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The closedness of some generalized curvature 2-forms on a Riemannian manifold II

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Abstract. In this paper we recall the closedness properties of generalized curvature 2-forms, which are said to be *Riemannian, Conformal, Projective, Concircular* and *Conharmonic curvature 2-forms*, given in [16]. Moreover, we extend the concept of recurrent generalized curvature tensor to the associated curvature 2-forms while generalizing some known results.

In particular, we introduce the recurrence of the Conformal curvature 2-form and give some interesting theorems. In the final section we focus on the closedness of the associated 2-forms for curvature-like tensors.

1. Introduction

Let M be a smooth n-dimensional Riemannian manifold endowed with the operator of covariant differentiation ∇ with respect to the metric g_{kl}. Let $R_{jkl}{}^m$ the Riemann curvature tensor of type (1,3). It satisfies the two Bianchi identities

$$R_{jkl}{}^m + R_{klj}{}^m + R_{ljk}{}^m = 0,$$

and

$$\nabla_i R_{jkl}{}^m + \nabla_j R_{kil}{}^m + \nabla_k R_{ijl}{}^m = 0.$$

The previous identities are valid in a torsion-free connection [14]. In this paper we define the Ricci tensor to be $R_{kl} = -R_{mkl}{}^m$ [28] and the scalar curvature

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$R = g^{ij}R_{ij}$. It is well known that in a metric connection the Ricci tensor is symmetric [14]. Contracting the second identity, we get $\nabla_m R_{jkl}{}^m = \nabla_k R_{jl} - \nabla_j R_{kl}$. From this, a Riemannian manifold is said to have a *harmonic curvature tensor* if $\nabla_m R_{jkl}{}^m = 0$ [3], or if the Ricci tensor is of *Codazzi type*([3], [9]).

Now, in the language of differential forms, there exist a *Riemannian curvature* 2*-form* associated to the Riemann curvature tensor, precisely one defines ([3], [14]):

$$\Omega_l^m = -\frac{1}{2}R_{jkl}{}^m dx^j \wedge dx^k. \tag{1.1}$$

Moreover, we may define another *curvature* 2*-form* associated to the divergence of the Riemann curvature tensor, that is [15]:

$$\Pi_l = \nabla_m R_{jkl}{}^m dx^j \wedge dx^k. \tag{1.2}$$

Finally, a *Ricci* 1*-form* associated to the Ricci tensor may be defined in the following way [22]:

$$\Lambda_l = R_{kl} dx^k. \tag{1.3}$$

Now we consider the class of curvature tensors $K_{jkl}{}^m$ with the usual symmetries of the Riemann curvature tensor satisfying the first Bianchi identity. Specifically, we admit a generalized curvature tensor satisfying the following relations (see [15] and [23]):

$$\begin{aligned} &\text{a)} \qquad K_{jkl}{}^m + K_{klj}{}^m + K_{ljk}{}^m = 0, \quad K_{jkl}{}^m = -K_{kjl}{}^m, \\ &\text{b)} \qquad \nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = B_{ijkl}{}^m, \end{aligned} \tag{1.4}$$

where $B_{ijkl}{}^m$ is a tensor source in the second Bianchi identity. Moreover, we may define also a completely covariant $(0,4)$-type tensor K with the following further properties [22]:

$$K_{jklm} = -K_{kjlm} = -K_{jkml}, \qquad K_{jklm} = K_{lmjk}. \tag{1.5}$$

In this way the contraction $K_{kl} = -K_{mkl}{}^m$ defines a symmetric generalized Ricci tensor [22]. It is worthwhile to see that in this general case the second Bianchi identity admits a nonzero source tensorial term B. An n-dimensional Riemannian manifold is said to be K flat if $K_{jkl}{}^m = 0$, K-symmetric if $\nabla_i K_{jkl}{}^m = 0$, and *K-harmonic* if $\nabla_m K_{jkl}{}^m = 0$ ([15]).

Now the *cuvature* 2*-form* associated to this tensor may be defined in the following manner:

$$\Omega_{(K)l}{}^m = K_{jkl}{}^m dx^j \wedge dx^k. \tag{1.6}$$

Consequently, the 2-form associated to the divergence of this tensor is defined as:

$$\Pi_{(K)l} = \nabla_m K_{jkl}{}^m dx^j \wedge dx^k. \tag{1.7}$$

If we consider the symmetric contraction $K_{kl} = -K_{mkl}{}^m$ a generalized *Ricci 1-form* may be defined [22] as:

$$\Lambda_{(K)l} = K_{kl} dx^k. \tag{1.8}$$

We have note that Conformal, Projective, Concircular, and Conharmonic curvature tensors given in [16] are built from the Riemann curvature tensor and the Ricci tensor. Thus, naturally, we may define the associated curvature 2-forms $\Omega_{(P)l}{}^m$, $\Omega_{(\tilde{C})l}{}^m$, $\Omega_{(N)l}{}^m$, which are said to be *Projective, Concircular, Conformal* and *Conharmonic curvature 2-forms* respectively, and the corresponding 2-forms associated to the divergence of such tensors $\Pi_{(P)l}$, $\Pi_{(\tilde{C})l}$, $\Pi_{(N)l}$. The closedness of such forms gives a great geometric importance which of makes specific restrictions on the Riemann curvature tensor and the Ricci tensor.

On the other hand, in [22] the authors defined a new notion of generalized curvature tensors $\bar{K}_{jkl}{}^m$, which are said to be *curvature like tensors*. These are built from a general curvature tensor $K_{jkl}{}^m$ and a symmetric tensor H_{kl}. With such kind of curvature-like tensors they investigated some geometric conditions which are equivalent to the second Bianchi identity in terms of the Ricci-like form and the associated curvature like forms.

In Section 2 the concept of recurrence from generalized curvature tensors can be extended to their tensor valued forms $\Omega_{(K)l}{}^m$ and some general Theorems will be pointed out.

In Section 3 we study the relation between *recurrent Conharmonic, Projective* and *Concircular 2-forms* while generalizing some results due to [11], [12], [19], [20], and [26]. In Section 4 recurrent Conformal curvature tensor valued 2-form $\Omega_{(C)l}{}^m$ will be investigated and some known results given in [15] and [21] could be extended. Finally in Section 5, we focus on the closedness of the associated 2-form for same curvature-like tensors given in [22].

2. Recurrent curvature 2-forms

Recurrent manifolds have been of great interest and were investigated by many geometers (see for example [12] or [13] for a compendium). In particular, Walker [27] studied manifolds for which the Riemann curvature tensor is recurrent, while Concircular recurrent manifolds were studied by Miyazawa [17], Conformally recurrent and Projectively recurrent by Adati and Miyazawa ([1], [2]).

In [15] the notion of K-recurrent manifold was introduced. A Riemannian manifold with generalized curvature tensor satisfying equation (2.16) in [16] is said to be K-recurrent $(KRM)_n$ if it is not K-flat and satisfies

$$\nabla_i K_{jkl}{}^m = \alpha_i K_{jkl}{}^m, \tag{2.1}$$

where α_i is a nonzero covector field. Thus $(KRM)_n$ manifolds include, as special cases, those which are Conformally recurrent, Projectively recurrent, Concircular recurrent, etc, (see [12] and [15]). In this section we extend the notion of K-recurrence from tensors to associated 2-forms $\Omega_{(K)l}{}^m$. Hereafter we consider n-dimensional non K-flat Riemannian manifolds.

Definition 2.1. Let M be an n-dimensional Riemannian manifold. The curvature 2-form $\Omega_{(K)l}{}^m = K_{jkl}{}^m dx^j \wedge dx^k$ is said to be recurrent if there exist a nonzero scalar 1-form α for which:

$$D\Omega_{(K)l}{}^m = \alpha \wedge \Omega_{(K)l}{}^m, \tag{2.2}$$

being $\alpha = \alpha_i dx^i$ the associated 1-form.

It is easy to see that the previous condition is a generalization of the notion of K-recurrence. In fact if we write equation (2.2) in local components, we have:

$$(\nabla_i K_{jkl}{}^m - \alpha_i K_{jkl}{}^m) dx^i \wedge dx^j \wedge dx^k = 0. \tag{2.3}$$

If $\alpha = 0$, we recover the closedness of $\Omega_{(K)l}{}^m$. The following theorem gives the meaning of this recurrence.

Theorem 2.1. *Let M be an n-dimensional Riemannian manifold. The curvature 2-form $\Omega_{(K)l}{}^m = K_{jkl}{}^m dx^j \wedge dx^k$satisfies condition (2.2) if and only if*

$$B_{ijkl}{}^m = \nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = \alpha_i K_{jkl}{}^m + \alpha_j K_{kil}{}^m + \alpha_k K_{ijl}{}^m. \tag{2.4}$$

From Theorem 2.1 we can assert the following two Corollaries:

Corollary 2.1. *Let M be an n-dimensional Riemannian manifold with closed curvature 2-form $\Omega_{(K)l}{}^m$ i.e. with $D\Omega_{(K)l}{}^m = 0$. Then the 2-form is recurrent if and only if $\alpha_i K_{jkl}{}^m + \alpha_j K_{kil}{}^m + \alpha_k K_{ijl}{}^m = 0$ for some $\alpha_i \neq 0$.*

Corollary 2.2. *Let M be an n-dimensional Riemannian manifold with recurrent curvature 2-form $\Omega_{(K)l}{}^m$ i.e. with $D\Omega_{(K)l}{}^m = \alpha \wedge \Omega_{(K)l}{}^m$. Then the 2-form is closed if and only if $\alpha_i K_{jkl}{}^m + \alpha_j K_{kil}{}^m + \alpha_k K_{ijl}{}^m = 0$ including $\alpha_i = 0$.*

Obviously if $\nabla_i K_{jkl}{}^m = \alpha_i K_{jkl}{}^m$, then condition (2.4) is satisfied. There are also other differential structures satisfying the same equation. For example we may consider the following one come from the same tensor K:

$$\begin{aligned}\nabla_i K_{jkl}{}^m &= 2\alpha_i K_{jkl}{}^m + 2\beta_i(\delta_j^m g_{kl} - \delta_k^m g_{jl}) + \beta_j(\delta_i{}^m g_{kl} - \delta_k^m g_{il}) \\ &\quad + \beta_k(\delta_j^m g_{il} - \delta_i^m g_{jl}) + \beta_l(\delta_j^m g_{ki} - \delta_k^m g_{ji}) + \beta^m(g_{ij}g_{kl} - g_{ik}g_{jl}), \quad (2.5)\end{aligned}$$

where α_i, β_i are nonzero covectors. This condition was originally proposed by Ewert-Krzemieniewski for the Riemann curvature tensor (see [10] equation 3). It is easy to see that for the previous structure equation (2.4) also holds for the formula (2.5). So the condition (2.4) is a proper generalization of the concept of a K-recurrent manifold.
When the previous Theorem 2.1 is applied to the Riemann *curvature tensor*, we have simply:

Corollary 2.3. *Let M be an n-dimensional Riemannian manifold. The closed tensor valued 2-form $\Omega_l^m = -\frac{1}{2}R_{jkl}{}^m dx^j \wedge dx^k$ satisfies condition (2.2) if and only if*

$$\alpha_i R_{jkl}{}^m + \alpha_j R_{kil}{}^m + \alpha_k R_{ijl}{}^m = 0. \quad (2.6)$$

We recall that condition (2.6) defines the so called B space studied by Venzi [25].

Now we focus on the notion of recurrence for the generalized Ricci 1-form $\Lambda_{(K)l} = K_{kl}dx^k$ where $K_{kl} = -K_{mkl}{}^m$. We state the following:

Definition 2.2. Let M be an n-dimensional Riemannian manifold. The generalized Ricci 1-form $\Lambda_{(K)l} = K_{kl}dx^k$ is said to be recurrent if there exist a nonzero scalar 1-form β for which:

$$D\Lambda_{(K)l} = \beta \wedge \Lambda_{(K)l}, \quad (2.7)$$

being $\beta = \beta_i dx^i$ the associated 1-form.

In local components the previous equation may be written in the form:

$$(\nabla_i K_{kl} - \beta_i K_{kl})dx^i \wedge dx^k = 0. \quad (2.8)$$

If $\beta = 0$, the closedness of the generalized Ricci 1-form is recovered. The following theorem explains the meaning of this kind of recurrence.

Theorem 2.2. *Let M be an n-dimensional Riemannian manifold. The generalized Ricci* 1*-form $\Lambda_{(K)l} = K_{kl}dx^l$ satisfies condition* (2.7) *if and only if*

$$\nabla_i K_{kl} - \nabla_k K_{il} = \beta_i K_{kl} - \beta_k K_{il}. \tag{2.9}$$

Remark 2.1. When K_{kl} becomes a Codazzi tensor [9], then $0 = \beta_i K_{kl} - \beta_k K_{il}$: transvecting with g^{kl} we get $\beta_i K = \beta^l K_{il}$. On the other hand, left multiplication by β^i gives simply $\beta_i \beta^i K_{kl} = \beta_k \beta^i K_{il} = \beta_k \beta_l K$. Thus for a given Riemannian manifold with recurrent generalized Ricci one form $\Lambda_{(K)l} = K_{kl}dx^k$, if K_{kl} is of Codazzi type, then it becomes of rank one.

It is very interesting to write now the previous Theorem 2.2 for the Ricci 1-form $\Lambda_l = R_{kl}dx^k$. We obtain the following condition of recurrence:

$$\nabla_m R_{kil}{}^m = \nabla_i R_{kl} - \nabla_k R_{il} = \beta_i R_{kl} - \beta_k R_{il}. \tag{2.10}$$

It is easy to see that if a manifold becomes Ricci recurrent, i.e. if the condition $\nabla_i R_{kl} = \beta_i R_{kl}$ [12] holds, then the previous equation is automatically satisfied. So the recurrence of the Ricci 1-form includes the concept of Ricci recurrent manifold as a special case. Moreover it is easy to see that equation (2.10) includes also other well known differential structures. For example one may consider Weakly Ricci Symmetric manifolds defined by the condition (see [5] and [15] for a compendium):

$$\nabla_i R_{kl} = A_i R_{kl} + B_k R_{il} + D_l R_{ik}, \tag{2.11}$$

with A, B and D are nonzero covector fields. These manifolds were introduced by Tamássy and Binh [24], and include relevant Robertson–Walker space times [8] and the perfect fluid space time [7]. One obtains immediately from the previous equation that the condition (2.10) is satisfied with $\beta_i = A_i - B_i$. Thus the recurrence of the Ricci 1-form may be of some interest in General Relativity.

Now it is possible to extend a remarkable result stated in [15] for Weakly Ricci Symmetric manifolds to the case of the recurrence of the Ricci 1-form. Take the covariant derivative ∇_j to (2.10), sum over cyclic permutations of indices j, k, i and use the Lovelock identity and the same equation (2.10). Then the following result is obtained:

$$\begin{aligned} &R_{jl}(\nabla_i\beta_k - \nabla_k\beta_i) + R_{kl}(\nabla_j\beta_i - \nabla_i\beta_j) + R_{il}(\nabla_k\beta_j - \nabla_j\beta_k) \\ &\quad = -(R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m). \end{aligned} \tag{2.12}$$

Let us suppose that the Ricci tensor is non-singular: thus there exist a $(2,0)$-type tensor $(R^{-1})^{sl}$ with the property $(R^{-1})^{sl}R_{jl} = \delta^s_j$: if β_i is closed, one gets simply

$R_{im}R_{jkl}{}^{m} + R_{jm}R_{kil}{}^{m} + R_{km}R_{ijl}{}^{m} = 0$. On the other hand, if the right hand side of equation (2) is zero, we may write:

$$R_{jl}(\nabla_i\beta_k - \nabla_k\beta_i) + R_{kl}(\nabla_j\beta_i - \nabla_i\beta_j) + R_{il}(\nabla_k\beta_j - \nabla_j\beta_k) = 0. \tag{2.13}$$

If the previous result is multiplied by $(R^{-1})^{sl}$, one easily obtains:

$$\delta_j^s(\nabla_i\beta_k - \nabla_k\beta_i) + \delta_k^s(\nabla_j\beta_i - \nabla_i\beta_j) + \delta_i^s(\nabla_k\beta_j - \nabla_j\beta_k) = 0. \tag{2.14}$$

Now we put $s = j$ and sum getting:

$$(n-2)(\nabla_i\beta_k - \nabla_k\beta_i) = 0. \tag{2.15}$$

Thus if suppose $n > 2$, we have that β_i is closed; if we recall that the form Π_l is closed if and only if $R_{im}R_{jkl}{}^{m} + R_{jm}R_{kil}{}^{m} + R_{km}R_{ijl}{}^{m} = 0$. A generalization of the similar result in [15] (Theorem 4.4) can be stated as follows:

Theorem 2.3. *Let M be an n-dimensional ($n > 2$) Riemannian manifold with recurrent Ricci 1-form $D\Lambda_l = \beta \wedge \Lambda_l$. If the Ricci tensor is non-singular, then β_i is closed if and only if $D\Pi_l = 0$.*

Finally we may focus on manifolds with recurrent Ricci 1-form having closed Weyl form. If (2.14) in [16] and (2.10) are taken in conjunction, one may write:

$$\beta_k R_{jl} - \beta_j R_{kl} = \frac{1}{2(n-1)}\Big[(\nabla_k R)g_{jl} - (\nabla_j R)g_{kl}\Big]. \tag{2.16}$$

Transvecting the previous equation with g^{kl} gives the $\beta_j R - \beta^m R_{jm} = \frac{1}{2}\nabla_j R$. Inserting this into (2.16), we get the following:

$$\beta_k R_{jl} - \beta_j R_{kl} = \frac{1}{2(n-1)}\Big[(\beta_k R - \beta^m R_{km})g_{jl} - (\beta_j R - \beta^m R_{jm})g_{kl}\Big]. \tag{2.17}$$

Now the previous result is multiplied by β^l to give simply $\beta_k\beta^l R_{jl} = \beta_j\beta^l R_{kl}$: a further multiplication by β^j brings $\beta_k\beta^j\beta^l R_{jl} = \beta^j\beta_j\beta^l R_{kl}$ that may be written in the form:

$$\beta^l R_{kl} = \frac{\beta_k\beta^j\beta^l R_{jl}}{\beta^j\beta_j} = t\beta_k \tag{2.18}$$

So we have found that β_k is an eigenvector of the Ricci tensor with eigenvalue t: this is the same result pointed out in [6]. Now (2.16) in [16] can be written in the following form:

$$\beta_j(R - t) = \frac{1}{2}\nabla_j R. \tag{2.19}$$

Now (2.17) is multiplied by β^j and (2.18) is used to give an expression of the Ricci tensor:

$$R_{kl} = \frac{\beta_k\beta_l}{\beta^j\beta_j}\Big[\frac{nt-R}{n-1}\Big] + g_{kl}\Big[\frac{R-t}{n-1}\Big]. \tag{2.20}$$

A manifold whose Ricci tensor is of the form (2.20) is said to be quasi Einstein [4]. We have thus shown that:

Theorem 2.4. *Let M be an n-dimensional Riemannian manifold with recurrent Ricci 1-form $D\Lambda_l = \beta \wedge \Lambda_l$. If the Weyl form is closed, then the manifold is quasi-Einstein.*

In next Sections 3 and 4, we will point out some applications of Theorem 2.1 to the Conharmonic, Concircular, Projective and Conformal recurrent 2-forms.

3. Recurrent conharmonic, projective and concircular 2-forms

In this section we give a generalization of some results given in [12]. We take into consideration of the condition (2.2) and its equivalent form (2.4) for the Conharmonic curvature tensor given in (1.14) in [16]. The *Conharmonic curvature 2-form*

$$\Omega_{(N)l}{}^m = N_{jkl}{}^m dx^j \wedge dx^k$$

is recurrent if and only if

$$\nabla_i N_{jkl}{}^m + \nabla_j N_{kil}{}^m + \nabla_k N_{ijl}{}^m = \alpha_i N_{jkl}{}^m + \alpha_j N_{kil}{}^m + \alpha_k N_{ijl}{}^m = B_{ijkl}{}^m, \tag{3.1}$$

where α_i is some nonzero covector. We recall that in this case the source term B takes the form:

$$\begin{aligned} B_{ijkl}{}^m = \frac{1}{n-2}\Big[&\delta_j^m(\nabla_i R_{kl} - \nabla_k R_{il}) + \delta_i^m(\nabla_k R_{jl} - \nabla_j R_{kl}) \\ &+ \delta_k^m(\nabla_j R_{il} - \nabla_i R_{jl}) + g_{il}(\nabla_j R_k^m - \nabla_k R_j^m) \\ &+ g_{jl}(\nabla_k R_i^m - \nabla_i R_k^m) + g_{kl}(\nabla_i R_j^m - \nabla_j R_i^m)\Big]. \end{aligned} \tag{3.2}$$

Taking equations (3.1) and (3.2) in conjunction with $m = j$ gives

$$\begin{aligned} &(n-2)\alpha_m N_{kil}{}^m + \alpha_i R g_{kl} - \alpha_k R g_{il} \\ &\qquad = (n-3)(\nabla_i R_{kl} - \nabla_k R_{il}) + \frac{1}{2}(g_{kl}\nabla_i R - g_{il}\nabla_k R). \end{aligned} \tag{3.3}$$

Now the previous result is transvected with g^{kl} to give after straightforward calculations:

$$(n-2)\alpha_i R = (n-2)\nabla_i R. \tag{3.4}$$

In the above calculations we have used the contraction $N_{mkl}{}^m = \frac{R}{n-2} g_{kl}$ and the following identity:

$$(n-2)\alpha_m g^{kl} N_{kil}{}^m = -\alpha_i R. \tag{3.5}$$

From the equation (3.4) if $n > 2$, we easily obtain that $\alpha_i = \frac{\nabla_i R}{R}$ and thus that α_i is a closed 1-form. Then we can give a generalization of the result given in [12] as follows:

Theorem 3.1. *Let M be an $n(\geq 3)$-dimensional Riemannian manifold with recurrent Conharmonic curvature 2-form. Then*

1) *if the scalar curvature is constant, it must vanish,*
2) *if the scalar curvature is not constant, the 1-form α_i is closed.*

Hereafter in this section we consider a manifold with recurrent Conharmonic 2-form which is also Ricci recurrent, i.e. satisfying the condition $\nabla_i R_{kl} = \alpha_i R_{kl}$ [12] with the same recurrence parameter of the curvature 2-form. In this case we may note that $N_{jkl}{}^m = R_{jkl}{}^m + U_{jkl}{}^m$and consequently that:

$$\nabla_i N_{jkl}{}^m = \nabla_i R_{jkl}{}^m + \alpha_i U_{jkl}{}^m. \tag{3.6}$$

Now from (3.1) it is easily inferred that:

$$\alpha_i R_{jkl}{}^m + \alpha_j R_{kil}{}^m + \alpha_k R_{ijl}{}^m = 0. \tag{3.7}$$

So we have shown that the 2-form $\Omega_l^m {=} {-}\frac{1}{2} R_{jkl}{}^m dx^j \wedge dx^k$ is recurrent, that is, $D\Omega_l^m = \alpha \wedge \Omega_l^m$. By the similar arguments we give a generalization of the results given in [12]:

Theorem 3.2. *Let M be an n-dimensional Ricci recurrent (i.e. $\nabla_i R_{kl} = \alpha_i R_{kl}$) Riemannian manifold. Then we have $D\Omega_{(N)l}{}^m = \alpha \wedge \Omega_{(N)l}{}^m$ if and only if $D\Omega_l^m = \alpha \wedge \Omega_l^m$ for the same recurrence parameter.*

Now we note that, from the definitions of *Conharmonic* and *Projective curvature tensors* in (1.12) and (1.14) in [16] the following relation holds

$$\begin{aligned} N_{jkl}{}^m &= P_{jkl}{}^m + \frac{1}{(n-2)(n-1)}(\delta_j^m R_{kl} - \delta_k^m R_{jl}) + \frac{1}{(n-2)}(R_j^m g_{kl} - R_k^m g_{jl}) \\ &= P_{jkl}{}^m + V_{jkl}{}^m. \end{aligned} \tag{3.8}$$

So if the manifold is Ricci recurrent, we have:

$$\nabla_i N_{jkl}{}^m = \nabla_i P_{jkl}{}^m + \alpha_i V_{jkl}{}^m. \tag{3.9}$$

From (3.1) it can be easily inferred that the following is true:

$$\nabla_i P_{jkl}{}^m + \nabla_j P_{kil}{}^m + \nabla_k P^m_{ijl} = \alpha_i P_{jkl}{}^m + \alpha_j P_{kil}{}^m + \alpha_k P_{ijl}{}^m. \tag{3.10}$$

So we have shown that the *Projective curvature* 2*-form* $\Omega_{(P)l}{}^m = P_{jkl}{}^m dx^j \wedge dx^k$ is recurrent, i.e. that $D\Omega_{(P)l}{}^m = \alpha \wedge \Omega_{(P)l}{}^m$. By the similar arguments we may state the following Theorem which generalizes a result in [12]:

Theorem 3.3. *Let M be an n-dimensional Ricci recurrent (i.e. $\nabla_i R_{kl} = \alpha_i R_{kl}$) Riemannian manifold. Then we have $D\Omega_{(N)l}{}^m = \alpha \wedge \Omega_{(N)l}{}^m$ if and only if $D\Omega_{(P)l}{}^m = \alpha \wedge \Omega_{(P)l}{}^m$ for the same recurrence parameter.*

Now one can write the following expression involving the *Concircular* and the *Projective curvature tensors holds* in (1.12) and (1.13) in [16]:

$$\begin{aligned}\tilde{C}_{jkl}{}^m &= P_{jkl}{}^m + \frac{R}{n(n-1)}(\delta^m_j g_{kl} - \delta^m_k g_{jl}) - \frac{1}{n-1}(\delta^m_j R_{kl} - \delta^m_k R_{jl}) \\ &= P_{jkl}{}^m + Q_{jkl}{}^m.\end{aligned} \tag{3.11}$$

So if the manifold is Ricci recurrent we have:

$$\nabla_i \tilde{C}_{jkl}{}^m = \nabla_i P_{jkl}{}^m + \alpha_i Q_{jkl}{}^m. \tag{3.12}$$

If the concircular curvature 2-form is recurrent, the following is true:

$$\nabla_i \tilde{C}_{jkl}{}^m + \nabla_j \tilde{C}_{kil}{}^m + \nabla_k \tilde{C}_{ijl}{}^m = \alpha_i \tilde{C}_{jkl}{}^m + \alpha_j \tilde{C}_{kil}{}^m + \alpha_k \tilde{C}_{ijl}{}^m. \tag{3.13}$$

Finally, by the same arguments used above we can state the following:

Theorem 3.4. *Let M be an n-dimensional Ricci recurrent (i.e. $\nabla_i R_{kl} = \alpha_i R_{kl}$) Riemannian manifold. Then we have $D\Omega_{(\tilde{C})l}{}^m = \alpha \wedge \Omega_{(\tilde{C})l}{}^m$ if and only if $D\Omega_{(P)l}{}^m = \alpha \wedge \Omega_{(P)l}{}^m$for the same recurrence parameter.*

4. Recurrent conformal 2-forms

In this section we study some recurrent conformal 2-forms on a Riemannian manifold. We recall that a manifold is conformally recurrent([1], [21]) when the conformal curvature tensor (1.9) in [16] satisfies the relation $\nabla_j C_{jkl}{}^m = \alpha_i C_{jkl}{}^m$ where α_i is a nonzero covector. In [21] Suh and Kwon studied *Conformally recurrent* semi-Riemannian manifolds with *harmonic conformal curvature tensor*, that is, with $\nabla_m C_{jkl}{}^m = 0$ (see also Besse [3]). In the Riemannian case they stated the following theorem ([21], Remark 3.3):

Theorem 4.1. *Let M be an $n(\geq 4)$-dimensional Riemannian manifold with Riemannian connection ∇. Assume that M is Conformally recurrent and has the harmonic Conformal curvature tensor. Then M is Conformally symmetric.*

Now if we consider the recurrent Conformal 2-form $D\Omega_{(C)l}{}^m = \alpha \wedge \Omega_{(C)l}{}^m$ on a Riemannian manifold, the general equation (2.4) can be written as:

$$\nabla_i C_{jkl}{}^m + \nabla_j C_{kil}{}^m + \nabla_k C_{ijl}{}^m = \alpha_i C_{jkl}{}^m + \alpha_j C_{kil}{}^m + \alpha_k C_{ijl}{}^m = B_{ijkl}{}^m. \quad (4.1)$$

Obviously, when $\nabla_i C_{jkl}{}^m = \alpha_i C_{jkl}{}^m$, the equation (4.1) is satisfied. If we take $i = m$ in the previous relation, we have:

$$\nabla_m C_{jkl}{}^m = \alpha_m C_{jkl}{}^m. \quad (4.2)$$

It is well known ([1]) that the source term for the second Bianchi identity for the Conformal curvature tensor may be written in the following form:

$$\begin{aligned} \nabla_i C_{jkl}{}^m + \nabla_j C_{kil}{}^m + \nabla_k C_{ijl}{}^m = \frac{1}{n-3}\Big[& \delta_j^m \nabla_p C_{kil}{}^p + \delta_k^m \nabla_p C_{ijl}{}^p \\ & + \delta_i^m \nabla_p C_{jkl}{}^p + g_{kl} \nabla_p C_{ji}{}^{mp} + g_{il} \nabla_p C_{kj}{}^{mp} + g_{jl} \nabla_p C_{ik}{}^{mp}\Big]. \end{aligned} \quad (4.3)$$

So we may state the following fundamental Lemma (see [22] Lemma 7.3):

Lemma 4.1. *Let M be an n-dimensional non-conformally flat Riemannian manifold. Then the conformal curvature 2-form $\Omega_{(C)l}{}^m$ is closed if and only if $\nabla_m C_{jkl}{}^m = 0$.*

Now if the manifold has harmonic Conformal curvature tensor, we have

$$D\Omega_{(C)l}{}^m = 0$$

and consequently:

$$\alpha_i C_{jkl}{}^m + \alpha_j C_{kil}{}^m + \alpha_k C_{ijl}{}^m = 0. \quad (4.4)$$

We show that in such a case $\alpha_i = 0$. Obviously we have also $\alpha_m C_{jkl}{}^m = 0$. Tranvecting equation (4.4) with α^i gives $\alpha^i \alpha_i C_{jkl}{}^m = 0$. On the other hand, transvecting with $C^{jkl}{}_m$ also gives $\alpha_i C_{jkl}{}^m C^{jkl}{}_m = 0$. So it becomes $\alpha_i = 0$. Thus we have obtained the following:

Theorem 4.2. *Let M be an n-dimensional non conformally flat Riemannian manifold with recurrent Conformal curvature 2-form $\Omega_{(C)l}{}^m$ and harmonic Conformal curvature tensor. Then the Conformal 2-form is closed with $\alpha_i = 0$.*

Considering K-recurrent manifolds in [15], the authors pointed out the following Theorem ([15] Theorem 3.10).

Theorem 4.3. *Let M be an n dimensional K recurrent Riemannian manifold i.e. satisfying equation* (2.16) *in* [16] *and the relation $\nabla_i K_{jkl}{}^m = \alpha_i K_{jkl}{}^m$, with α_i closed. Then the following relation holds:*

$$R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = -\frac{1}{A}\nabla_m B_{ijkl}{}^m. \tag{4.5}$$

In [15] the authors claimed that for the cases $K = C$, $\tilde{C}$, P and N from the previous equation one gets $R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = 0$. This gives the closedness of the corresponding forms $\Pi_{(K)l}$. Now we will extend this relation to the case of recurrent Conformal 2-form

$$D\Omega_{(C)l}{}^m = \alpha \wedge \Omega_{(C)l}{}^m$$

with closed recurrence parameter. Lovelock's identity (2.14) in [16] is thus written for the *Conformal curvature tensor*:

$$\nabla_i\nabla_m C_{jkl}{}^m + \nabla_j\nabla_m C_{kil}{}^m + \nabla_k\nabla_m C_{ijl}{}^m = -\frac{n-3}{n-2}(R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m). \tag{4.6}$$

Now we recall that $\nabla_m C_{jkl}{}^m = \alpha_m C_{jkl}{}^m$ and $\nabla_i C_{jkl}{}^m + \nabla_j C_{kil}{}^m + \nabla_k C_{ijl}{}^m = \alpha_i C_{jkl}{}^m + \alpha_j C_{kil}{}^m + \alpha_k C_{ijl}{}^m$. So the left hand side of previous equation may be written in the form:

$$(\nabla_i\alpha_m)C_{jkl}{}^m + (\nabla_j\alpha_m)C_{kil}{}^m + (\nabla_k\alpha_m)C_{ijl}{}^m + \alpha_m(\alpha_i C_{jkl}{}^m + \alpha_j C_{kil}{}^m + \alpha_k C_{ijl}{}^m). \tag{4.7}$$

Now the divergence of $\alpha_i C_{jkl}{}^m + \alpha_j C_{kil}{}^m + \alpha_k C_{ijl}{}^m = B_{ijkl}{}^m$ is taken to give straightforwardly:

$$\nabla_m B_{ijkl}{}^m = (\nabla_m\alpha_i)C_{jkl}{}^m + (\nabla_m\alpha_j)C_{kil}{}^m + (\nabla_m\alpha_k)C_{ijl}{}^m + \alpha_m(\alpha_i C_{jkl}{}^m + \alpha_j C_{kil}{}^m + \alpha_k C_{ijl}{}^m). \tag{4.8}$$

If the closedness of the recurrence parameter is taken into account, we can write finally:

$$\nabla_m B_{ijkl}{}^m = -\frac{n-3}{n-2}(R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m). \tag{4.9}$$

We have thus proved the following result:

Theorem 4.4. *Let M be an n-dimensional Riemannian manifold with the recurrent Conformal tensor valued 2-form $D\Omega_{(C)l}{}^{m} = \alpha \wedge \Omega_{(C)l}{}^{m}$. If the recurrence parameter is closed, then* (4.9) *holds.*

Now the following Lemma about the source term of the second Bianchi identity for the *Conformal curvature tensor* is stated:

Lemma 4.2. *The divergence of the source term in the second Bianchi identity for the Conformal curvature tensor takes the form:*

$$\nabla_m B_{ijkl}{}^{m} = -\frac{1}{n-2}(R_{im}R_{jkl}{}^{m} + R_{jm}R_{kil}{}^{m} + R_{km}R_{ijl}{}^{m}). \tag{4.10}$$

Proof. We recall that in the case of Conformal curvature tensor the source term B takes the form:

$$\begin{aligned} B_{ijkl}{}^{m} = {} & \frac{1}{n-2}\Big[\delta_j^m(\nabla_i R_{kl} - \nabla_k R_{il}) + \delta_i^m(\nabla_k R_{jl} - \nabla_j R_{kl}) \\ & + \delta_k^m(\nabla_j R_{il} - \nabla_i R_{jl}) + g_{il}(\nabla_j R_k^m - \nabla_k R_j^m) \\ & + g_{jl}(\nabla_k R_i^m - \nabla_i R_k^m) + g_{kl}(\nabla_i R_j^m - \nabla_j R_i^m)\Big] \\ & - \frac{1}{(n-1)(n-2)}\Big[\delta_j^m(\nabla_i R g_{kl} - \nabla_k R g_{il}) + \delta_i^m(\nabla_k R g_{jl} - \nabla_j R g_{kl}) \\ & + \delta_k^m(\nabla_j R g_{il} - \nabla_i R g_{jl})\Big]. \end{aligned} \tag{4.11}$$

Taking the covariant derivative ∇_m of the previous equation and recalling that $\nabla_l \nabla_m R_{jk}{}^{lm} = 0$ (see [14] and [15]), we obtain:

$$\nabla_m B_{ijkl}{}^{m} = -\frac{1}{n-2}(\nabla_i \nabla_m R_{jkl}{}^{m} + \nabla_j \nabla_m R_{kil}{}^{m} + \nabla_k \nabla_m R_{ijl}{}^{m}). \tag{4.12}$$

□

Now Lovelock's identity is used to conclude. If we take into account both Theorem 4.4 and Lemma 4.2, we have $R_{im}R_{jkl}{}^{m} + R_{jm}R_{kil}{}^{m} + R_{km}R_{ijl}{}^{m} = 0$ and thus may state the following:

Theorem 4.5. *Let M be an n-dimensional Riemannian manifold with the recurrent conformal curvature 2-form i.e. $D\Omega_{(C)l}{}^{m} = \alpha \wedge \Omega_{(C)l}{}^{m}$. If the recurrence parameter is closed, then $D\Pi_{(C)l} = 0$.*

Now we may consider a Riemannian manifold endowed with the differential structure (2.5) written simply for the Riemann tensor, that is,

$$\nabla_i R_{jkl}{}^m = 2\alpha_i R_{jkl}{}^m + 2\beta_i(\delta_j^m g_{kl} - \delta_k^m g_{jl}) + \beta_j(\delta_i^m g_{kl} - \delta_k^m g_{il}) + \beta_k(\delta_j^m g_{il} - \delta_i^m g_{jl}) + \beta_l(\delta_j^m g_{ki} - \delta_k^m g_{ji}) + \beta^m(g_{ij}g_{kl} - g_{ik}g_{jl}). \quad (4.13)$$

From (2.4) we may infer that $\alpha_i R_{jkl}{}^m + \alpha_j R_{kil}{}^m + \alpha_k R_{ijl}{}^m = 0$ and thus that $D\Omega_l^m = \alpha \wedge \Omega_l^m$. This structure was studied in [10]. In the same reference it was pointed out that this structure satisfies $\nabla_i C_{jkl}{}^m = \alpha_i C_{jkl}{}^m$ and thus the conformal curvature 2-form is recurrent. So if the covector α_i is closed, we may infer that:

Theorem 4.6. *Let M be an n-dimensional Riemannian manifold endowed with the differential structure* (4.13) *with closed α_i. Then $D\Pi_{(C)l} = 0$.*

5. Curvature 2 forms originating from curvature-like tensors

In Section 2 in [16] we scrutinized the closedness properties of tensor valued 2-forms originating from curvature tensors (Conformal, Projective, Conharmonic, Concircular) derived from the Riemann curvature tensor and satisfying the equation (2.16) in [16]. In this section we extend such properties to curvature-like tensors $\bar{K}_{jkl}{}^m$ built from a generalized curvature tensor $K_{jkl}{}^m$ satisfying the equation (1.4) with null source term $B_{jkl}{}^m$ (see [22] and [23]). In particular, we recall that the contraction $K_{kl} = -K_{mkl}{}^m$ is symmetric. It is possible to write a Lovelock's identity also in this case. We state the following:

Lemma 5.1. (Generalized Lovelock's identity) *Let M be an n dimensional Remannian manifold. Then the divergence of any curvature tensor $K_{jkl}{}^m$ satisfying the second Bianchi identity with zero source term obeys the following identity:*

$$\nabla_i\nabla_m K_{jkl}{}^m + \nabla_j\nabla_m K_{kil}{}^m + \nabla_k\nabla_m K_{ijl}{}^m = -(K_{im}R_{jkl}{}^m + K_{jm}R_{kil}{}^m + K_{km}R_{ijl}{}^m). \quad (5.1)$$

Proof. Contracting the second Bianchi identity $\nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = 0$ we get $\nabla_m K_{jkl}{}^m = \nabla_k K_{jl} - \nabla_j K_{kl}$. Now the covariant derivative ∇_i is applied to the previous expression, a sum over cyclic permutations of the indices i, j, k is performed obtaining:

$$\nabla_i\nabla_m K_{jkl}{}^m + \nabla_j\nabla_m K_{kil}{}^m + \nabla_k\nabla_m K_{ijl}{}^m = (\nabla_i\nabla_k - \nabla_k\nabla_i)K_{jl} + (\nabla_j\nabla_i - \nabla_i\nabla_j)K_{kl} + (\nabla_k\nabla_j - \nabla_j\nabla_k)K_{il}. \quad (5.2)$$

Finally, the Ricci and the first Bianchi identities for the Riemann curvature tensor are used. □

We consider now a new curvature tensor $\bar{K}_{jkl}{}^m$, named curvature-like tensor, built from $K_{jkl}{}^m$ with the remarkable property:

$$\nabla_m \bar{K}_{jkl}{}^m = C\nabla_m K_{jkl}{}^m + D\big[(\nabla_j \Psi) b_{kl} - (\nabla_k \Psi) b_{jl}\big] + E\big[\nabla_j H_{kl} - \nabla_k H_{jl}\big], \quad (5.3)$$

where C, D, E are constants, Ψ an arbitrary scalar function, b_{kl} a symmetric $(0,2)$-type Codazzi tensor [9] and H_{kl} a symmetric $(0,2)$-type tensor. In this case Lovelock's identity is not more invariant under this change. Nevertheless it is possible to show that:

Theorem 5.1. *Let M be an n-dimensional Riemannian manifold having a curvature like tensor $\bar{K}_{jkl}{}^m$ originating from a tensor $K_{jkl}{}^m$ satisfying $\nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = 0$, and with the property* (5.3). *Then*

$$\begin{aligned}\nabla_i \nabla_m \bar{K}_{jkl}{}^m + \nabla_j \nabla_m \bar{K}_{kil}{}^m + \nabla_k \nabla_m \bar{K}_{ijl}{}^m &= (EH_{im} - CK_{im}) R_{jkl}{}^m \\ +(EH_{jm} - CK_{jm}) R_{kil}{}^m + (EH_{km} - CK_{km}) R_{ijl}{}^m. & \quad (5.4)\end{aligned}$$

Proof. The covariant derivative ∇_i is applied to the equation (5.3), a sum over cyclic permutations of indices i, j, k is performed to obtain:

$$\begin{aligned}&\nabla_i \nabla_m \bar{K}_{jkl}{}^m + \nabla_j \nabla_m \bar{K}_{kil}{}^m + \nabla_k \nabla_m \bar{K}_{ijl}{}^m \\ &= C(\nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m) \\ &+D\big[(\nabla_j \psi)(\nabla_i b_{kl} - \nabla_k b_{il}) + (\nabla_k \psi)(\nabla_j b_{il} - \nabla_i b_{jl}) + (\nabla_i \psi)(\nabla_k b_{il} - \nabla_i b_{kl})\big] \\ &+E\big[(\nabla_k \nabla_i - \nabla_i \nabla_k) H_{jl} + (\nabla_i \nabla_j - \nabla_j \nabla_i) H_{kl} + (\nabla_j \nabla_k - \nabla_k \nabla_j) H_{il}\big]. \quad (5.5)\end{aligned}$$

Now the properties of Codazzi tensor are taken into consideration, Ricci identity and equation (5.1) are then used. □

It is interesting to consider the curvature 2-form associated to the divergence of $\bar{K}_{jkl}{}^m$, that is,

$$\Pi_{(\bar{K})l} = \nabla_m \bar{K}_{jkl}{}^m dx^j \wedge dx^k. \quad (5.6)$$

The generalized Lovelock's identity allows us to discuss some general conditions which gives the closedness of the above 2-form. In fact, we may state the following remarkable:

Theorem 5.2. *Let M be an n-dimensional Riemannian manifold having a curvature like tensor $\bar{K}_{jkl}{}^m$ originating from a tensor $K_{jkl}{}^m$ satisfying $\nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = 0$, and with the property* (5.3). *Then the curvature 2-form $\Pi_{(\bar{K})l} = \nabla_m \bar{K}_{jkl}{}^m dx^j \wedge dx^k$ is closed if and only if*

$$\begin{aligned}(EH_{im} - CK_{im}) R_{jkl}{}^m + (EH_{jm} - CK_{jm}) R_{kil}{}^m & \\ + (EH_{km} - CK_{km}) R_{ijl}{}^m = 0. & \quad (5.7)\end{aligned}$$

Nevertheless, if we consider that the tensor H_{kl} is subjected to the condition:

$$F\Big[\nabla_j H_{kl} - \nabla_k H_{jl}\Big] = D\Big[(\nabla_j \Psi) b_{kl} - (\nabla_k \Psi) b_{jl}\Big], \tag{5.8}$$

being F a constant, then the Lovelock's identity is still unchanged. A symmetric $(0,2)$-type tensor H_{kl} satisfying the previous equation is called *generalized Weyl tensor*. Moreover, using the covariant derivative ∇_i on (5.8), summing over cyclic permutations of indices i, j, k and taking into account of the properties of Codazzi tensors and of the Ricci identity, we obtain easily:

$$H_{im} R_{jkl}{}^{m} + H_{jm} R_{kil}{}^{m} + H_{km} R_{ijl}{}^{m} = 0. \tag{5.9}$$

In this case the previous Theorem 5.2 takes the follwing

Theorem 5.3. *Let M be an n-dimensional Riemannian manifold with a curvature like-tensor $\bar{K}_{jkl}{}^{m}$ originating from a tensor $K_{jkl}{}^{m}$ satisfying $\nabla_i K_{jkl}{}^{m} + \nabla_j K_{kil}{}^{m} + \nabla_k K_{ijl}{}^{m} = 0$, and with the property (5.3). If H_{kl} is a generalized Weyl tensor, then $\Pi_{(\bar{K})l} = \nabla_m \bar{K}_{jkl}{}^{m} dx^j \wedge dx^k$ is closed if and only if:*

$$K_{im} R_{jkl}{}^{m} + K_{jm} R_{kil}{}^{m} + K_{km} R_{ijl}{}^{m} = 0. \tag{5.10}$$

In reference [22], [23] the authors studied a new curvature-like tensor built exactly in this way. We define it as follows:

$$\begin{aligned} \bar{K}_{(C)jkl}{}^{m} = K_{jkl}{}^{m} &+ \frac{1}{n-2}(\delta_j^m H_{kl} - \delta_k^m H_{jl} + H_j^m g_{kl} - H_k^m g_{jl}) \\ &- \frac{H}{(n-1)(n-2)}(\delta_j^m g_{kl} - \delta_k^m g_{jl}), \end{aligned} \tag{5.11}$$

where $K_{jkl}{}^{m}$ is a generalized curvature tensor satisfying(1.4), H_{kl} a symmetric (0,2) tensor with the property $\nabla^m H_{km} = \frac{1}{2}\nabla_k H$ where $H = g^{kl} H_{kl}$ being a sort of curvature scalar. The authors called it *Conformal curvature-like tensor*. Then it can be easily checked that (5.3) holds for such a curvature-like tensor as follows:

$$\begin{aligned} \nabla_m \bar{K}_{(C)jkl}{}^{m} = \nabla_m K_{jkl}{}^{m} &+ \frac{n-3}{2(n-1)(n-2)}\Big[(\nabla_j H) g_{kl} - (\nabla_k H) g_{jl}\Big] \\ &+ \frac{1}{n-2}\Big[\nabla_j H_{kl} - \nabla_k H_{ji}\Big]. \end{aligned} \tag{5.12}$$

It is now easy to see that if H_{kl} is the Weyl tensor in [23] which satisfies the condition:

$$\nabla_j H_{kl} - \nabla_k H_{jl} = \frac{1}{2(n-1)}\Big[\nabla_j H g_{kl} - \nabla_k H g_{jl}\Big]. \tag{5.13}$$

Then the 2-form associated to the divergence of the Conformal curvature like tensor is closed if and only if $K_{im}R_{jkl}{}^m + K_{jm}R_{kil}{}^m + K_{km}R_{ijl}{}^m = 0$. This condition does not depend on the tensor H_{kl}.

The same procedure may be pursued for other curvature-like tensors defined in [22]. This gives the same algebraic closedness conditions for the form $\Pi_{(\bar{K})l}$. The *Conharmonic curvature-like tensor* is defined as [22]:

$$\bar{K}_{(N)jkl}{}^m = K_{jkl}{}^m + \frac{1}{n-2}(\delta_j^m H_{kl} - \delta_k^m H_{jl} + H_j^m g_{kl} - H_k^m g_{jl}). \tag{5.14}$$

The property (5.3) can be checked as follows:

$$\begin{aligned}\nabla_m \bar{K}_{(N)jkl}{}^m = \nabla_m K_{jkl}{}^m &+ \frac{1}{2(n-2)}\Big[(\nabla_j H)g_{kl} - (\nabla_k H)g_{jl}\Big] \\ &+ \frac{1}{n-2}\Big[\nabla_j H_{kl} - \nabla_k H_{jl}\Big].\end{aligned} \tag{5.15}$$

Again it is now easy to see that if H_{kl} is the Weyl tensor [23], then the 2-form associated to the divergence of the Conharmonic curvature-like tensor is closed if and only if $K_{im}R_{jkl}{}^m + K_{jm}R_{kil}{}^m + K_{km}R_{ijl}{}^m = 0$.

The *Projective curvature-like tensor* is defined as [22]:

$$\bar{K}_{(P)jkl}{}^m = K_{jkl}{}^m + \frac{1}{n-1}(\delta_j^m H_{kl} - \delta_k^m H_{jl}). \tag{5.16}$$

Also in this case one easily gets:

$$\nabla_m \bar{K}_{(P)jkl}{}^m = \nabla_m K_{jkl}{}^m + \frac{1}{n-1}\Big[\nabla_j H_{kl} - \nabla_k H_{ji}\Big]. \tag{5.17}$$

Again it is now easy to see that if H_{kl} is the Weyl tensor [23], then the 2-form associated to the divergence of the Projective curvature-like tensor is closed if and only if $K_{im}R_{jkl}{}^m + K_{jm}R_{kil}{}^m + K_{km}R_{ijl}{}^m = 0$.

Finally the *Concircular curvature-like tensor* is defined as [22]:

$$\bar{K}_{(\tilde{C})jkl}{}^m = K_{jkl}{}^m + \frac{H}{n(n-1)}(\delta_j^m g_{kl} - \delta_k^m g_{jl}). \tag{5.18}$$

This case is particular, because one has simply:

$$\nabla_m \bar{K}_{(\tilde{C})jkl}{}^m = \nabla_m K_{jkl}{}^m + \frac{1}{n(n-1)}\Big[\nabla_j H g_{kl} - \nabla_k H g_{jl}\Big]. \tag{5.19}$$

Thus the Concircular curvature-like tensor is closed if and only if $K_{im}R_{jkl}{}^m + K_{jm}R_{kil}{}^m + K_{km}R_{ijl}{}^m = 0$. Thus the class of curvature-like tensors, having H_{kl}

as Weyl tensor, becomes a simple and general closedness condition for the 2-form associated to the divergence of each tensor.

Before completing this section, it is worthwhile to mention an example in which the equation (5.10) is satisfied.

In order to do this, let us consider a manifold whose curvature-like tensor $\bar{K}_{jkl}{}^m$ is recurrent, that is:

$$\nabla_i \bar{K}_{jkl}{}^m = \lambda_i \bar{K}_{jkl}{}^m, \tag{5.20}$$

where we suppose that the covector λ_i is closed. In general the second Bianchi identity for the curvature like tensor $\bar{K}_{jkl}{}^m$ will have a source term $\bar{B}_{ijkl}{}^m$

$$\nabla_i \bar{K}_{jkl}{}^m + \nabla_j \bar{K}_{kil}{}^m + \nabla_k \bar{K}_{ijl}{}^m = \bar{B}_{ijkl}{}^m. \tag{5.21}$$

Now noting that $\nabla_m \bar{K}_{jkl}{}^m = \lambda_m \bar{K}_{jkl}{}^m$, then the left side of (5.1) takes the form:

$$(\nabla_i \lambda_m)\bar{K}_{jkl}{}^m + (\nabla_j \lambda_m)\bar{K}_{kil}{}^m + (\nabla_k \lambda_m)\bar{K}_{ijl}{}^m + \lambda_m(\lambda_i \bar{K}_{jkl}{}^m + \lambda_j \bar{K}_{kil}{}^m + \lambda_k \bar{K}_{ijl}{}^m). \tag{5.22}$$

Now the divergence of (5.21) written in the form $\lambda_i \bar{K}_{jkl}{}^m + \lambda_j \bar{K}_{kil}{}^m + \lambda_k \bar{K}_{ijl}{}^m = \bar{B}_{ijkl}{}^m$ is taken to give:

$$\begin{aligned} \nabla_m \bar{B}_{ijkl}{}^m = (\nabla_m \lambda_i)\bar{K}_{jkl}{}^m + (\nabla_m \lambda_j)\bar{K}_{kil}{}^m + (\nabla_m \lambda_k)\bar{K}_{ijl}{}^m \\ + \lambda_m(\lambda_i \bar{K}_{jkl}{}^m + \lambda_j \bar{K}_{kil}{}^m + \lambda_k \bar{K}_{ijl}{}^m). \end{aligned} \tag{5.23}$$

If the closedness of the recurrence parameter is considered, that is, $\nabla_i \lambda_m = \nabla_m \lambda_i$, then the left side of (5.1) becomes $\nabla_m \bar{B}_{ijkl}{}^m$. From this we assert the following.

Theorem 5.4. *Let M be an n-dimensional Riemannian manifold having a recurrent curvature like tensor $\nabla_i \bar{K}_{jkl}{}^m = \lambda_i \bar{K}_{jkl}{}^m$, with the property* (5.3)*. If the covectorλ_i is closed, then*

$$\begin{aligned} (EH_{im} - CK_{im})R_{jkl}{}^m + (EH_{jm} - CK_{jm})R_{kil}{}^m \\ + (EH_{km} - CK_{km})R_{ijl}{}^m = \nabla_m \bar{B}_{ijkl}{}^m. \end{aligned} \tag{5.24}$$

In [22] the authors pointed out that if the tensor H_{kl} becomes the Weyl tensor, the Conformal curvature like tensor $\bar{K}_{(C)jkl}{}^m$ in (5.11) satisfies the second Bianchi identity with vanishing source term $\bar{B}_{ijkl}{}^m$. Thus by Theorem 5.4 and Theorem 5.2 (see (5.7)) we conclude that the curvature 2-form $\Pi_{(\bar{K})l} = \nabla_m \bar{K}_{jkl}{}^m dx^j \wedge dx^k$ (associated to the divergence of such tensor) is closed. Then this gives an example in which the equation (5.10) is satisfied.

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