Publ. Math. Debrecen
82/1 (2013), 155–162
DOI: 10.5486/PMD.2013.5351

Projective equivalence between an (α, β)-metric and a Randers metric

By YAOYONG YU (Hangzhou) and YING YOU (Hangzhou)

Abstract. In this paper, we study the projective equivalence between an (α, β)-metric and a Randers metric constructed with conformal Riemannian metrics and different one forms.

1. Introduction

Two Finsler metrics on a manifold are called (pointwise) projectively equivalent if they have the same geodesics as point sets in some neighborhood of any point. The first author and Y. Shen have studied the projective equivalence between two Randers metrics in [10]. Then the authors in [2] have studied the projective equivalence between a quadratic metric and Randers metrics. But for the projective equivalence between a general (α, β)-metric and a Randers metric, it may be difficult. If a Finsler metric F is projective equivalent to a Minkowski metric, then it is called projectively flat. In this case, the geodesics of F is some segments of straight lines as point sets in a local coordinates. Douglas metric is a more general class of Finsler metrics than the projectively flat metrics (See Definition 2.1 below).

(α, β)-metrics are a class of Finsler metrics constructed by some Riemannian metrics and one forms including Randers metric as a spacial case. It is in [5] which used some equations to characterize the (α, β)-metrics of Douglas type. Noticing that a projectively flat Finsler metric must be of Douglas type, we can

Mathematics Subject Classification: 53B40, 53C60.
Key words and phrases: Finsler manifold, projective change, (α, β)-metric, Randers metric.
The second author is the corresponding author.
This paper is supported by NSFC(11026105) and NSFC(10901043).

study the projective equivalence between two (α, β)-metrics using the techniques similar as [5].

The purpose of this paper is to study the projective equivalence between an (α, β)-metric and a Randers metric constructed with conformal Riemannian metrics and different one forms.

2. Preliminaries

For a given Finsler metric $F = F(x, y)$, the geodesics of F statisy the following ODEs:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{1}{4} g^{il}\{[F^2]_{x^m y^l} y^m - [F^2]_{x^l}\}.$$

Definition 2.1. Let

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\Big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\Big), \tag{2.1}$$

where G^i are the spray coefficients of F. The tensor $D := D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

Two Finsler metrics F and $\bar{F}$ are projecively equivalent if and only if in some local coordinates $\{x^i\}$ on M there exists a homogenous function $P(x, y)$ of degree one on $TM \backslash \{0\}$ such that

$$G^i(x, y) = \bar{G}^i(x, y) + P(x, y) y^i, \tag{2.2}$$

where $G^i(x, y)$ and $\bar{G}^i(x, y)$ are the spray coefficients of the corresponding Finsler metrics F and $\bar{F}$.

There is an important class of Finsler metrics called (α, β)-metrics which are defined by a Riemannian metric $\alpha = \sqrt{a_{ij} y^i y^j}$ and a one form $\beta = b_i y^i$. They are expressed in the form

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a positive C^∞ function on $(-b_o, b_o)$ satisfying $\phi(0) = 1$. It can be proved that F is a positive definite Finsler metric for any α and β with $||\beta||_\alpha < b_o$ if and only if ϕ satisfies [14]

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_o). \tag{2.3}$$

If $\phi = 1 + s$, the Finsler metric $F = \alpha + \beta$ is called Randers metric which occurs naturally in Physics and Biology applications.

Let $\nabla\beta = b_{i|j} dx^i \otimes dx^j$ be covariant derivative of β with respect to α.

Denote

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. Let $s_j := b^i s_{ij}$, $s^i_j := a^{il} s_{lj}$, $s_0 = s_i y^i$, $s^i_0 := s^i_j y^j$ and $r_{00} := r_{ij} y^i y^j$.

The geodesic coefficients G^i of F and geodesic coefficients G^i_α of α are related as follows

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1} y^i\}, \tag{2.4}$$

where

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

$$Q = \frac{\phi'}{\phi - s\phi'}, \qquad \Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

It is known that Douglas metrics can be also characterized by the following equations

$$G^i y^j - G^j y^i = \frac{1}{2}(\Gamma^i_{kl} y^j - \Gamma^j_{kl} y^i) y^k y^j.$$

From (2.4) we can easily obtain

$$\alpha Q(s^i_0 y^j - s^j_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i) = \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l. \tag{2.5}$$

The authors in [5] used some equations to characterize the (α, β)-metrics of Douglas type.

Lemma 2.2. *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric satisfying*

(a) *β is not parallel with respect to α;*

(b) *$db \neq 0$ everywhere or $b =$ const.;*

(c) *F is not of Randers type.*

Then F is a Douglas metric if and only if in some local coordinates the following equations hold

$$(1 + (k_1 + k_2 s^2)s^2 + k_3 s^2)\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'), \tag{2.6}$$

$$b_{i|j} = 2\tau[(1 + k_1 b^2)\alpha_{ij} + (k_2 b^2 + k_3) b_i b_j], \tag{2.7}$$

where $b^i := a^{ij} b_j$, $b := \|\beta\|_\alpha$, $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α, $\tau = \tau(x)$ is a scalar function and ξ is a one form, k_1, k_2, k_3 are constants.

Using (2.5), we can give an equivalent condition of two (α, β)-metrics having the same Douglas tensor.

Lemma 2.3. *$F = \alpha\phi(\beta/\alpha)$ and $\bar{F} = \bar{\alpha}\bar{\phi}(\bar{\beta}/\bar{\alpha})$ have the same Douglas tensor iff*

$$\begin{aligned} &\alpha Q(s^i_0 y^j - s^j_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i) \\ &\quad - \left\{\bar{\alpha}\bar{Q}(\bar{s}^i_0 y^j - \bar{s}^j_0 y^i) + \bar{\Psi}(-2\bar{Q}\bar{\alpha}\bar{s}_0 + \bar{r}_{00})(\bar{b}^i y^j - \bar{b}^j y^i)\right\} \\ &= \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l, \end{aligned} \tag{2.8}$$

where G^i_{kl} are scalar functions on M.

Proof. Noticing the definition 2.1, we can also define a Douglas spray rather than a Douglas metric. Let $H^i := G^i - \bar{G}^i$ be coefficients of a 'new' spray. If F and $\bar{F}$ have the same Douglas tensor, then H^i are the coefficients of a Douglas spray. Using (2.5), we get the result (2.8). □

Lemma 2.4. *If ϕ satisfies $\frac{Q(s)-Q(0)}{s} = k = \mathrm{const.}$, then*

$$\phi = Ce^{\int_0^s \frac{\phi'(0)+kt}{1+\phi'(0)t+kt^2} dt}$$

wiht C a positive constant.

Proof. Note that $Q(0) = \phi'(0)$, one can easily verify this lemma. □

3. Two metrics have the same Douglas tensor

In this section, we suppose that the two metrics have the same Douglas tensor.

Lemma 3.1 ($n \geq 3$). *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric with $\phi \neq Ce^{\int_0^s \frac{\phi'(0)+kt}{1+\phi'(0)t+kt^2}dt}$ for any constant $C > 0$ and k, $\bar{F} = \bar{\alpha} + \bar{\beta}$ be a Randers metric, where α and $\bar{\alpha}$ are two conformal Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms. Then they have the same Douglas tensor if and only if both F and $\bar{F}$ are Douglas metrics.*

Proof. If both F and $\bar{F}$ are Douglas metrics, they have the same Douglas tensor. We suppose F and $\bar{F}$ have the same Douglas tensor. For Randers metric, we can compute that $\bar{Q} = 1$ and $\bar{\Psi} = 0$, then (2.8) gives

$$\alpha Q(s^i_0 y^j - s^j_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i) - \bar{\alpha}(\bar{s}^i_0 y^j - \bar{s}^j_0 y^i) = \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l. \quad (3.1)$$

The above equation is valid under any local coordinates of M. Since α and $\bar{\alpha}$ are conformal, we can choose a coordinates (x^i) such that $\bar{\alpha} = \lambda(x)\alpha$ for a positive scalar function $\lambda(x)$. Then we have

$$\alpha Q(s^i_0 y^j - s^j_0 y^i) + \Psi(-2Q\alpha s_0 + r_{00})(b^i y^j - b^j y^i) - \lambda(x)\alpha(\bar{s}^i_0 y^j - \bar{s}^j_0 y^i) = \frac{1}{2}(G^i_{kl} y^j - G^j_{kl} y^i) y^k y^l, \quad (3.2)$$

For a fixed point $x \in M$, we change the coordinates (y^i) of T_xM as $(s, y^a) \to (y^i)$ by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\tilde{\alpha} \qquad y^a = y^a,$$

where $2 \leq a \leq n$, $1 \leq i \leq n$. Then we have $\alpha = \frac{b}{\sqrt{b^2-s^2}}\tilde{\alpha}$ and $\beta = \frac{bs}{\sqrt{b^2-s^2}}\tilde{\alpha}$, where $\tilde{\alpha} = \sqrt{\sum_2^n (y^a)^2}$.

Using this special coordinates, we are going to simplify the above equation (3.2).

For $i = a$, $j = b$, (3.2) is equivalent to

$$bQ(\tilde{s}^a_0 y^b - \tilde{s}^b_0 y^a) - \lambda b(\tilde{\bar{s}}^a_0 y^b - \tilde{\bar{s}}^b_0 y^a) = \frac{s}{2}\{(\tilde{G}^b_{10} + \tilde{G}^b_{01})y^a - (\tilde{G}^a_{10} + \tilde{G}^a_{01})y^b)\} \quad (3.3)$$

and

$$bs\tilde{\alpha}^2 Q(s^a_1 y^b - s^b_1 y^a) - \lambda bs\tilde{\alpha}^2(\bar{s}^a_1 y^b - \bar{s}^b_1 y^a) = \frac{1}{2}\{(\tilde{G}^a_{00} y^b - \tilde{G}^b_{00} y^a)(b^2 - s^2) + s^2\tilde{\alpha}^2(G^a_{11} y^b - G^b_{11} y^a)\} \quad (3.4)$$

where $\tilde{\bar{s}}_0^a := \bar{s}_b^a y^b$, $\tilde{G}_{10}^a := G_{1b}^a y^b$.

Letting $s = 0$ in (3.4), we get

$$\tilde{G}_{00}^a y^b - \tilde{G}_{00}^b y^a = 0.$$

Plugging into (3.4) yields

$$bQ(s_1^a y^b - s_1^b y^a) - \lambda b(\bar{s}_1^a y^b - \bar{s}_1^b y^a) = \frac{1}{2} s(G_{11}^a y^b - G_{11}^b y^a) \tag{3.5}$$

Letting $s = 0$ in (3.5) we get

$$bQ(0)(s_1^a y^b - s_1^b y^a) = \lambda b(\bar{s}_1^a y^b - \bar{s}_1^b y^a). \tag{3.6}$$

Plugging into (3.5) yields

$$b(Q(s) - Q(0))(s_1^a y^b - s_1^b y^a) = \frac{1}{2} s(G_{11}^a y^b - G_{11}^b y^a) \tag{3.7}$$

If $(Q(s) - Q(0))/s \neq \text{const.}$, then

$$s_1^a y^b - s_1^b y^a = 0.$$

Thus, $s_1^a = 0$. Plugging $s_1^a y^b - s_1^b y^a = 0$ into (3.6) yields

$$\bar{s}_1^a y^b - \bar{s}_1^b y^a = 0.$$

Therefore, we have $\bar{s}_1^a = 0$.

Letting $s = 0$ in (3.3), we get

$$bQ(0)(\tilde{s}_0^a y^b - \tilde{s}_0^b y^a) = \lambda b(\tilde{\bar{s}}_0^a y^b - \tilde{\bar{s}}_0^b y^a). \tag{3.8}$$

Plugging into (3.3) yields

$$b(Q(s) - Q(0))(\tilde{s}_0^a y^b - \tilde{s}_0^b y^a) = \frac{s}{2}\{(\tilde{G}_{10}^b + \tilde{G}_{01}^b) y^a - (\tilde{G}_{10}^a + \tilde{G}_{01}^a) y^b)\} \tag{3.9}$$

If $(Q(s) - Q(0))/s \neq \text{const.}$, then

$$\tilde{s}_0^a y^b - \tilde{s}_0^b y^a = 0.$$

Thus, $s_c^a \delta_d^b - s_c^b \delta_d^a = 0$. Summing the index $b = d$ yields $(n-2)s_c^a = 0$. Noting that $n > 2$, then $s_c^a = 0$. Plugging into (3.8) yields $\tilde{\bar{s}}_0^a y^b - \tilde{\bar{s}}_0^b y^a = 0$. Similarly, we get $\bar{s}_c^a = 0$.

Therefore, we have proved $\bar{s}_j^i = 0$, i.e. $\bar{\beta}$ is closed. In this case, $\bar{F}$ is a Douglas metric and so is F. $\square$

4. Main result

Theorem 4.1 ($n \geq 3$)**.** *Let $F = \alpha\phi(\beta/\alpha)$ be an (α,β)-metric satisfying*

(a) *β is not parallel with respect to α;*

(b) *$db \neq 0$everywhere or $b = \text{const.}$;*

(c) *F is not of Randers type and $\phi \neq Ce^{\int_0^s \frac{\phi'(0)+kt}{1+\phi'(0)t+kt^2}dt}$ for any constant $C > 0$ and k.*

Let $\bar{F} = \bar{\alpha}+\bar{\beta}$ be a Randers metric, where α and $\bar{\alpha}$ are two conformal Riemannian metrics, β and $\bar{\beta}$ are two nonzero one forms. Then they are projectively equivalent if and only if in some local coordinates the following equations hold

$$(1 + (k_1 + k_2 s^2)s^2 + k_3 s^2)\phi'' = (k_1 + k_2 s^2)(\phi - s\phi'), \tag{4.1}$$

$$G^i_\alpha = G^i_{\bar{\alpha}} + \xi y^i - \tau(k_1\alpha^2 + k_2\beta^2)b^i, \tag{4.2}$$

$$b_{i|j} = 2\tau[(1 + k_1 b^2)\alpha_{ij} + (k_2 b^2 + k_3)b_i b_j], \tag{4.3}$$

$$d\bar{\beta} = 0, \tag{4.4}$$

where $b^i := a^{ij}b_j$, $b := \|\beta\|_\alpha$, $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α, $\tau = \tau(x)$ is a scalar function and ξ is a one form, k_1, k_2, k_3 are constants.

Proof. If we suppose F and $\bar{F}$ are projectively equivalent, then they have the same Douglas tensor. Then from Lemma 4.1 we can get that they are both Douglas metrics. It is known that Randers matric is a Douglas metric if and only if $d\bar{\beta} = 0$. Meanwhile, by Lemma 2.2 one can see that an (α,β)-metric is a Douglas metric if and only if (4.1) and (4.3) hold. Plugging (4.1) and (4.3) into (2.4) gives (4.2).

If (4.1)–(4.4) hold, then plugging these equations into (2.4) and using (2.2), we can easily get the projectively equivalence of F and $\bar{F}$. □

References

[1] S. Bácsó and M. Matsumoto, Projective change between Finsler spaces with (α,β)-metric, *Tensor, N.S.* **55** (1994), 252–257.

[2] N. Cui and Y. Shen, Projective change between two classes of (α,β)-metrics, *Diff. Geom. Appl.* **27**(4) (2009), 566–573.

[3] M. Hashiguchi and Y. Ichijyō, Randers spaces with rectilinear geodesics, *Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chen.)* **13** (1980), 33–40.

[4] L. Zhou, A local classification of a class of (α,β) metrics with constant flag curvature, *Differ. Geom. Appl.* **28**, no. 2 (2010), 170–193.

[5] B. Li, Y. Shen and Z. Shen, On a class of Douglas metrics in Finsler geometry, *Studia Sci. Math. Hungarica* **46**(3) (2009), 355–365.

[6] M. Matsunoto and X. Wei, Projective changes of Finsler spaces of constant curvature, *Publ. Math. Debrecen* **44** (1994), 175–181.

[7] X. Mo, Z. Shen and C. Yang, Some constructions of projective flat Finsler metrics, *Science in China, Series A* **43** (2006), 703–714.

[8] H. Park and Y. Lee, Projective changes between a Finsler space with (α,β)-metric and the associated Riemannian metric, *Tensor* **24** (1984), 163–188.

[9] A. Rapcsák, Über die bahntreuen Abbildungen metrisher Räume, *Publ. Math. Debrecen* **8** (1961), 285–290.

[10] Y. Shen and Y. Yu, On projective related Randers metric, *Int. J. Math.* **19**(5) (2008), 503–520.

[11] Z. Shen, Differential Geometry of Spray and Finsler Spaces, *Kluwer Academic Publishers*, 2001.

[12] Z. Shen, On projectively flat (α,β)-metrics, *Can. Math. Bull.* **52**(1) (2009), 132–144.

[13] Z. Shen and G. C. Yildirim, On a class of projectively flat metrics with constant flag curvatrue, *Can. J. Math.* **60**(2) (2008), 443–456.

[14] Z. Shen, Landsberg Curvature, S-Curvature and Riemann Curvature, in 'A Sampler of Finsler Geometry' MSRI series, *Cambridge University Press*, 2004.

YAOYONG YU
SCHOOL OF SCIENCE
HANGZHOU DIANZI UNIVERSITY
HANGZHOU 310018
P.R. CHINA

E-mail: yuyaoyong@hdu.edu.cn

YING YOU
SCHOOL OF SCIENCE
HANGZHOU NORMAL UNIVERSITY
HANGZHOU 310018
P.R. CHINA

E-mail: youying1024@126.com

(Received October 4, 2011)