D}_ih_{jm} = 0$.

On the indicatrix-inhomogeneous level of study of the space $\mathcal{F}^N$ the characteristic representation reads $S_{nmij} = C(h_{nj}h_{mi} - h_{ni}h_{mj})$ with $C = C(x)$, whence from $\mathcal{D}_iS_n{}^k{}_{jm} = 0$ it would follow that $C_i = 0$, where $C_i = \partial C/\partial x^i$. In turn, since $\mathcal{C}_{\text{Ind.}} = 1 - C$ (such an equality can be found in Section 5.8 of [1]) and $\mathcal{C}_{\text{Ind.}} = H^2$, the conclusion $C_i = 0$ implies $H_i = 0$, which returns us to the indicatrix-homogeneous case.

Hence, to lift the connection from the indicatrix-homogeneous grounds to the indicatrix-inhomogeneous level, we are compelled to propose an extended preservation law which does not require $\mathcal{D}_i {S_n}^k{}_{jm} = 0$. In this connection, the use of the normalized angle seems to be the most natural proposal. To this end we introduce a *characteristic indicatrix scale factor* $R(x)$ in each tangent space to normalize the angle. If the volume $V_{\mathcal{I}_x}$ of the Finslerian indicatrix $\mathcal{I}_x \subset T_xM$ is finite, it is attractive to obtain the scale by the help of the equality

$$V_{\mathcal{I}_x} = C_1(R(x))^{N-1}, \qquad C_1 = \text{const}\,. \tag{1.25}$$

The scale factor $R(x)$ appeared in this way has the clear geometrical meaning of the *radius of the indicatrix* supported by $p.\ x$.

In this respect, there is the deep qualitative distinction of the Finsler geometry from the Riemannian geometry. Namely, in the latter geometry we have simply $V_{\mathcal{I}_x} = \text{const}$, whence $R = \text{const}$. In the Finsler geometry, the value of $V_{\mathcal{I}_x}$ may vary from point to point of the background manifold M, in which case the R may be a function of x.

Accordingly, we replace the above angle $\alpha_{\{x\}}(y_1, y_2)$ by the *normalized angle*

$$\alpha^{\{H(x)\}}_{\{x\}}(y_1, y_2) = H(x)\alpha_{\{x\}}(y_1, y_2), \qquad y_1, y_2 \in T_xM, \tag{1.26}$$

where we have introduced the function $H(x) = 1/R(x)$, to use the preservation law

$$d_i\alpha^{\{H(x)\}}_{\{x\}}(y_1, y_2) = 0 \tag{1.27}$$

instead of $d_i\alpha_{\{x\}}(y_1, y_2) = 0$ formulated in (1.13). The law (1.27) can be written in the *recurrent form*

$$d_i\alpha + \frac{1}{H}H_i\alpha = 0. \tag{1.28}$$

The d_i is the operator (1.9) and $H_i = \partial H/\partial x^i$.

Since the angle $\alpha_{\{x\}}(y_1, y_2)$ is measured by the indicatrix arc length, it seems quite natural to normalize the angle by means of the characteristic scale factor of indicatrix, according to (1.26), before placing the angle under the action of the separable operator d_i.

To deduce the tensorial implications of the recurrent preservation law (1.28), it proves being of great help to apply the coincidence-limit method [13] of studying geodesics. Namely, for the function $E = (1/2)\alpha^2$ from (1.28) we obtain the following E-*equation*

$$\frac{\partial E}{\partial x^i} + N^k{}_{1i}\frac{\partial E}{\partial y_1^k} + N^k{}_{2i}\frac{\partial E}{\partial y_2^k} = -\frac{2}{H}H_iE,$$

where $N^k{}_{1i} = N^k{}_i(x, y_1)$ and $N^k{}_{2i} = N^k{}_i(x, y_2)$. By differentiating this E-equation with respect to y_1 and y_2 and, then, applying the coincidence limit operation $y_2 \to y_1$ to the resultant expressions, it is possible to arrive at the following general conclusion: *In any Finsler space the assumption* $d_i\alpha + (1/H)H_i\alpha = 0$ *entails the equality*

$$\mathcal{D}_ih_{mn} = \frac{2}{F}h_{mn}d_iF - \frac{2}{H}H_ih_{mn}. \tag{1.29}$$

This equality has been derived in Appendix E of [12] in all detail by performing long substitutions (see (E.37) in Appendix E in [12]).

By differentiating the equality (1.29) with respect to y^j, it is possible to obtain the coefficients $N^k{}_{imn}$. In this way, when $d_iF = 0$ is valid, simple direct evaluations yield the representation

$$N^k{}_{imn} = \frac{2}{H}H_i\frac{1}{F}l^kh_{mn} - \mathcal{D}_iC^k{}_{mn}, \tag{1.30}$$

which extends the previous (1.20). The symmetry property for these coefficients reads now

$$N^t{}_{imj}g_{tn} - \frac{2}{H}H_i\frac{1}{F}h_{mj}l_n = N^t{}_{imn}g_{tj} - \frac{2}{H}H_i\frac{1}{F}h_{mn}l_j.$$

Instead of (1.16) we obtain

$$FN^k{}_{inm}l_k = \frac{2}{H}H_ih_{mn}. \tag{1.31}$$

The condition $\mathcal{D}_iS_n{}^k{}_{jm} = 0$ indicated in (1.24) is now extended, namely the above representation (1.30) entails

$$\mathcal{D}_iS_n{}^k{}_{jm} = -\frac{2}{H}H_i\left(h_j^kh_{mn} - h_m^kh_{jn}\right). \tag{1.32}$$

From (1.29) it follows that whenever $d_iF = 0$ we have

$$\mathcal{D}_ig_{mn} = -\frac{2}{H}H_ih_{mn}. \tag{1.33}$$

The equality (1.33) suggests us to introduce the *total connection coefficients*

$$T^k{}_{im} = -N^k{}_{im} - \frac{1}{H} H_i h^k_m, \tag{1.34}$$

so that the *deflection tensor*

$$\Delta^k{}_{im} \stackrel{\text{def}}{=} -N^k{}_{im} - T^k{}_{im} \tag{1.35}$$

is non-vanishing as far as $H_i \neq 0$, namely

$$\Delta^k{}_{im} = \frac{1}{H} H_i h^k_m. \tag{1.36}$$

It follows that

$$T^k{}_{im} y^m = -N^k{}_{im} y^m \equiv -N^k{}_i, \qquad l_k T^k{}_{im} = -l_k N^k{}_{im}. \tag{1.37}$$

There arises the *total covariant derivative* $\mathcal{T}$ showing the properties

$$\mathcal{T}_i F = 0, \qquad \mathcal{T}_i l_m = 0, \qquad \mathcal{T}_i l^m = 0, \tag{1.38}$$

and the *metricity*

$$\mathcal{T}_i g_{nm} = 0, \tag{1.39}$$

where

$$\mathcal{T}_i F \stackrel{\text{def}}{=} d_i F, \qquad \mathcal{T}_i l_m \stackrel{\text{def}}{=} d_i l_m - T^h{}_{im} l_h, \qquad \mathcal{T}_i l^m \stackrel{\text{def}}{=} d_i l^m + T^m{}_{ih} l^h, \tag{1.40}$$

and

$$\mathcal{T}_i g_{nm} \stackrel{\text{def}}{=} d_i g_{nm} - T^h{}_{im} g_{hn} - T^h{}_{in} g_{hm}. \tag{1.41}$$

In all the previous formulas started with (1.26), the $H(x)$ was an arbitrary smooth function not related anyhow to the indicatrix curvature, and the constancy of the indicatrix curvature was not implied.

If the indicatrix $\mathcal{I}_x \subset T_x M$ of a Finsler space is a space of constant curvature at any supporting point $x \in M$, we say that the Finsler space is the $\mathcal{F}^N$-*space*, where $N \geq 3$ is the dimension of the space. In such spaces we can take the positive function $H(x)$ from the equality $\mathcal{C}_{\text{Ind.}}(x) \equiv (H(x))^2$, where $\mathcal{C}_{\text{Ind.}}(x)$ is the value of the curvature of the indicatrix $\mathcal{I}_x$. At any point $x \in M$ of the space $\mathcal{F}^N$ this function $H = H(x)$ naturally introduces the scale factor $R(x) = 1/H(x)$ in the tangent Riemannian space $\mathcal{R}_{\{x\}}$ supported by the point. It is the function $H(x)$ that we use in the normalized angle (1.26) when treating the space $\mathcal{F}^N$, that is, the angle is normalized by the help of the square root of the indicatrix curvature.

When we use the recurrent preservation law supplemented by the condition $\mathcal{D}_iF=0$, from (1.29) we have $\mathcal{D}_ih_{mn}=-(2/H)H_ih_{mn}$. Applying the covariant derivative $\mathcal{D}_i$ to the tensor $S_n{}^k{}_{ij}=C(h_{nj}h^k_i-h_{ni}h^k_j)$ and noting that $C=1-H^2$, after short evaluations we obtain the derivative $\mathcal{D}_iS_n{}^k{}_{jm}$ which is just equivalent to the derivative indicated in (1.32).

Thus, the following proposition is valid.

Proposition 1.2. *The recurrent-type preservation* (1.28) *of the angle, that is,* $d_i\alpha+(1/H)H_i\alpha=0$*, complies with the indicatrix-inhomogeneous Finsler space* $\mathcal{F}^N$ *at any smooth* $H=H(x)$ *obtainable from the identification* $\mathcal{C}_{\mathrm{Ind.}}=H^2$.

The observations motivate us to go to the preservation law (1.27) which is not separable from the standpoint of the indicatrix-arc angle $\alpha_{\{x\}}(y_1,y_2)$, whenever $H\neq const$.

The coefficients $N^m{}_n$ shown in (2.31) don't involve explicitly the gradients H_n. If, however, we expand the partial derivatives $\partial/\partial x^n$ which enter the right-hand part of (2.31), the coefficients will break down into two parts:

$$N^m{}_n=N^{\mathrm{I}m}{}_n+\breve{N}^m{}_n,\qquad \breve{N}^m{}_n=\breve{N}^mH_n. \tag{1.42}$$

Here, the first part $N^{\mathrm{I}m}{}_n$ are the coefficients of the indicatrix-homogeneous case (given by the formula (2.30) in [10], and by the formula (2.36) in [11]) in which the constant H has been merely replaced by arbitrary $H(x)$, and the vector field $\breve{N}^m$ does not involve any gradient of $H(x)$. We may say that the coefficients $N^m{}_n$ are of the *linear dependence* on the gradient H_n.

The entailed coefficients $N^k{}_{mn}$ are given by the representation (2.34) which is applicable to any indicatrix-inhomogeneous Finsler space $\mathcal{F}^N$. It is also possible to evaluate explicitly the derivative coefficients $N^k{}_{mni}=\partial N^k{}_{mn}/\partial y^i$. The required evaluations (which have been presented in detail in [12]) lead to the validity of the representation (1.30) in the $\mathcal{F}^N$-space with an arbitrary smooth function $H(x)$, provided that $d_nF=0$ is assumed.

Having evaluated the coefficients $N^k{}_{mn}$, we obtain from (1.34) the total connection coefficients $T^k{}_{im}$ thereby solving the problem of finding the connection in the $\mathcal{F}^N$-space at the indicatrix-inhomogeneous level.

2. Indicatrix of constant curvature

When $N\geq 4$, to elucidate the conformal properties of the tangent Riemannian space $\mathcal{R}_{\{x\}}$ we should use the formula (1.22), which proposes us a convenient and simple representation for the respective curvature tensor $\{\widehat{R}_n{}^m{}_{ij}(x,y)\}$, and construct the Weyl tensor W_{ijmn} in the space $\mathcal{R}_{\{x\}}$, so that

$$F^2 W_{ijmn} = S_{ijmn} - \frac{1}{N-2}(S_{im}g_{jn} + S_{jn}g_{im} - S_{in}g_{jm} - S_{jm}g_{in}) + \frac{1}{(N-1)(N-2)}\breve{S}(g_{im}g_{jn} - g_{in}g_{jm}), \tag{2.1}$$

where $S_{ijmn} = g_{jh}S_i{}^j{}_{mn}$, $S_{im} = g^{jn}S_{ijmn}$, and $\breve{S} = g^{im}S_{im}$. Contracting the tensor two times by the unit vector l^n yields directly $(N-2)F^2 W_{ijmn}l^n l^j = -S_{im} + (1/(N-1))\breve{S}h_{im}$, where $h_{im} = g_{im} - l_i l_m$. Therefore, in any dimension $N \geq 4$ the vanishing $W_{ijmn} = 0$ is tantamount to the representation

$$S_{nmij} = C(h_{nj}h_{mi} - h_{ni}h_{mj}). \tag{2.2}$$

It is known (see Section 5.8 in [1]) that the indicatrix is a space of constant curvature if and only if the tensor S_{nmij} fulfills the representation (2.2), in which case $C = C(x)$ (that is, the factor C is independent of y). The respective indicatrix curvature value $\mathcal{C}_{\text{Ind.}}$ is given by

$$\mathcal{C}_{\text{Ind.}} = 1 - C. \tag{2.3}$$

Next, in the dimension $N = 3$ the tensor W_{ijmn} vanishes identically and, therefore, the equality

$$S_{ijmn} = L(h_{im}h_{jn} - h_{in}h_{jm}) \quad \text{with} \quad L = \frac{1}{2}\breve{S} \tag{2.4}$$

holds, where L may depend on y. In terms of the tensor $C_{im} = (S_{im} - (\breve{S}/4)g_{im})/F^2$ of the Cotton–York type, the tangent Riemannian space $\mathcal{R}_{\{x\}}$ of a three-dimensional Finsler space is conformally flat if and only if the vanishing

$$\mathcal{S}_n C_{im} - \mathcal{S}_m C_{in} = 0 \tag{2.5}$$

holds, where $\mathcal{S}$ denotes the Riemannian covariant derivative operative in the space $\mathcal{R}_{\{x\}}$. Noting that (2.4) entails $S_{im} = Lh_{im}$, denoting $L_n = \partial L/\partial y^n$, and taking into account the property $\mathcal{S}_n g_{im} = 0$, we obtain the equality

$$\mathcal{S}_n C_{im} - \mathcal{S}_m C_{in} = \frac{1}{F^2}\left(L_n\left(h_{im} - \frac{1}{2}g_{im}\right) - L_m\left(h_{in} - \frac{1}{2}g_{in}\right)\right), \tag{2.6}$$

whence (2.5) holds iff $L_n = 0$, that is when $L = L(x)$ (more detail can be found in Appendix B in [12]).

These observations are summed up in

Proposition 2.1. *In an arbitrary Finsler space of any dimension $N \geq 3$ the tangent Riemannian space $\mathcal{R}_{\{x\}}$ is conformally flat if and only if the indicatrix $\mathcal{I}_x$ is a space of constant curvature.*

The question arises: What is the form of the conformal multiplier in the space $\mathcal{R}_{\{x\}}$ under study? To deduce the required conclusions we can start with the tensor $u_{ij} = z(x,y)(c_1(x))^{-2}F^{-2a(x)}g_{ij}$, where z is a smooth positive function to be tested relative to dependence on y and $\{c_1(x),\ a(x)\}$ are positive functions; the inequality $1 > a(x) > 0$ is implied.

Denoting $u_{ijk} = \partial u_{ij}/\partial y^k$, we construct the coefficients $Z_{ijk} = (u_{kji} + u_{iki} - u_{ijk})/2$ and, then, the respective Christoffel symbols $Z^m{}_{ij} = u^{mh}Z_{ijh}$, where the components u^{mh} are reciprocal to the u_{mh}, namely $u^{mh} = (1/z)F^{2a}g^{mh}(c_1)^2$. In this way we obtain merely

$$Z^m{}_{ij} = \left(-\frac{a}{F}l_i + \frac{1}{2z}z_i\right)\delta^m_j + \left(-\frac{a}{F}l_j + \frac{1}{2z}z_j\right)\delta^m_i - \left(-\frac{a}{F}l^m + \frac{1}{2z}g^{mk}z_k\right)g_{ij} + C^m{}_{ij}, \tag{2.7}$$

where $z_i = \partial z/\partial y^i$. With these coefficients, we are able to evaluate the respective curvature tensor

$$\widetilde{R}_n{}^m{}_{ij} = \frac{\partial Z^m{}_{ni}}{\partial y^j} - \frac{\partial Z^m{}_{nj}}{\partial y^i} + Z^h{}_{ni}Z^m{}_{hj} - Z^h{}_{nj}Z^m{}_{hi}. \tag{2.8}$$

This tensor vanishes iff the scalar p^2 with $p = \left[z(x,y)(c_1(x))^{-2}F^{-2a(x)}\right]^{-1/2}$ is indeed the conformal multiplier. From $\widetilde{R}_n{}^m{}_{ij} = 0$ we can obtain some expression for the tensor S_{nmij}. Assuming the zero-degree homogeneity of the function $z(x,y)$ with respect to the argument y, which entails the identity $z_il^i = 0$, and considering the implications of the identity $S_{nmij}l^ml^j = 0$, we arrive at the conclusion that $\widetilde{R}_n{}^m{}_{ij} = 0$ is equivalent to the representation

$$\begin{aligned} S_{nmij} = {} & a(2-a)(h_{nj}h_{mi} - h_{ni}h_{mj}) + F^2\frac{1}{2z^2}\big(z_hg^{hs}z_s\big)(h_{nj}h_{mi} - h_{ni}h_{mj}) \\ & + \frac{a-1}{2z}\Big(z_n(l_ih_{mj} - l_jh_{mi}) - z_m(l_ih_{nj} - l_jh_{ni}) \\ & + l_n(z_ih_{mj} - z_jh_{mi}) - l_m(z_ih_{jn} - z_jh_{in})\Big)F \end{aligned} \tag{2.9}$$

(all the involved evaluations have been explicitly presented in Appendix C in [12]).

In Finsler geometry the tensor S_{nmij} possesses the property $S_{nmij}l^i = 0$. The above representation (2.9) shows that we meet the property iff we fulfill the equation $(a-1)(z_n h_{mj} - z_m h_{nj}) = 0$. Noting $a \neq 1$, we obtain $z_n = 0$, which means that the function z is independent of y. Without any loss of generality we can take $z = 1$.

Thus we have proved the following proposition.

Proposition 2.2. *If the tangent Riemannian space $\mathcal{R}_{\{x\}}$ is conformally flat, then the dependence of the conformal multiplier on the variable y is presented by the power of the Finsler metric function F, such that*

$$g_{\{x\}}(y) = p^2 u_{\{x\}}(y), \quad p = c_1(x)\,(F(x,y))^{a(x)}, \quad c_1(x) > 0, \tag{2.10}$$

with a curvatureless tensor $u_{\{x\}}(y)$, where the positive zeroth degree homogeneity of the conformal multiplier on the variable y and the inequality $1 > a(x) > 0$ were preassigned.

Since the curvature tensor $\widetilde{R}_n{}^m{}_{ij}$ constructed from the tensor $u_{\{x\}}(y) = \{u_{mn}(x,y)\}$ in accordance with the rule (2.8) vanishes identically, there must exist the transformation represented locally by means of the functions

$$\bar{y}^i = y^i(x,y) \tag{2.11}$$

upon which the tensor u_{mn} becomes a Euclidean metric tensor in each tangent Riemannian space $\mathcal{R}_{\{x\}}$. That is, the equality

$$u_{mn} = a_{ij}\bar{y}^i_m \bar{y}^j_n \tag{2.12}$$

must be valid, where $\bar{y}^i_n = \partial \bar{y}^i / \partial y^n$ and $a_{mn} = a_{mn}(x)$.

It is natural to assume that the functions (2.11) are positively homogeneous with respect to the argument y. From (2.10) it follows that the tensor u_{kh} is homogeneous of the degree $(-2a)$ with respect to y. Therefore, the homogeneity degree H of the transformation (2.11) must be given by $H = 1 - a$. Thus,

$$\bar{y}^i(x, ky) = k^H \bar{y}^i(x,y), \qquad k > 0. \tag{2.13}$$

With $z = 1$ the above representation (2.9) reduces to
$S_{nmij} = a(2-a)(h_{nj}h_{mi} - h_{ni}h_{mj})$. Recollecting (2.3), we just obtain $\mathcal{C}_{\text{Ind.}} = 1 - a(2-a) \equiv (1-a)^2$. Since the difference $1-a$ is equal to H, the identification $\mathcal{C}_{\text{Ind.}} = H^2$ is valid.

Making the choice $c_1 = 1/H$, from (2.10) we obtain the equality

$$S(x,\bar{y}) = (F(x,y))^{H(x)} \tag{2.14}$$

which complies with the correspondence of the indicatrix to the Euclidean sphere under the transformation (2.11); $S(x,\bar{y}) = \sqrt{a_{mn}(x)\bar{y}^m\bar{y}^n}$. Then, from (2.10) we may conclude that the conformal multiplier p^2 is constructed from the Finsler metric function F according to

$$p = \frac{1}{H(x)} \left(F(x,y)\right)^{1-H(x)}. \tag{2.15}$$

We take $1 > H > 0$ for definiteness, the extension of the approach to other values of H being a straightforward task.

The transformation (2.11) gives rise to the deformation

$$y = \mathbf{C}(x,\bar{y}), \quad y,\bar{y} \in T_xM. \tag{2.16}$$

Introducing also the *deformation tensor*

$$C^i_m = p\bar{y}^i_m, \tag{2.17}$$

we can conclude that the following important property is valid:

$$\mathcal{F}^N = \mathbf{C}\cdot\mathcal{R}^N : g_{mn} = C^i_m C^j_n a_{ij} \tag{2.18}$$

(use the transformation (2.12) together with the equality $g_{mn} = p^2 u_{mn}$ ensued from (2.10)). The zero-degree homogeneity

$$C^i_m(x,ky) = C^i_m(x,y), \qquad k > 0, \tag{2.19}$$

holds (for any admissible y), together with the identity $C^i_m(x,y)y^m = (F(x,y))^{1-H}\bar{y}^i$. The deformation is unholonomic:

$$\frac{\partial C^i_m}{\partial y^n} - \frac{\partial C^i_n}{\partial y^m} \neq 0. \tag{2.20}$$

The vanishing appears if only the factor $p = F^{1-H}/H$ is independent of the vectors y, that is, when $H = 1$ (which is the proper Riemannian case).

The converse transformation

$$\bar{y} = \mathbf{C}^{-1}(x,y) : t^i = t^i(x,y), \quad t^n \equiv \bar{y}^n, \tag{2.21}$$

is $(1/H)$-homogeneous, so that $y^i(x,kt) = k^{1/H}y^i(x,t)$ with $k > 0$ (for any admissible t). The identity $y^i_n t^n = (1/H)y^i$ holds, where $y^i_n = \partial y^i/\partial t^n$. We obtain the reciprocal deformation tensor $\widetilde{C}^n_m = (1/p)y^n_m$, so that $\widetilde{C}^n_i C^i_m = \delta^n_m$, with C^i_m introduced in (2.17).

The respective two-vector angle $\alpha_{\{x\}}(y_1, y_2)$ proves to be obtainable from the angle $\alpha^{\text{Riem}}_{\{x\}}(y_1, y_2)$ operative in the associated Riemannian space $\mathcal{R}^N = (M, S)$, namely the simple equality

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{H(x)} \alpha^{\text{Riem}}_{\{x\}}(\bar{y}_1, \bar{y}_2) \tag{2.22}$$

(see (II.2.51)–(II.2.52) in [12], or (2.27)–(2.28) in [11]) is valid. To give reasons for this equality, let us denote by $l^i = y^i/F(x, y)$ and $L^i = t^i/S(x, t)$ the components of the Finslerian and Riemannian unit vectors $l = \{l^i\}$ and $L = \{L^i\}$, which respectively possess the properties $F(x, l) = 1$ and $S(x, L) = 1$. Since in virtue of the equality (2.14) the indicatrices of the spaces $\mathcal{F}^N = (M, F)$ and $\mathcal{R}^N = (M, S)$ are in correspondence under the deformation $\mathbf{C}$, we may apply the transformation (2.16) to the unit vectors: $l = \mathbf{C} \cdot L : l^i = y^i(x, L)$. On the other hand, from (2.10) and (2.12) it follows that $g_{mn}(x, l) = (1/H(x))^2 a_{ij}(x) t^i_m(x, l) t^j_n(x, l)$. Therefore, under the transformation $l = \mathbf{C} \cdot L$ we have $g_{mn}(x, l) dl^m dl^n = (1/H(x))^2 a_{ij}(x) dL^i dL^j$. No support vector enters the right-hand part of the last equality, whence (2.22) is valid.

The definition

$$U^i \stackrel{\text{def}}{=} \frac{1}{S} \bar{y}^i \equiv \frac{1}{F^H} \bar{y}^i \tag{2.23}$$

introduces the normalized vector, which is obviously unit: $U_i U^i = 1$, and $U_i = a_{ij} U^j$. The zero-degree homogeneity $U^i(x, ky) = U^i(x, y)$ with $k > 0$ holds (for any admissible t), entailing the identity $U^i_n y^n = 0$ with

$$U^i_n = \frac{\partial U^i}{\partial y^n} = \frac{1}{F^H} t^i_n - \frac{1}{F} H U^i l_n, \tag{2.24}$$

where $t^i_n = \partial t^i / \partial y^n$. It follows that

$$F^H U^h_s y^k_h = h^k_s, \qquad F^H U^i_k y^k_t = \delta^i_t - U^i U_t, \qquad U_i U^i_n = 0. \tag{2.25}$$

The identity

$$U_i \left(\frac{\partial U^i}{\partial x^n} + L^i{}_{kn} U^k \right) = 0 \tag{2.26}$$

is obviously valid, where $L^i{}_{nk}$ are the Riemannian connection coefficients (1.12).

The representation (2.22) of the angle takes on the simple form

$$\alpha_{\{x\}}(y_1, y_2) = \frac{1}{H(x)} \arccos \lambda, \quad \text{with} \quad \lambda = a_{mn}(x) U^m_1 U^n_2, \tag{2.27}$$

where $U_1^m = U^m(x, y_1)$ and $U_2^m = U^m(x, y_2)$.

When the recurrent preservation $d_i\alpha + (1/H)H_i\alpha = 0$ proposed by (1.28) is applied to the angle shown in (2.27), we obtain simply

$$d_i\lambda = 0, \tag{2.28}$$

where d_i is the separable operator (1.9). That is, the recurrent preservation law formulated for the Finsler $\mathcal{F}^N$*-space angle* $\alpha_{\{x\}}$ given by (2.27) is tantamount to the preservation law for the Euclidean angle $\alpha_{\{x\}}^{\mathrm{Riem}} = \arccos\lambda$, whence to the separable preservation law (2.28).

The right-hand part in the formula $\lambda = a_{mn}(x)U_1^m U_2^n$ is such that the law $d_i\lambda = 0$ gets valid provided we impose the condition

$$\mathcal{D}_n U^i = 0 \tag{2.29}$$

on field $U^i = U^i(x, y)$, where we introduced the covariant derivative

$$\mathcal{D}_n U^i = d_n U^i + L^i{}_{nk} U^k. \tag{2.30}$$

Since $d_n U^i = \partial U^i/\partial x^n + N^k{}_n U_k^i$, we can arrive at the conclusion that in the $\mathcal{F}^N$-space the coefficients $N^m{}_n$ can unambiguously be found from the equation $d_n\left(H(x)\alpha_{\{x\}}(y_1, y_2)\right) = 0$ to be explicitly given by the representation

$$N^m{}_n = -y_i^m F^H\left(\frac{H}{F}U^i\frac{\partial F}{\partial x^n} + \frac{\partial U^i}{\partial x^n} + \left(a^i{}_{nk} + S^i{}_{nk}\right)U^k\right) + l^m d_n F \tag{2.31}$$

(we refer to (II.3.12) in [12]). Here, the equality $L^m{}_{ij} = a^m{}_{ij} + S^m{}_{ij}$ indicated in (1.12) has been used.

Whenever $d_n F = 0$, the representation (2.31) takes on the form

$$N^m{}_n = -l^m\frac{\partial F}{\partial x^n} - y_i^m F^H\left(\frac{\partial U^i}{\partial x^n} + \left(a^i{}_{nk} + S^i{}_{nk}\right)U^k\right), \tag{2.32}$$

where the identity $y_n^i t^n = (1/H)y^i$ indicated below (2.21) has been taken into account. These coefficients $N^m{}_n$ present the *general solution* to the equations $d_n\left(H(x)\alpha_{\{x\}}(y_1, y_2)\right) = 0$ and $d_n F = 0$, so that no problem of uniqueness of connection coefficients may be questioned. The torsion tensor $S^i{}_{kn}$ is the only freedom in the right-hand part of (2.32).

The evaluations performed in Section II.3 of [12] have led us also to the representation

$$N^m{}_n = d_n^{\mathrm{Riem}} y^m(x, t) + \frac{1}{H}H_n y^m \ln F \tag{2.33}$$

(this is the formula (II.3.29) in [12]) which is alternative to (2.32); here, $y^m = y^m(x,t)$ are the functions (2.16).

The representations (2.31)–(2.33) involve the gradient H_n and are applicable to any indicatrix-inhomogeneous Finsler space $\mathcal{F}^N$.

The coefficients $N^k{}_{mn}$ can be evaluated from (2.32) to explcitly read

$$N^k{}_{mn} = -\frac{1}{F}h^k_n\frac{\partial F}{\partial x^m} - l^k\frac{\partial l_n}{\partial x^m} - C^k{}_{ns}N^s{}_m + \frac{1}{F}\left(l_n h^k_s - (1-H)l^k h_{ns}\right)N^s{}_m - y^k_h F^H\left(\frac{\partial U^h_n}{\partial x^m} + L^h{}_{ms}U^s_n\right) \quad (2.34)$$

(this was the content of Proposition II.3.4 in [12]). With these coefficients, the validity of the representation (1.30) for the entailed coefficients $N^k{}_{imn}$ can be verified (we address the reader to Proposition II.3.5 in [12]).

3. Application to Finsleroid space

Among possible metric functions $F(x,y)$ of the Finsler space $\mathcal{F}^N$ there is the remarkable example, to be denoted by $K(x,y)$, which reveals the following important properties: the indicatrix is closed and axially symmetric, and Finsler metric tensor is positive-definite.

Below, we make the notation change $H(x) \to h(x)$.

The scalar $g(x)$ obtained through

$$h(x) = \sqrt{1 - \frac{g^2(x)}{4}}, \quad \text{with} \quad -2 < g(x) < 2, \quad (3.1)$$

plays the role of the characteristic parameter. It follows that $g_i = -(4/h)gh_i$, where $g_i = \partial g/\partial x^i$ and $h_i = \partial h/\partial x^i$.

The Finsleroid space (M,K) is constructed starting with a Riemannian space (M,S), where $S = \sqrt{a_{ij}(x)y^iy^j}$ is the Riemannian metric function and $a_{ij}(x)$ is a positive-definite Riemannian metric tensor. Namely, we assume that in addition to a Riemannian metric $\sqrt{a_{ij}(x)y^iy^j}$ the manifold M admits a non-vanishing 1-form $b = b_i(x)y^i$ of the unit length: $a_{ij}(x)b^i(x)b^j(x) = 1$, where $b^i(x) = a^{ij}(x)b_j(x)$. The tensor $a^{ij}(x)$ is reciprocal to $a_{ij}(x)$, so that $a_{ij}a^{jn} = \delta^n_i$, where δ^n_i stands for the Kronecker symbol. We need also the quadratic form

$$B = b^2 + gbq + q^2 \equiv \left(b + \frac{1}{2}gq\right)^2 + h^2q^2, \quad (3.2)$$

where $q=\sqrt{r_{mn}y^my^n}$ with $r_{mn}=a_{mn}-b_mb_n$, so that $a_{ij}(x)y^iy^j=b^2+q^2$.

We shall also use the scalar

$$\chi=\frac{1}{h}\Bigl(-\arctan\frac{G}{2}+\arctan\frac{L}{hb}\Bigr),\qquad \text{if }\ b\ge 0;$$
$$\chi=\frac{1}{h}\Bigl(\pi-\arctan\frac{G}{2}+\arctan\frac{L}{hb}\Bigr),\quad \text{if }\ b\le 0, \tag{3.3}$$

with the function $L=q+(g/2)b$ fulfilling the identity $L^2+h^2b^2=B$.

The definition range $0\le\chi\le\pi/h$ is sufficient to describe all the tangent space. The normalization in (3.3) is such that $\chi|_{y=b}=0$. The quantity (3.3) can conveniently be written as $\chi=f/h$ with the function $f=\arccos(A(x,y)/\sqrt{B(x,y)})$, where $A=b+(1/2)gq$, ranging as follows: $0\le f\le\pi$. The Finsleroid-axis vector b^i relates to the value $f=0$, and the opposed vector $-b^i$ relates to the value $f=\pi$:

$$f=0\ \sim\ y=b;\qquad f=\pi\ \sim\ y=-b. \tag{3.4}$$

With these ingredients, we construct the Finsler metric function

$$K=\sqrt{B}\,J,\quad\text{where}\quad J=\mathrm{e}^{-\frac{1}{2}g\chi}. \tag{3.5}$$

The normalization is such that $K(x,b(x))=1$ (notice that $q=0$ at $y^i=b^i$). The positive (not absolute) homogeneity holds: $K(x,\gamma y)=\gamma K(x,y)$ for any $\gamma>0$ and all admissible (x,y).

The entailed components $y_i=(1/2)\partial K^2/\partial y^i$ of the covariant tangent vector $\hat{y}=\{y_i\}$ can be found in the simple form $y_i=\bigl(a_{ij}y^j+gqb_i\bigr)\,J^2$.

The determinant of the respective Finslerian metric tensor $g_{ij}=\partial y_i/\partial y^j$ is given by the formula $\det(g_{ij})=J^{2N}\det(a_{ij})$ and, therefore, is everywhere positive.

The remarkable property $A^iA_i=N^2g^2/4$ is valid, where $A_i=KC_i{}^j{}_j$, so that the contraction A^iA_i is independent of vectors $y\in T_xM$.

Within any tangent space T_xM, the indicatrix $\mathcal{IF}^{PD}_{g;\{x\}}=\{y\in\mathcal{FF}^{PD}_{g;\{x\}}:y\in T_xM,K(x,y)=1\}$ bounds the convex body $\mathcal{FF}^{PD}_{g;\{x\}}=\{y\in\mathcal{FF}^{PD}_{g;\{x\}}:y\in T_xM,K(x,y)\le 1\}$ around the origin $0\in T_xM$. This body extends the Riemannian notion of unit ball. We call the body the *Finsleroid.* The direction of the vector $b^i(x)$ in the tangent space T_xM has the clear geometrical meaning of the axis of the Finsleroid $\mathcal{FF}^{PD}_{g;\{x\}}\subset T_xM$. The Finsleroid can be regarded as rotund around this direction. The Finsleroid is *not* symmetric under reflection through the origin $0\in T_xM$, for we have $K(x,-y)\ne K(x,y)$ in general.

We call $K(x, y)$ the $\mathcal{FF}_g^{PD}$-*Finsleroid metric function*, obtaining the $\mathcal{FF}_g^{PD}$-*Finsler space* $\mathcal{FF}_g^{PD} = \{M;\ a_{ij}(x);\ b_i(x);\ g(x);\ K(x,y)\}$ which we call the *Finsleroid space* for short. The upperscript "PD" means "positive-definite".

On the punctured tangent bundle $TM \setminus 0$, the metric function K is smooth globally of the class C^2 regarding the y-dependence and the Finsleroid metric tensor g_{ij} is positive definite. Because of the identity $r_{mn}b^n = 0$, the function is not of the class C^3. Indeed, the scalar $q = \sqrt{r_{mn}y^m y^n}$ is zero when $y = b$ or $y = -b$, that is, in the directions of the north pole or the south pole of the Finsleroid. The third derivatives of the function $K(x, y)$ with respect to y involve the fraction $1/q$ which gives rise to the *pole singularities* when $q = 0$.

On the b-slit tangent bundle $\mathcal{T}_b M = TM \setminus 0 \setminus b \setminus -b$ (obtained by deleting out in $TM \setminus 0$ all the directions which point along, or oppose, the directions given rise to by the 1-form b), the function K is smooth of the class C^∞ regarding the y-dependence.

The metric function K has been first appeared in the paper [14] in which a broad class of Finsler metrics whose indicatrices are spaces of constant curvature has been found. The consideration in the paper was referred to a fixed tangent space. In [15], the metric function K was used to geometrize the tangent bundle of a smooth manifold in a positive-definite way. The terminology "Finsleroid" has been introduced in the work [16], in which the two-vector angle has been found in process of investigation of appropriate geodesics. In [16] the angle was given by the representation which completely agrees with the representation

$$\alpha_{\{x\}}^{\text{Finsleroid}}(y_1, y_2) = \frac{1}{h(x)} \alpha_{\{x\}}^{\text{Riem}}(\bar{y}_1, \bar{y}_2)$$

coming from the formula (2.22) derived in the present paper.

Now, we are able to elucidate the structure of the coefficients $N^k{}_m$ in the proper Finsleroid case. According to the formula (6.26) of [7], the quantity $U^i = (1/K^h)\bar{y}^i$ which enters the representation (2.32) of the coefficients can explicitly be given by

$$U^i = \left[hv^i + \left(b + \frac{1}{2} gq \right) b^i \right] \frac{1}{\sqrt{B}}, \tag{3.6}$$

where $v^i = y^i - bb^i$. With the deformation tensor C^i_m evaluated by the help of this field U^i, the validity of the deformation property (2.18) can readily be verified.

From (3.6) we have

$$\frac{\partial U^i}{\partial g} = -\frac{g}{4h} v^i \frac{1}{\sqrt{B}} + \frac{1}{2} q b^i \frac{1}{\sqrt{B}} - \frac{1}{2B} U^i q b,$$

or

$$\frac{\partial U^i}{\partial g}=-\frac{g}{4h^2}U^i+\frac{g}{4h^2}(b+\frac{1}{2}gq)b^i\frac{1}{\sqrt{B}}+\frac{1}{2}qb^i\frac{1}{\sqrt{B}}-\frac{1}{2B}U^iqb.$$

Since $K^hy^m_iU^i=(1/h)y^m$ (a consequence of the homogeneity involved) and

$$K^hy^m_ib^i=\left[b^m+\frac{1}{B}\left(\frac{1}{h}\left(b+\frac{1}{2}gq\right)-b-gq\right)y^m\right]\sqrt{B}$$

(this is the formula (D.12) of [7]), we can evaluate the contraction

$$\begin{aligned}K^hy^m_i\frac{\partial U^i}{\partial g}&=-\frac{g}{4h^2}\frac{1}{h}y^m-\frac{1}{2B}\frac{1}{h}qby^m+\frac{g}{4h^2}\left(b+\frac{1}{2}gq\right)b^m\\&+\frac{g}{4h^2}\frac{1}{B}\frac{1}{h}\left(b+\frac{1}{2}gq\right)^2y^m-\frac{g}{4h^2}\frac{1}{B}\left(b+\frac{1}{2}gq\right)(b+gq)y^m\\&+\frac{1}{2}q\left[b^m+\frac{1}{B}\left(\frac{1}{h}\left(b+\frac{1}{2}gq\right)-b-gq\right)y^m\right].\end{aligned}$$

Using the equality (3.2) together with the representation

$$A^m=\frac{N}{2}g\frac{1}{qK}\left[q^2b^m-(b+gq)v^m\right]\equiv KC^{mn}{}_n$$

(indicated by (A.27) in [7]), we come to

$$K^hy^m_i\frac{\partial U^i}{\partial g}=\frac{1}{h^2}\frac{q}{B}\left(q+\frac{1}{2}gb\right)\frac{K}{Ng}A^m. \tag{3.7}$$

Therefore, in the Finsleroid case the object $\{N^k{}_i\}$ proposed by (2.32) is the sum

$$N^k{}_i=N^{\mathrm{I}k}{}_i+\breve{N}^k{}_i,\qquad \breve{N}^k{}_i=\breve{N}^kg_i, \tag{3.8}$$

where

$$\breve{N}^k=-\frac{1}{h^2}\frac{q}{B}\left(q+\frac{1}{2}gb\right)\frac{K}{Ng}A^k-\frac{1}{2}\bar{M}y^k \tag{3.9}$$

with $\bar{M}$ coming from $\partial K^2/\partial g=\bar{M}K^2$. The torsion tensor $S^k{}_{ij}=S^k{}_{ij}(x)$ has been neglected. The $N^{\mathrm{I}k}{}_i$ are the coefficients (6.53) of [7] (they can also be found in [10], [11]), namely,

$$\begin{aligned}N^{\mathrm{I}k}{}_i=&\left[\left(b-\frac{1}{h}\left(b+\frac{1}{2}gq\right)\right)\eta^{kj}\right.\\&\left.+\left(\frac{1}{q^2}v^k\left(b-\frac{1}{h}(b+gq)\right)+\left(\frac{1}{h}-1\right)b^k\right)y^j\right]\nabla_ib_j-a^k{}_{ij}y^j,\end{aligned} \tag{3.10}$$

where $\eta^{kn} = a^{kn} - b^k b^n - (1/q^2) v^k v^n$, so that $y_k \eta^{kn} = b_k \eta^{kn} = 0$. The designation ∇_i stands for the Riemannian covariant derivative constructed with the help of the Riemannian Christoffel symbols $a^k{}_{ij} = a^k{}_{ij}(x)$ appeared in the background Riemannian space (M, S), so that $\nabla_i b_j = \partial b_j / \partial x^i - a^k{}_{ij} b_k$. The coefficients (3.10) don't involve the gradient g_i.

The $N^{\mathrm{I}k}{}_i$ are the coefficients $N^k{}_i$ obtained when the condition $h = \mathrm{const}$ which specifies the indicatrix-homogeneous case is postulated.

The coefficients $N^{\mathrm{I}k}{}_{im} = \partial N^{\mathrm{I}k}{}_i / \partial y^m$ are known from [7], namely

$$N^{\mathrm{I}k}{}_{im} = -\Big((1-h)b_m + \frac{g}{2q} v_m\Big) \frac{1}{h} a^{kj} \nabla_i b_j - \frac{g}{2qh} \eta^k_m y^j \nabla_i b_j \\ - \left(\frac{g}{2q} v^k - (1-h) b^k\right) \frac{1}{h} \nabla_i b_m - a^k{}_{nm} \tag{3.11}$$

(see (6.49) in [7]). For the coefficients $\breve{N}^k{}_{im} = \partial \breve{N}^k{}_i / \partial y^m$ the representation

$$\breve{N}^k{}_{im} = \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q} - 2h^2\right) \frac{2}{Ng} A_m l^k + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(1 + \frac{1}{2} g \frac{b}{q}\right) \left(\frac{b}{q} + g\right) h^k_m \\ + \frac{1}{h^2} g_i \frac{q^2}{2B} \left(\frac{b}{q} + \frac{1}{2} g\right) \frac{2}{Ng} \frac{2}{Ng} A_m A^k - \frac{1}{2} g_i \bar{M} h^k_m + \frac{1}{K} l_m \breve{N}^k{}_i \tag{3.12}$$

is obtained (which was shown in Appendix A in [12]).

Thus the full coefficients $N^k{}_{im} = N^{\mathrm{I}k}{}_{im} + \breve{N}^k{}_{im}$ are completely and explicitly known.

Using (3.12) we find straightforwardly that

$$y_k \frac{\partial^2 \breve{N}^k{}_i}{\partial y^m \partial y^n} = \frac{2}{h} h_i h_{mn}. \tag{3.13}$$

For the coefficients $\breve{N}^k{}_{imn} = \partial \breve{N}^k{}_{im} / \partial y^n$ the representation

$$\breve{N}^k{}_{imn} = -\frac{g}{2h^2} g_i \frac{1}{K} h_{mn} l^k - \frac{1}{gh^2} g_i \frac{1}{K} A^k{}_{mn} \tag{3.14}$$

can explicitly be derived, where $A^k{}_{mn} = K C^k{}_{mn}$. After that, it is possible to evaluate the sum $N^k{}_{imn} = N^{\mathrm{I}k}{}_{imn} + \breve{N}^k{}_{imn}$, where $N^{\mathrm{I}k}{}_{imn} = \partial N^{\mathrm{I}k}{}_{im} / \partial y^n$. The result reads simply

$$N^k{}_{imn} = \frac{2}{h} h_i \frac{1}{K} l^k h_{mn} - \frac{1}{K} \mathcal{D}_i A^k{}_{mn} \tag{3.15}$$

(we address the reader to Appendix A in [12]). This method establishes the validity of the representation (1.30) in the proper Finsleroid case.

4. Conclusions: Problem solved

It is possible to meet the opinion that, in contrast to the Riemannian case, in the Finsler geometry there is no canonical connection, so various Finsler connections should be developed on the axiomatic tensorial level. At the same time, in the Riemannian geometry the canonical connection, called ordinarily the Levi Civita connection, is a straightforward implication from the angle, so the following problem deserves the attentive consideration.

Connection Problem. Generate the Finsler connection from the two-vector angle.

The definition of the two-vector angle $\alpha_{\{x\}}(y_1, y_2)$ by the help of the length of the geodesic arc on the indicatrix seems to be the most natural Finslerian generalization of the angle introduced in Riemannian spaces by the help of the Euclidean notion of angular measure. Like to Riemannian case, in Finsler space this length, whence the angle $\alpha_{\{x\}}(y_1, y_2)$, is free of any vector of support. Actually, the refined definition for this angle has been formulated in the beginning of the section "7. Definitions of Angle" of the book [1]. However, applications of such an angle to Finsler spaces have not been developed. The main difficulty is due to the loss of possibility to have a tensorial representation for the angle. "A finite angle will be obtained by integrating the expression (7.2) over a finite arc" (p. 31 of [1]), where (7.2) was the representation of the infinitesimal piece of the angle. One should deal with an integral measure of the angle. Whence, in the respect of applications to various wide classes of Finsler spaces, such an angle is complicated and implicit object. In Riemannian space, the integration can readily be performed, yielding simply $\alpha^{\text{Riem}}_{\{x\}}(y_1, y_2) = a_{mn}(x) y_1^m y_2^n / S_1 S_2$, where $S_1 = \sqrt{a_{mn}(x) y_1^m y_1^n}$ and $S_2 = \sqrt{a_{mn}(x) y_2^m y_2^n}$.

The essential simplicity can be met at in particular Finsler spaces. In the $\mathcal{F}^N$-space the angle is expressible in the simple tensorial form $\alpha_{\{x\}}(y_1, y_2) = \big(1/H(x)\big) \alpha^{\text{Riem}}_{\{x\}}(\bar{y}_1, \bar{y}_2)$. This observation has permitted us to obtain a complete solution of the above Connection Problem in the $\mathcal{F}^N$-space of an arbitrary dimension $N \geq 3$, assuming that the indicatrix curvature value $\mathcal{C}_{\text{Ind.}}$ belongs to the range $(0, 1)$. The treatment was local in both the base manifold and the tangent space. The smoothness of the class C^2 in the base manifold and of the class C^5 in the tangent space was implied. The obtained representations (2.32) and (1.34) for the coefficients $N^m{}_n$ and $T^k{}_{im}$ completely determine the connection $\mathcal{F}N$ and then the total covariant derivative $\mathcal{T}$. The length preservation $\mathcal{T}_i F = 0$ and the metricity $\mathcal{T}_i g_{mn} = 0$ are keeping fine. The connection is uniquely determined up to the torsion tensor of the associated Riemannian space.

Can the obtained connection $\mathcal{F}N$ be regarded as being *canonical* for the Finsler space $\mathcal{F}^N$?

In Section 2 the connection $\mathcal{F}N$ was obtained by means of solving the separable preservation law of the normalized angle. It is instructive to follow the *converse method*, basing on the stipulation that the sought Finslerian connection be the deformation of the Riemannian linear connection. The method involves several important steps.

First of all, it is *intuitively* obvious that due to the conformally flat nature of the tangent Riemannian spaces $\mathcal{R}_{\{x\}}$, the space $\mathcal{F}^N$ induces a Riemannian metric tensor $a_{ij}(x)$ on the background manifold M, which in turns should induce the angle from the Riemannian space $(M, a_{ij}(x))$ in the space $\mathcal{F}^N$. There arises the idea that the connection $\mathcal{F}N$ comes in the $\mathcal{F}^N$-space by following the same path. What is the Riemannian image of the connection $\mathcal{F}N$ in the space $(M, a_{ij}(x))$? Maybe the metrical linear Riemannian connection $\mathcal{R}L$ of the space $(M, a_{ij}(x))$, namely, $\mathcal{R}L = \{L^m{}_j, L^m{}_{ij}\}$ where $L^m{}_j = -L^m{}_{ji}y^i$ and $L^m{}_{ij} = L^m{}_{ij}(x)$ are the coefficients which are shown in (1.12)? Let us adopt the last possibility to proceed.

We have seen in Section 2 that the Finsler space $\mathcal{F}^N$ under study is obtained from the Riemannian space $\mathcal{R}^N$ by means of the deformation $y = \mathbf{C}(x, \bar{y})$, which properties were listed in (2.17)–(2.20). The Finsler angle α can be regarded as the result of this deformation, namely $\alpha = \mathbf{C} \cdot \alpha^{\mathrm{Riem}}$, where α means the angle $\alpha_{\{x\}}(y_1, y_2) = (1/H(x))\alpha^{\mathrm{Riem}}_{\{x\}}(\bar{y}_1, \bar{y}_2)$.

Let us assume that the sought connection $\mathcal{F}N = \{N^m{}_i, T^m{}_{ij}\}$ for the space $\mathcal{F}^N$ is also produced by such a deformation, namely $\mathcal{F}N = \mathbf{C} \cdot \mathcal{R}L$. Denoting by $\mathcal{T}$ the covariant derivative which is constructed with the help of the connection coefficients $N^m{}_i$ and $T^m{}_{ij}$, we can set forth the natural requirement that the $\mathbf{C}$-deformation be $\mathcal{T}$-*covariant constant*, that is, $\mathcal{T} \cdot \mathbf{C} = 0$.

In terms of local coordinates the last condition reads

$$\mathcal{T}_n C^m_k = 0, \tag{4.1}$$

where

$$\mathcal{T}_n C^m_k = d_n C^m_k - T^h{}_{nk} C^m_h + L^m{}_{nl} C^l_k. \tag{4.2}$$

Here, C^m_k is the deformation tensor introduced in (2.17).

As long as this condition is valid, from the relation $g_{mn} = C^i_m C^j_n a_{ij}$ (indicated in (2.18)) it ensues that the metricity $\mathcal{T}_n g_{ij} = 0$ holds in the space $\mathcal{F}^N$ because of the Riemannian metricity

$$\frac{\partial a_{ij}}{\partial x^n} - L^h{}_{in} a_{hj} - L^h{}_{jn} a_{hi} = 0.$$

With the help of the tensor $\widetilde{C}^n_m = (1/p)y^n_m$ introduced below (2.21) we can solve the above equation (4.1) with respect to the coefficients $T^h{}_{nk}$, obtaining

$$T^h{}_{nk} = \widetilde{C}^h_j \left(d_n C^j_k + L^j{}_{nl} C^l_k \right). \tag{4.3}$$

Here, d_n is the operator (1.2). Finally, we set forth the natural assumption that $\mathcal{T} y = 0$ (this reads $N^h{}_k + T^h{}_{nk} y^k = 0$ with respect to local coordinates) and contract (4.3) by y^k. The result is simple $d_n \left(F^{1-H} \bar{y}^j \right) + F^{1-H} L^j{}_{nl} \bar{y}^l = 0$, where the identity $C^i_m(x,y) y^m = (F(x,y))^{1-H} \, \bar{y}^i$ indicated below (2.19) has been taken into account. The two nullifications $\mathcal{T} y = 0$ and $\mathcal{T}_n g_{ij} = 0$ obviously entail $d_n F = 0$. Whence we have

$$d_n \bar{y}^j + L^j{}_{nl} \bar{y}^l = H_n \bar{y}^j \ln F. \tag{4.4}$$

Using here (2.23) together with the first member of (2.25) we are led to the conclusion that the equality (4.4) is equivalent to the representation (2.32) for the coefficients $N^m{}_n$. If we differentiate (4.4) with respect to y^k to obtain the object $d_n \bar{y}^j_k$ and, then, insert the object in (4.3), we just find the representation (1.34) for the connection coefficients $T^k{}_{im}$. So the metrical Finsler connection $\mathcal{F}N$ has been completely determined in the space $\mathcal{F}^N$. On inserting the obtained coefficients $N^m{}_n$ in the separable equation (1.27) of preservation of the normalized angle, it is easy to observe that the equation is fulfilled. Thus the converse method works fine!

If we apply the developed theory to the Finsleroid space, we obtain the metric connection of the smoothness class C^∞ regarding the y-dependence on all the b-slit tangent bundle $\mathcal{T}_b M = TM \setminus 0 \setminus b \setminus -b$.

It is possible to extend the content of Section 4 of the previous indicatrix-homogeneous study [10], [11] to make it possible to perform the comparison between the commutators of the Finsler covariant derivative $\mathcal{T}$ arisen in the space $\mathcal{F}^N$ and the commutators of the Riemannian covariant derivative ∇ introduced in the associated Riemannian space, *not* assuming $H = \text{const}$, such that $H(x)$ can be an arbitrary smooth function of x. In this way, the associated curvature tensor $\rho_k{}^n{}_{ij}$ can be derived. The respective evaluations have been presented in detail in Section II.5 in [12], where various important properties of the tensor have been elucidated.

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