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Isometries on positive operators of unit norm

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Abstract. Let $p > 1$ be a real number. We describe the structure of surjective isometries of the space of all positive operators in the unit sphere of the von Neumann–Schatten p-class. In the finite dimensional case we extend the obtained result for 'a priori' nonsurjective transformations.

1. Introduction and statement of the results

The problem of describing the morphisms of a given structure appears in many parts of mathematics and plays a crucial role in most of the cases. The natural morphisms of metric spaces are the isometries, they have a vast literature. Concerning results on linear isometries of normed spaces we refer to the two volume set [2], [3]. In the case when a linear structure is not present, the problem of isometries becomes much more difficult. Such spaces are investigated in numerous areas of mathematics. In the Hilbert space formalism of quantum mechanics several nonlinear structures appear. Among them one of the most important is the set of density operators. On this set several metrics are studied, e.g. the Bures metric and those coming from the trace norm or the Hilbert-Schmidt norm which are special cases of the p-norms. In [7] the authors have determined the structure of surjective isometries of the space of density operators with respect to the metric induced by the 1-norm. The corresponding result can be found also in [5] as Theorem 2.4.4. Motivated by that result, in this paper we describe the

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general form of the isometries of the space of all positive operators in the unit sphere of the von Neumann–Schatten p-class under the assumption $p > 1$.

In what follows, we fix the basic notation and definitions used throughout the paper. The symbol H stands for a complex Hilbert space and $B(H)$ signifies the space of bounded linear operators on H. Moreover, we denote by $|A|$ the absolute value of the operator $A \in B(H)$. Let $p \geq 1$ be a real number. The symbol $C_p(H)$ signifies the set of those elements A of $B(H)$ which have the property that for each orthonormal basis $\{\varphi_i\}_{i\in I}$ in H the series $\sum_{i\in I}\langle |A|^p\varphi_i, \varphi_i\rangle$ is convergent. We define the p-norm of an operator $A \in C_p(H)$ by the formula

$$\|A\|_p = (\operatorname{tr}|A|^p)^{\frac{1}{p}} \ ,$$

where tr denotes the trace functional. The pair $(C_p(H), \|.\|_p)$ is actually a normed space, usually called the von Neumann–Schatten p-class (see, e.g. [9]). We denote by d_p the metric induced by the p-norm and $C_p(H)_1^+$ stands for the set of those positive operators in $C_p(H)$ which have unit p-norm. The elements of $C_1(H)_1^+$ are called density operators.

As we have already mentioned, the surjective isometries of $C_1(H)_1^+$ have been investigated in [7]. As for the case when the surjectivity of transformations is not assumed, in [6] the general form of the isometries of $C_1(H)_1^+$ endowed with d_p is given for any $p \geq 1$ under the condition $\dim H < \infty$. Since the p-norm corresponds to the space $C_p(H)$ rather than $C_1(H)$ it is a natural problem to describe the structure of the isometries of the space $C_p(H)_1^+$ equipped with d_p $(p > 1)$. Turning to our results, let p be a real number greater than 1. We remark that using the strict convexity of $C_p(H)$ (c.f. [4, Theorem 2.4.]) it can be shown that any isometry of the space of all positive operators in $C_p(H)$ is affine. However, concerning isometries of $C_p(H)_1^+$ we cannot formulate such assertion since this set is not convex. It is an important feature of the p-norms that they are unitarily invariant meaning that for any $A \in C_p(H)$ and unitary operators U, V on H the operator UAV belongs to $C_p(H)$ and $\|UAV\|_p = \|A\|_p$ $(p \geq 1)$. Similar assertion holds for antiunitary operators. We deduce that for any unitary or antiunitary operator U on H the transformation $A \mapsto UAU^*$ $(A \in C_p(H)_1^+)$ is a surjective isometry with respect to d_p $(p \geq 1)$. The following result tells that if $p > 1$ then the reverse statement is also true: all surjective isometries are necesseraly of that form.

Theorem 1. *Let $p > 1$ be a real number and suppose that $\phi : C_p(H)_1^+ \to C_p(H)_1^+$ is a surjective isometry with respect to d_p. Then we have either a unitary or an antiunitary operator U on H such that ϕ is of the form*

$$\phi(A) = UAU^* \quad (A \in C_p(H)_1^+). \tag{1}$$

Referring to Theorem 1 and [5, Theorem 2.4.4.], we obtain that for any $p \geq 1$ the surjective isometries of $(C_p(H)_1^+, d_p)$ can be written in the form (1). In the finite dimensional case, we can extend the previous statement for the isometries of $C_p(H)_1^+$ which are not assumed to be surjective.

Theorem 2. *Suppose that* $\dim H < \infty$ *and let* $p > 1$ *be a real number. If* $\phi : C_p(H)_1^+ \to C_p(H)_1^+$ *is an isometry with respect to* d_p*, then we have either a unitary or an antiunitary operator* U *on* H *such that* ϕ *is of the form* (1).

Theorem 2 and the case $p = 1$ of [6, Theorem 1] tell us that if H is finite dimensional, then for any $p \geq 1$ the isometries of $(C_p(H)_1^+, d_p)$ have the form (1).

2. Proofs

Throughout this section we shall use the following notation. The collection of rank-1 projections on H is denoted by $P_1(H)$. In the case when $n = \dim H < \infty$ for any self-adjoint operator (or equivalently $n \times n$ complex Hermitian matrix) $T : H \to H$ the symbol $(\lambda_i(T))_{i=1}^n$ signifies the increasing sequence of the eigenvalues of T (counted according to their multiplicities). We say that the self-adjoint operators $A, B \in B(H)$ are orthogonal if $AB = 0$, which is equivalent to the property that they have mutually orthogonal ranges.

In what follows, we collect some assertions which will be needed in this section. It is a straightforward consequence of the definition of p-classes that any finite rank operator in $B(H)$ belongs to $C_p(H)$ and that any element of $C_p(H)$ is compact ($p \geq 1$). We recall a formula concerning p-norms. Namely, if $p \geq 1$ and $A \in C_p(H)$ is self-adjoint, then

$$\|A\|_p = \left(\sum_{i \in I} |\lambda_i|^p \right)^{\frac{1}{p}},$$

where $(\lambda_i)_{i \in I}$ is a sequence consisting of the nonzero eigenvalues of A (counted according to multiplicities). The simple observation below can be proved by the Cauchy–Schwarz inequality.

($*$) Suppose that $\dim H < \infty$ and let $A \in B(H)$ be a positive operator. Then for any unit vector $x \in H$ we have $\langle Ax, x \rangle \leq \lambda_n(A)$ and equality holds if and only if x is an eigenvector of A corresponding to $\lambda_n(A)$.

Our arguments are based on a so-called identification lemma which reads as follows.

Lemma. *Suppose that* $\dim H < \infty$ *and let* $p, \gamma \geq 1$ *be fixed real numbers. If* $A, B \in C_p(H)_1^+$ *are such that the equality*

$$d_p(A, \gamma P) = d_p(B, \gamma P)$$

holds for any $P \in P_1(H)$, *then* $A = B$.

Proof. In the following argument we suppose that $n = \dim H > 1$. Let $T \in C_p(H)_1^+$ and define the function $f_T : P_1(H) \to \mathbb{R}$ by

$$f_T(P) = d_p(T, \gamma P)^p \quad (P \in P_1(H)).$$

Moreover denote by $\mathcal{M}_T$ the eigensubspace of T corresponding to $\lambda_n(T)$. We are going to show that f_T uniquely determines $\lambda_n(T)$ and $\mathcal{M}_T$. To do this, we assert that

$$\min f_T(P_1(H)) = (\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p$$

and f_T attains its minimum exactly for those rank-1 projections on H which project into $\mathcal{M}_T$. For the proof, first observe that since $\|T\|_p^p = \sum_{i=1}^n \lambda_i(T)^p = 1$, we have

$$f_T(P) = (\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p \tag{2}$$

for any $P \in P_1(H)$ whose range is included in $\mathcal{M}_T$.

We have to prove that $(\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p$ is a lower bound of the range of f_T. To this end, let $P \in P_1(H)$. We learn from [1, Theorem 9.7] that if $\|.\|$ is a unitarily invariant norm on the space of $n \times n$ complex matrices and R, S are Hermitian matrices, then

$$\| \operatorname{diag}(\lambda_n(R), \dots, \lambda_1(R)) - \operatorname{diag}(\lambda_n(S), \dots, \lambda_1(S))\| \leq \|R - S\|,$$

where diag(.) denotes the diagonal matrix whose diagonal is the given sequence. We have seen in the introduction that the p-norm is unitarily invariant, thus the previous inequality yields

$$(\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p = \sum_{i=1}^n |\lambda_i(T) - \lambda_i(\gamma P)|^p \leq \|T - \gamma P\|_p^p = f_T(P).$$

This together with equality (2) implies that

$$\min f_T(P_1(H)) = (\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p \tag{3}$$

and f_T attains its minimum for any element of $P_1(H)$ which projects into $\mathcal{M}_T$.

In what follows let P be a rank-1 projection on H which minimizes f_T. We have to show that P projects into $\mathcal{M}_T$. To do this, first we give a lower bound for $f_T(P)$. According to [4, Lemma 2.2.], for any linear operator K on H and orthonormal basis $\{\varphi_i\}_{i=1}^n$ in H we have $\sum_{i=1}^n |\langle K\varphi_i, \varphi_i\rangle|^p \le \|K\|_p^p$. Let x be a unit vector in $\operatorname{rng} P$, where rng denotes the range of operators. Moreover choose an orthonormal basis $\{\varphi_i\}_{i=1}^n$ in H such that $\varphi_1 = x$. Apply the preceding assertion for this basis and for the operator $K = T - \gamma P$ in order to get

$$\begin{aligned}(\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p = f_T(P) = \|T - \gamma P\|_p^p \ge \\ |\langle Tx, x\rangle - \gamma|^p + \sum_{i=2}^n \langle T\varphi_i, \varphi_i\rangle^p.\end{aligned} \tag{4}$$

Since T is positive, by $(*)$ one has $\langle Tx, x\rangle \le \lambda_n(T) \le 1 \le \gamma$. Referring to inequality (4), we thus get

$$\sum_{i=2}^n \langle T\varphi_i, \varphi_i\rangle^p \le 1 - \lambda_n(T)^p. \tag{5}$$

We emphasize that this relation is valid for all orthonormal bases $\{\varphi_i\}_{i=2}^n$ in $(\operatorname{rng} P)^\perp$, where $^\perp$ denotes the orthogonal complement of subspaces in H.

Now define $Q = I - P$ and $\mathcal{L} = \operatorname{rng} Q$. It is clear that

$$\sum_{i=2}^n \langle T\varphi_i, \varphi_i\rangle^p = \sum_{i=2}^n \langle QTQ\varphi_i, \varphi_i\rangle^p$$

for all orthonormal bases $\{\varphi_i\}_{i=2}^n \subset \mathcal{L}$. It is obvious that there is an orthonormal basis in $\mathcal{L}$ consisting of eigenvectors of $(QTQ)|_{\mathcal{L}}$. Inserting such a basis in inequality (5) and using the last displayed equality we obtain

$$1 - \lambda_n(T)^p \ge \operatorname{tr}\left((QTQ)|_{\mathcal{L}}\right)^p. \tag{6}$$

It is clear that $(QTQ)^p$ is 0 on $\mathcal{L}^\perp$. Therefore, using the fact that the p-norm is invariant under taking adjoints (see [4, Theorem 1.3.]), we get that

$$\begin{aligned}&\operatorname{tr}\left((QTQ)|_{\mathcal{L}}\right)^p = \operatorname{tr}(QTQ)^p|_{\mathcal{L}} = \operatorname{tr}(QTQ)^p = \left\|\sqrt{T}Q\right\|_{2p}^{2p} \\ &= \left\|Q\sqrt{T}\,\right\|_{2p}^{2p} = \operatorname{tr}\left(\sqrt{T}Q\sqrt{T}\,\right)^p = \operatorname{tr}\left(T - \sqrt{T}P\sqrt{T}\,\right)^p = \left\|T - \sqrt{T}P\sqrt{T}\,\right\|_p^p.\end{aligned} \tag{7}$$

Let $N = \sqrt{T}P\sqrt{T}$. By [1, Theorem 9.7], equation (7) and inequality (6), we infer

$$\sum_{i=1}^n |\lambda_i(T) - \lambda_i(N)|^p \le \|T - N\|_p^p \le 1 - \lambda_n(T)^p. \tag{8}$$

Observe that N is an element of $P_1(H)$ multiplied by a scalar which is easily seen to be $\operatorname{tr} N$. Hence

$$\lambda_n(N) = \operatorname{tr} N = \langle Tx, x\rangle$$

and by (8) it follows that

$$|\lambda_n(T) - \langle Tx, x\rangle|^p + 1 - \lambda_n(T)^p \leq 1 - \lambda_n(T)^p.$$

Now we deduce $\langle Tx, x\rangle = \lambda_n(T)$. Using $(*)$, we conclude that x is an eigenvector of T corresponding to $\lambda_n(T)$, which means that P projects into $\mathcal{M}_T$.

To sum up, we obtain that f_T attains its minimum exactly for those elements of $P_1(H)$ which project into $\mathcal{M}_T$. This implies that f_T uniquely determines $\mathcal{M}_T$. We assert that from f_T the number $\lambda_n(T)$ can also be recovered. Indeed, it is clear that the function $\lambda \mapsto (\gamma - \lambda)^p + 1 - \lambda^p$ ($\lambda \in]0, 1]$) is strictly decreasing and thus injective. Hence, from the value $(\gamma - \lambda_n(T))^p + 1 - \lambda_n(T)^p$ we can recover $\lambda_n(T)$ and then by (3) it follows that f_T uniquely determines also $\lambda_n(T)$.

To complete the proof we use induction on $\dim H$. It is obvious that Lemma holds for 1-dimensional complex Hilbert spaces. Suppose now that it is valid for any space of dimension at most $n - 1$ and that A, B are operators satisfying the conditions of Lemma. By what we have proved so far, we see that the maximal eigenvalues of A and B are the same and this holds also for the corresponding eigensubspaces $\mathcal{M}_A, \mathcal{M}_B$. This yields that $A|_{\mathcal{M}_A} = B|_{\mathcal{M}_A}$. If $\mathcal{M}_A = \mathcal{M}_B = H$ or $\lambda_n(A) = \lambda_n(B) = 1$, then both of A and B is 0 on $\tilde{\mathcal{M}} = \mathcal{M}_A^{\perp}$, therefore $A = B$. Otherwise, we easily infer

$$d_p\left(A|_{\tilde{\mathcal{M}}}, \gamma P|_{\tilde{\mathcal{M}}}\right) = d_p\left(B|_{\tilde{\mathcal{M}}}, \gamma P|_{\tilde{\mathcal{M}}}\right) \quad (P \in P_1(H),\ \operatorname{rng} P \subset \tilde{\mathcal{M}}).$$

It is obvious that $0 < \|A|_{\tilde{\mathcal{M}}}\|_p = \|B|_{\tilde{\mathcal{M}}}\|_p \leq 1$, and the last displayed equality implies that for every $P \in P_1(H)$ projecting into $\tilde{\mathcal{M}}$ we have

$$d_p\left(\frac{1}{\|A|_{\tilde{\mathcal{M}}}\|_p} A|_{\tilde{\mathcal{M}}}, \frac{\gamma}{\|A|_{\tilde{\mathcal{M}}}\|_p} P|_{\tilde{\mathcal{M}}}\right) = d_p\left(\frac{1}{\|A|_{\tilde{\mathcal{M}}}\|_p} B|_{\tilde{\mathcal{M}}}, \frac{\gamma}{\|A|_{\tilde{\mathcal{M}}}\|_p} P|_{\tilde{\mathcal{M}}}\right).$$

Clearly, $\gamma / \|A|_{\tilde{\mathcal{M}}}\|_p \geq 1$, and

$$\frac{1}{\|A|_{\tilde{\mathcal{M}}}\|_p} A|_{\tilde{\mathcal{M}}}, \frac{1}{\|A|_{\tilde{\mathcal{M}}}\|_p} B|_{\tilde{\mathcal{M}}} \in C_p(\tilde{\mathcal{M}})_1^+.$$

Moreover, it is trivial that when P runs through the set of all rank-1 projections on H projecting into $\tilde{\mathcal{M}}$, then $P|_{\tilde{\mathcal{M}}}$ runs through the set $P_1(\tilde{\mathcal{M}})$. Thus, by the inductive hypothesis it follows that $(1/\|A|_{\tilde{\mathcal{M}}}\|_p) A|_{\tilde{\mathcal{M}}} = (1/\|A|_{\tilde{\mathcal{M}}}\|_p) B|_{\tilde{\mathcal{M}}}$ and this together with the previous argument yields $A = B$. The proof of Lemma is complete. $\square$

We remark that in the case when $p = \gamma = 1$, Lemma has a nice geometrical content. To see this, we recall the well-known fact that the set of density operators acting on H is convex and its extreme points are the elements of $P_1(H)$. Having this assertion in mind, the mentioned case of Lemma can be interpreted as follows. Considering $C_1(H)_1^+$ as a metric space equipped with d_1, each density operator on H is uniquely determined by its distances to the extreme points of $C_1(H)_1^+$. We now turn to the proof of our first result.

Proof of Theorem 1. We begin this part with a characterization of orthogonality of operators in terms of their distance. Namely, let $A, B \in C_p(H)_1^+$. Then we have

$$AB = 0 \iff d_p(A, B) = 2^{1/p}. \tag{9}$$

In the proof of [6, Theorem 1] a similar equivalence has been verified. It is easy to see that most parts of the corresponding argument are valid in the present case as well. Namely, if $d_p(A, B) = 2^{1/p}$, then there is an orthonormal basis $\{e_i\}_{i \in I}$ in H whose members are common eigenvectors of A and B, and $|\lambda_i - \mu_i|^p = \max\{\lambda_i^p, \mu_i^p\}$, where $\lambda_i = \langle Ae_i, e_i\rangle$, respectively $\mu_i = \langle Be_i, e_i\rangle$ is the eigenvalue of A, respectively B corresponding to e_i $(i \in I)$. It follows that for any $i \in I$ at least one of the numbers λ_i and μ_i is 0 and hence $AB = 0$. Conversely, if the latter equality holds, then it is easy to check that $d_p(A, B) = 2^{1/p}$. Now we conclude that ϕ preserves orthogonality in both directions, i.e. for any $A, B \in C_p(H)_1^+$ we have

$$AB = 0 \iff \phi(A)\phi(B) = 0.$$

It is clear that ϕ is bijective. In the proof of [8, Theorem 4], using only the bijectivity and the orthogonality preserving property of a certain transformation on $C_1(H)_1^+$, the author showed that it preserves the elements of $P_1(H)$ in both directions. It is easy to check that the corresponding argument in [8] is valid in this case as well, thus we get that for any $A \in C_p(H)_1^+$ one has $A \in P_1(H)$ if and only if $\phi(A) \in P_1(H)$. It follows that $\phi|_{P_1(H)} : P_1(H) \to P_1(H)$ is bijective. The proof of [6, Theorem 1] contains a formula for the distance between rank-1 projections on a finite dimensional complex Hilbert space. The argument which the authors used to derive that formula does not in fact require finite dimensionality, therefore it holds for any Hilbert space. Namely, we have

$$d_p(P, Q) = 2^{\frac{1}{p}} \sqrt{1 - \operatorname{tr} PQ} \quad (P, Q \in P_1(H)). \tag{10}$$

We infer that

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \quad (P, Q \in P_1(H)).$$

In the mathematical foundations of quantum mechanics the trace of the product of rank-1 projections is usually called transition probability, so the latter equality means that $\phi|_{P_1(H)}$ preserves this quantity. A well-known theorem of Wigner describes the structure of those bijective transformations on the set of rank-1 projections on a Hilbert space which preserve transition probability. Applying this statement (c.f. p. 12 in [5]) we get that there exists either a unitary or an antiunitary operator U on H such that

$$\phi(P) = UPU^* \quad (P \in P_1(H)).$$

Let $A \in C_p(H)_1^+$ be an operator of finite rank. Referring to the preceding paragraph, we deduce that for any $P \in P_1(H)$ the equality $d_p(\phi(A), UPU^*) = d_p(A, P)$ holds true. Since the p-norm is unitarily invariant, it follows that one has $d_p(U^*\phi(A)U, P) = d_p(A, P)$ for each $P \in P_1(H)$. Using the equivalence (9), this yields that a rank-1 projection on H is orthogonal to $\psi(A) = U^*\phi(A)U$ if and only if it is orthogonal to A. We infer that $\operatorname{rng}\psi(A) = \operatorname{rng} A$. It is clear that when P runs through the set of those elements in $P_1(H)$ which project into $\operatorname{rng} A$, the operator $P|_{\operatorname{rng} A}$ runs through $P_1(\operatorname{rng} A)$. The previous observations imply that for any $P \in P_1(\operatorname{rng} A)$ the equality $d_p\left(\psi(A)|_{\operatorname{rng} A}, P\right) = d_p\left(A|_{\operatorname{rng} A}, P\right)$ holds. Then Lemma applies and we obtain that $\psi(A)|_{\operatorname{rng} A} = A|_{\operatorname{rng} A}$ and hence $\phi(A) = UAU^*$.

To complete the proof let $A \in C_p(H)_1^+$. By [4, Lemma 5.2.], the collection of finite rank operators of $B(H)$ is dense in $C_p(H)$, which easily yields that the finite rank elements of $C_p(H)_1^+$ are dense in that space. Since ϕ is continuous, it follows by the preceding paragraph that $\phi(A) = UAU^*$ and then we are done. □

We now present the proof of our second statement.

Proof of Theorem 2. First, observe that by (9) the transformation ϕ preserves orthogonality in both directions. Let $n = \dim H$. Elementary considerations show that for any $A \in C_p(H)_1^+$ we have $A \in P_1(H)$ exactly when A is contained in a collection of n pairwise orthogonal operators in $C_p(H)_1^+$. We conclude that ϕ maps the set $P_1(H)$ into itself. Referring to (10), we obtain that $\phi|_{P_1(H)} : P_1(H) \to P_1(H)$ preserves the transition probability. There is a nonsurjective version of Wigner's theorem, c.f. [5, Theorem 2.1.4.]. In the present case it yields that we have a unitary or an antiunitary operator U on H such that

$$\phi(P) = UPU^* \quad (P \in P_1(H)).$$

Now let $A \in C_p(H)_1^+$. Just as in the previous proof, we deduce that $d_p(U^*\phi(A)U, P) = d_p(A, P)$ holds for any $P \in P_1(H)$. Then Lemma applies and we get that $U^*\phi(A)U = A$ and this completes the proof of Theorem 2. □

3. Remarks

We have seen that in the finite dimensional case the conclusion of Theorem 1 holds for any isometry. As for infinite dimensional spaces, concerning arbitrary isometries we cannot expect such a regular form like (1). To see this, consider the following example. Assume that $\dim H = \infty$. Then one can construct two subspaces H_1, H_2 in H such that the direct sum of them is H and $1 \leq \dim H_2 < \infty$. Denote by P the projection of H onto H_1. It is clear that the Hilbert dimensions of H and H_1 are the same, therefore there is a unitary transformation V which maps H onto H_1. Now for a given $p \geq 1$ define $\phi : C_p(H)_1^+ \to B(H)$ by $\phi(A) = VAV^*P$, i.e. $\phi(A)$ is the operator which equals VAV^* on H_1 and 0 on H_2 ($A \in C_p(H)_1^+$). By the unitary invariance of p-norms, $VAV^* \in C_p(H_1)$ and $\|VAV^*\|_p = \|A\|_p$ ($A \in C_p(H)$), therefore it is easy to see that ϕ maps $C_p(H)_1^+$ into itself and it is an isometry with respect to d_p. Observe that the range of ϕ does not contain any rank-1 projection P on H which projects into H_2. This shows that ϕ is not surjective and hence it cannot be written in the form (1).

We conclude the paper with an application of the argument in the proof of Theorem 2. The notion of p-norms is usually extended also for the case $p = \infty$ by defining $\|A\|_\infty$ to be the operator norm of $A \in B(H)$. Observe that for any unitary-antiunitary operator U on H the transformation $A \mapsto UAU^*$ ($A \in C_1(H)_1^+$) is an isometry of $C_1(H)_1^+$ equipped with the metric which comes from the operator norm. As we have already mentioned, in [6] the authors described the structure of the isometries of $C_1(H)_1^+$ with respect to d_p ($p \geq 1$) under the assumption $\dim H < \infty$. Using an argument similar to the one in the proof of Theorem 2 it can be shown that the conclusion of [6, Theorem 1] holds also in the remaining case, $p = \infty$. Namely, we have the following assertion. Suppose that $\dim H < \infty$ and let $\phi : C_1(H)_1^+ \mapsto C_1(H)_1^+$ be an isometry with respect to the metric induced by the operator norm. Then

$$\phi(A) = UAU^* \quad (A \in C_1(H)_1^+),$$

where U is a unitary-antiunitary operator on H. We omit the details of the corresponding proof.

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