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Odd solutions of $\sigma(n) - 2n = 2$ have at least six distinct prime factors

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Abstract. In this paper we prove that odd integers n with $\sigma(n) = 2(n+1)$ have at least six distinct prime factors.

1. Introduction

A positive integer n is called perfect if $\sigma(n) = 2n$, where $\sigma(n)$ is the divisor sum $\sigma(n) = \sum_{d|n,\ d\geq 1} d$. By Euclid and Euler, it is known that all the even perfect numbers are of the form $2^{p-1}(2^p - 1)$ such that $2^p - 1$ is a Mersenne prime. On the other hand, no odd perfect number is known up to now. Many necessary conditions for the existence of an odd perfect number have been found. For example, Euler showed that an odd perfect number n has prime factorization:

$$n = p_0^{e_0} p_1^{2e_1} \dots p_k^{2e_k}, \quad p_0 \equiv e_0 \equiv 1 \mod 4.$$

Let $\omega(n)$ be the number of distinct prime factors of n. In [10], Nielsen proved that $\omega(n) \geq 9$ for n being odd perfect.

Analogous to the perfect numbers, a natural number n is called quasi-perfect (resp. almost-perfect) if $\sigma(n) - 2n = 1$ (resp. -1). The only known almost-perfect numbers are powers of 2 (p. 74 of [6]). For quasi-perfect numbers, Cattaneo [3] showed that they are odd squares. But still none of them is found. For n being

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quasi-perfect numbers, Abbott, Aull, Brown and Suryanarayana [1] proved $\omega(n) \geq 5$; Kishore [9] proved $\omega(n) \geq 6$; Hagis and Cohen [7] proved $\omega(n) \geq 7$.

Generally, concerning equations $\sigma(n) - 2n = 2k$ $(k \geq 0)$, Tripathi (p. 148 of [11]), raised the following problem: "Let $k \geq 0$. If $\sigma(n) = 2(n+k)$, then n must be even." However, the answer for general k is negative (see Proposition 2.10 below). But for the interesting case $k = 1$, the answer seems to be affirmative. In this paper, we will give some evidence in this direction. The main result is the following theorem.

Theorem 1.1. *If n is an odd integer with $\sigma(n) - 2n = 2$, then $\omega(n) \geq 6$.*

The main tools of the proof are the results of Dickson [4] and of Kishore [9] on primitive non-deficient numbers (for the definition of primitive non-deficient numbers, see the beginning of Section 2).

2. Main results

In the sequel, we call a natural number n near-perfect if $\sigma(n) - 2n = 2$ for convenience.

A positive integer n is called abundant, perfect, or deficient according as

$$\sigma(n) > 2n, \ = 2n, \ < 2n.$$

If $d_1, \dots, d_k$ are the divisors of n, then the divisors of sn include $sd_1, \dots, sd_k$ and 1 for $s > 1$. Therefore

$$\sigma(sn) > s\sigma(n) \quad (s > 1),$$

so that any greater multiple of an abundant number or perfect number is abundant.

A non-deficient number will be called primitive if it is not a multiple of a smaller non-deficient number. The above discussion shows that perfect numbers are primitive. A well-known result of Dickson [4] says that there are only finitely many odd primitive non-deficient numbers having fixed number of distinct prime factors.

Proposition 2.1. *Near-perfect numbers are primitive abundant.*

Proof. Let n be near-perfect. If n is not primitive, then it has a non-deficient divisor m such that $m \neq n$. Replacing m by its multiple if necessary, we can assume $p = n/m$ is a prime and m is still non-deficient. If $n = p^t$, then

$$\frac{\sigma(n)}{n} = 1 + \frac{1}{p} + \cdots + \frac{1}{p^t} \leq 1 + \frac{1}{2} + \cdots + \frac{1}{2^t} < 2.$$

This contradicts the fact that n is abundant. So n cannot be a prime power. Let q be another prime divisor of n. We have

$$\sigma(n) \geq q + 1 + \sum_{p|d|n} d = q + 1 + p \sum_{d'|m} d' = q + 1 + p\sigma(m) \geq q + 1 + 2pm \geq 2n + 3.$$

This contradicts to the fact $\sigma(n) = 2n + 2$. So n is primitive. □

Remark 2.2. The above proposition does not hold for general k since for $k = 2$, $n = 12$ satisfies $\sigma(n) - 2n = 2k$. But 12 is not primitive since $6 = 1 + 2 + 3$ is perfect.

Tripathi [11] observed $2^{e-1}(2^e - 3)$ with $2^e - 3$ being a prime is an even near-perfect number. In the following, we prove its converse under some condition.

Proposition 2.3. *Let n have two distinct prime factors. Then n is near-perfect if and only if n is of the form $n = 2^{e-1}(2^e - 3)$, where $2^e - 3$ is a prime.*

Proof. The "if" part is a direct verification. For the "only if" part, let n be a near-perfect number with prime factorization $n = p^e q^t$. If n is odd, we have

$$\begin{aligned} 2 \leq \frac{\sigma(n)}{n} &= \frac{(1 - p^{-e-1})(1 - q^{-t-1})}{(1 - p^{-1})(1 - q^{-1})} < \frac{1}{(1 - p^{-1})(1 - q^{-1})} \\ &\leq \frac{1}{(1 - 3^{-1})(1 - 5^{-1})}. \end{aligned}$$

This is a contradiction. Hence, we can assume $p = 2$ and $n = 2^e q^t$.

Assume $t > 1$. Then we have

$$\begin{aligned} \sigma(n) - 2(n + 1) &= (2^{e+1} - 1)(1 + q + \cdots + q^t) - 2^{e+1}q^t - 2 \\ &= -q^t + (2^{e+1} - 1)(q + \cdots + q^{t-1}) + (2^{e+1} - 3) = 0. \end{aligned}$$

So $q|(2^{e+1} - 3)$. Since $2^{e+1} - 3 > 0$, we have $2^{e+1} - 3 \geq q$. Hence

$$\sigma(n) - 2(n + 1) \geq -q^t + (q + 2)(q + \cdots + q^{t-1}) + q > 0.$$

The above contradiction tells us that $t = 1$, $n = 2^e q$. Calculating the divisor sum of n again, we get

$$0 = \sigma(n) - 2(n + 1) = (2^{e+1} - 1)(1 + q) - 2^{e+1}q - 2 = 2^{e+1} - q - 3.$$

Thus $q = 2^{e+1} - 3$ and $n = 2^e(2^{e+1} - 3)$. □

Remark 2.4. Unlike perfect numbers, not all even near-perfect numbers have two distinct prime factors. Using Mathematica 7.0, we searched the near-perfect numbers up to 10^{10}. They are:

$$20,\ 104,\ 464,\ 650,\ 1952,\ 130304,\ 522752,\ 8382464,\ 134193152.$$

We get $650 = 2\times 5^2\times 13$ has three prime factors and all the others have two prime factors. Thus, the condition "n has two distinct prime factors" in Proposition 2.3 is necessary.

Euler showed that an odd perfect number n has prime factorization:

$$n = p_0^{e_0} p_1^{2e_1} \dots p_k^{2e_k},\ p_0 \equiv e_0 \equiv 1 \mod 4.$$

For near perfect numbers, we have the following analogous proposition.

Proposition 2.5. *Assume n is an even near-perfect number. Then the prime factorization of n is*

$$n = 2^e p_1^{e_1} p_2^{2e_2} \dots p_k^{2e_k}, \quad p_1 \equiv e_1 \equiv 1 \mod 4.$$

Proof. Let $2^e || n$. Let $m = n/2^e$. Since

$$\sigma(n) = \sigma(2^e)\sigma(m) = (2^{e+1}-1)\sigma(m) = 2^{e+1}m + 2,$$

we get $2||\sigma(m)$.

For p being an odd prime, $\sigma(p^t) = 1 + p + \dots + p^t$ is odd if and only if t is even.

Let $p_1, \dots, p_k$ be the distinct prime factors of n. From the above discussion, there is only one prime with odd exponent. Without loss generality, we can assume it is p_1. Write $n = 2^e p_1^{e_1} p_2^{2e_2} \dots p_k^{2e_k}$ with e_1 being odd. Then $2||\sigma(p_1^{e_1})$.

If $p_1 \equiv 3 \mod 4$, then $4|(1+p_1)|\sigma(p_1^{e_1})$. So $p_1 \equiv 1 \mod 4$.

If $e_1 \equiv 3 \mod 4$, then $4|(1+p_1+p_1^2+p_1^3)|\sigma(p_1^{e_1})$. So $e_1 \equiv 1 \mod 4$.

This completes the proof. □

Example 2.6. $650 = 2 \times 13 \times 5^2$ is an even near-perfect number with the special prime 13 congruent to 1 modulo 4.

The following lemma will be used in the proof of Theorem 1.1.

Lemma 2.7. *Let n be an odd near-perfect number. Then n is relatively prime to $\sigma(n)$. Assume $p^e||n$. If $3|n$, then*

$$e \not\equiv 2 \text{ mod } 3 \text{ for } p \equiv 1 \text{ mod } 3;\ e \not\equiv 1 \text{ mod } 2 \text{ for } p \equiv 2 \text{ mod } 3.$$

If $5|n$, *then*

$$e \not\equiv 4 \bmod 5 \text{ for } p \equiv 1 \bmod 5;\ e \not\equiv 3 \bmod 4 \text{ for } p \equiv 2 \bmod 5;$$

$$e \not\equiv 3 \bmod 4 \text{ for } p \equiv 3 \bmod 5;\ e \not\equiv 1 \bmod 2 \text{ for } p \equiv 4 \bmod 5.$$

Proof. Since $\sigma(n) - 2n = 2$ and n is odd, we have $(\sigma(n), n) = 1$.

If $3|n$, then $3 \nmid \sigma(n)$. By assumption, $\sigma(p^e)|\sigma(n)$, so $3 \nmid \sigma(p^e)$.

If $p \equiv 1 \mod 3$, then $\sigma(p^e) = 1 + p + p^2 + \cdots + p^e \equiv e + 1 \mod 3$. So $e \not\equiv 2 \mod 3$.

If $p \equiv 2 \mod 3$, then $\sigma(p^e) \equiv 1 + 2 + 2^2 + \cdots + 2^e \equiv 2^{e+1} - 1 \mod 3$. Since $2^2 \equiv 1 \mod 3$, we have $2 \nmid (e + 1)$. So $e \not\equiv 1 \mod 2$.

The case $5|n$ can be proved similarly. So we omit the details. □

Fixing a positive integer s, Dickson [4] proved that there are only finitely many odd primitive non-deficient numbers having s distinct prime factors. He also listed all primitive non-deficient odd numbers with four or fewer distinct prime factors. By Proposition 2.1, near perfect numbers are primitive abundant. So we can check Dickson's [4] tables to show whether there exists an odd near-perfect numbers with four or fewer distinct prime factors.

However, in the procedure, we found there are some mistakes in Dickson's table of odd primitive non-deficient numbers having 4 distinct prime factors. Originally, we write a computer programme to correct the table. Then, the referee tell us that the errata of Dickson's table already exist (see [5] and [8]). For the convenience of reader, we still decide to relist the entire table at the end of this paper. Our table coincides with the combination of Dickson's table and errata in Ferrier [5] and Herzog [8].

Lemma 2.8. *If n is an odd near-perfect number, then $\omega(n) \geq 5$.*

Proof. By Proposition 2.1, n is primitive abundant. Assume $\omega(n) \leq 4$. By the Theorem of Dickson (p. 417 of [4]), we have $\omega(n) \geq 3$. If $\omega(n) = 3$, then n is in the table of [4] (p. 417). If $\omega(n) = 4$, then n is in the table A of the last section. Using Lemma 2.7, lots of numbers in the tables can be excluded quickly. We list the remaining possible values of n below.

1) $\omega(n) = 3$:

$$3^2 5^2 7,\quad 3^5 5^2 13;$$

2) $\omega(n) = 4$:

$$3^3 7^1 11^2 13^1,\quad 3^6 7^1 11^2 17^2,\quad 3^4 7^3 11^2 23^2,\quad 3^6 7^3 13^1 17^2,\quad 3^6 7^4 13^3 19^3,$$

$$3^7 7^3 13^4 19^4,\quad 3^8 7^3 13^3 19^3,\quad 3^1 5^2 7^1 13^1,\quad 3^2 5^2 11^2 43^1,\quad 3^2 5^2 11^2 61^1,$$

$$3^4 5^2 13^1 p^1 \ \ (97 \le p \le 373,\ p \equiv 1 \bmod 3,\ p \not\equiv 4 \bmod 5),\ \ 3^5 5^2 17^2 61^1,$$
$$3^5 5^2 17^2 67^1,\ \ 3^5 5^2 17^2 73^1,\ \ 3^4 5^4 17^2 103^1,\ \ 3^5 5^4 17^2 151^1,\ \ 3^5 5^4 17^2 157^1,$$
$$3^5 5^4 17^2 163^1,\ \ 3^5 5^6 17^2 181^1,\ \ 3^6 5^4 17^2 181^1,\ \ 3^6 5^4 17^2 193^1,\ \ 3^9 5^4 17^2 223^1,$$
$$3^8 5^6 17^2 239^2,\ \ 3^{10} 5^6 17^2 241^1,\ \ 3^4 5^4 23^2 41^2,\ \ 3^5 5^4 23^2 43^1,\ \ 3^5 5^4 29^2 31^1.$$

Direct computation shows that these numbers are not near-perfect. This is a contradiction, which shows that $\omega(n) \ge 5$. □

In [9], Kishore proved that $|\sigma(n)/n - 2| > 10^{-14}$ for all odd integers n with five distinct prime factors. Moreover, he gave a table of odd primitive abundant numbers n with five distinct prime factors for which $2 < \sigma(n)/n < 2 + 2/10^{10}$. Using Kishore's table, finally, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.8, we only need to show $\omega(n) \ne 5$. Otherwise, assume $\omega(n) = 5$. If $n > 10^{10}$, then $2 < \sigma(n)/n = 2 + (2/n) < 2 + 2/10^{10}$. Since n is primitive abundant, it is in the Table 2 of [9]. Using Lemma 2.7 again, we get that the possible values of n are

$$3^{24} 5^{12} 17^6 257^4 65521,\ \ 3^{21} 5^{10} 17^4 257^4 62563,\ \ 3^{20} 5^8 17^6 257^4 63397,$$
$$3^{17} 5^4 17^2 227^2 44281,\ \ 3^{16} 5^{10} 17^4 257^4 62533,\ \ 3^{16} 5^8 17^8 257^4 63377^2,$$
$$3^{12} 5^8 17^4 257^2 58271^2,\ \ 3^{11} 5^8 17^6 257^2 56453^2,\ \ 3^{10} 5^{12} 17^8 257^2 47701,$$
$$3^8 5^{16} 17^8 257^4 15137^2,\ \ 3^7 7^3 13^4 19^3 p \ \ (1193683 \le p \le 1193783).$$

Direct computation shows that these numbers are not near-perfect. Therefore, we have $n \le 10^{10}$. However, using Mathematica 7.0, we find that there is no odd near-perfect number $n \le 10^{10}$. This is a contradiction, which yields $\omega(n) \ge 6$. □

Remark 2.9. To verify that there is no odd near-perfect number less than 10^{10}, the computation by Mathematica 7.0, which is performed on a PC with CPU of 2.93 GHz and RAM of 3.00 GB, takes about 240 hours.

For general k, Tripathi (p. 148 of [11]), raised the following problem: "Let $k \ge 0$. If $\sigma(n) = 2(n + k)$, then n must be even." The following proposition gives a negative answer to his problem.

Proposition 2.10. *Let $p_1 < \cdots < p_i < \dots$ be all the odd prime numbers. If at least one of $e_1, \dots, e_s$ is odd, and s is sufficiently large, then there exists a positive integer k such that $\sigma(p_1^{e_1} \cdot \cdots \cdot p_s^{e_s}) = 2(p_1^{e_1} \cdot \cdots \cdot p_s^{e_s} + k)$.*

PROOF. Let $1 \le i \le s$ be chosen such that e_i is odd. Then $2|\sigma(p_i^{e_i})$, so $2|\sigma(p_1^{e_1} \dots p_s^{e_s})$.

For $n = p_1^{e_1} \dots p_s^{e_s}$, we have

$$\frac{\sigma(n)}{n} \ge \left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right)\dots\left(1 + \frac{1}{p_s}\right)$$

and the product on the right side diverges as $s \to \infty$. Therefore, if s is sufficiently large, we get that $\sigma(n) > 2n$.

Hence, for such n, the number $\sigma(n) - 2n$ is positive and even so it equals $2k$ for some positive integer k. $\square$

However, for fixed k, it seems that the odd numbers n such that $\sigma(n) = 2n+2k$ are rare. For the case $k = 1, 2, 4$, there are no such odd n for $n \le 10^{10}$. For the case $k = 3$, there is only three odd integers 8925, 32445, 442365 for $n \le 10^{10}$ such that $\sigma(n) = 2n + 6$. The computations are all done by Mathematica 7.0, on a PC with CPU of 2.93 GHz and RAM of 3.00 GB. We raise the following conjecture.

Conjecture 2.11. *For a fixed nonnegative integer k, the number of odd solutions of the equation $\sigma(n) = 2(n + k)$ is finite.*

The above conjecture is in accordance with the well-known conjecture asserting that there is no odd perfect number.

Finally, we list odd primitive non-deficient numbers with four distinct prime factors in the following table. All the numbers are divided into several groups. In each group, the numbers are arranged in alphabet order. E.g., in the group 3, 7, 11, $[13, 23]$, the numbers are all of the form $3^a 7^b 11^c p^d$, with $13 \le p \le 23$ being a prime. Moreover, the numbers $3^a 7^b 11^c p^d$ are arranged according to the alphabet order of vectors (p, a, b, c, d).

Table A

(The odd primitive non-deficient numbers with four distinct prime factors.)

$3, 7, 11, [13, 23]:$	$3^4 7^1 11^1 13^1$	$3^3 7^3 11^2 19^2$	$3^4 7^4 11^3 23^1$
$3^2 7^2 11^2 13^3$	$3^3 7^2 11^2 17^1$	$3^3 7^3 11^3 19^1$	$3^5 7^2 11^2 23^2$
$3^2 7^2 11^3 13^2$	$3^3 7^3 11^1 17^2$	$3^4 7^2 11^1 19^2$	$3^5 7^2 11^4 23^1$
$3^2 7^3 11^2 13^2$	$3^4 7^2 11^1 17^1$	$3^4 7^2 11^2 19^1$	$3^5 7^3 11^2 23^1$
$3^3 7^1 11^1 13^2$	$3^5 7^1 11^3 17^2$	$3^4 7^3 11^1 19^1$	$3^6 7^2 11^2 23^1$
$3^3 7^1 11^2 13^1$	$3^6 7^1 11^2 17^2$	$3^5 7^2 11^1 19^1$	
$3^3 7^2 11^1 13^1$	$3^3 7^2 11^3 19^2$	$3^4 7^3 11^2 23^2$	

$3, 7, 13, [17, 19]$:	$3^2 5^1 7^1 31^1$	$3^2 5^2 11^1 43^2$	$3^6 5^1 11^2 83^1$
$3^4 7^2 13^3 17^3$	$3^2 5^1 7^1 37^1$	$3^2 5^2 11^2 43^1$	$3^2 5^3 11^2 89^1$
$3^4 7^3 13^2 17^2$	$3^2 5^1 7^1 41^1$	$3^2 5^3 11^1 43^1$	$3^6 5^1 11^2 89^2$
$3^5 7^2 13^2 17^2$	$3^2 5^1 7^1 43^1$	$3^3 5^1 11^2 43^2$	$3^6 5^1 11^3 89^1$
$3^5 7^3 13^2 17^1$	$3^2 5^1 7^1 47^1$	$3^3 5^1 11^3 43^1$	$3^7 5^1 11^2 89^1$
$3^6 7^2 13^3 17^1$	$3^2 5^1 7^1 53^1$	$3^4 5^1 11^1 43^1$	$3^2 5^3 11^2 97^1$
$3^6 7^3 13^1 17^2$	$3^2 5^1 7^1 59^1$	$3^2 5^2 11^2 47^1$	$3^7 5^1 11^3 97^2$
$3^7 7^2 13^2 17^1$	$3^2 5^1 7^1 61^1$	$3^2 5^3 11^1 47^1$	$3^7 5^1 11^4 97^1$
$3^6 7^4 13^3 19^3$	$3^2 5^1 7^1 67^1$	$3^4 5^1 11^2 47^1$	$3^8 5^1 11^3 97^1$
$3^7 7^3 13^4 19^4$	$3^2 5^1 7^1 71^1$	$3^5 5^1 11^1 47^1$	$3^2 5^3 11^2 101^1$
$3^7 7^3 13^5 19^3$	$3^2 5^1 7^1 73^1$	$3^2 5^2 11^2 53^1$	$3^2 5^3 11^2 103^1$
$3^7 7^4 13^3 19^2$	$3^2 5^1 7^1 79^1$	$3^2 5^3 11^1 53^1$	$3^2 5^3 11^2 107^2$
$3^8 7^3 13^3 19^3$	$3^2 5^1 7^1 83^1$	$3^4 5^1 11^2 53^1$	$3^2 5^3 11^3 107^1$
$3^8 7^4 13^2 19^3$	$3^2 5^1 7^1 89^1$	$3^6 5^1 11^1 53^2$	$3^2 5^4 11^2 107^1$
$3^{10} 7^3 13^4 19^2$	$3^2 5^1 7^1 97^1$	$3^7 5^1 11^1 53^1$	$3^2 5^3 11^3 109^1$
	$3^2 5^1 7^1 101^1$	$3^2 5^2 11^2 59^1$	$3^2 5^4 11^2 109^1$
$3, 5, 7, [11, 103]$:	$3^2 5^1 7^1 103^1$	$3^2 5^3 11^1 59^2$	$3^2 5^3 11^3 113^1$
$3^1 5^1 7^1 11^2$		$3^2 5^4 11^1 59^1$	$3^2 5^4 11^2 113^1$
$3^1 5^1 7^2 11^1$	$3, 5, 11, [13, 139]$:	$3^4 5^1 11^2 59^1$	$3^2 5^4 11^3 127^1$
$3^1 5^2 7^1 11^1$	$3^2 5^1 11^1 13^1$	$3^2 5^2 11^2 61^1$	$3^2 5^5 11^2 127^1$
$3^2 5^1 7^1 11^1$	$3^2 5^1 11^1 17^1$	$3^2 5^4 11^1 61^1$	$3^2 5^4 11^3 131^1$
$3^1 5^1 7^2 13^1$	$3^2 5^1 11^2 19^1$	$3^4 5^1 11^2 61^1$	$3^2 5^4 11^4 137^2$
$3^1 5^2 7^1 13^1$	$3^2 5^2 11^1 19^1$	$3^2 5^2 11^4 67^2$	$3^2 5^5 11^3 137^1$
$3^2 5^1 7^1 13^1$	$3^3 5^1 11^1 19^1$	$3^2 5^3 11^2 67^1$	$3^2 5^5 11^3 139^1$
$3^1 5^2 7^1 17^1$	$3^2 5^2 11^1 23^1$	$3^4 5^1 11^2 67^2$	
$3^2 5^1 7^1 17^1$	$3^3 5^1 11^1 23^1$	$3^4 5^1 11^3 67^1$	$3, 5, 13, [17, 383]$:
$3^1 5^2 7^2 19^1$	$3^2 5^2 11^1 29^1$	$3^5 5^1 11^2 67^1$	$3^2 5^2 13^1 17^1$
$3^1 5^3 7^1 19^1$	$3^3 5^1 11^1 29^1$	$3^2 5^3 11^2 71^1$	$3^3 5^1 13^1 17^1$
$3^2 5^1 7^1 19^1$	$3^2 5^2 11^1 31^1$	$3^4 5^1 11^4 71^2$	$3^2 5^2 13^1 19^1$
$3^1 5^2 7^2 23^1$	$3^3 5^1 11^1 31^1$	$3^5 5^1 11^2 71^1$	$3^3 5^1 13^1 19^1$
$3^2 5^1 7^1 23^1$	$3^2 5^2 11^1 37^1$	$3^2 5^3 11^2 73^1$	$3^2 5^2 13^1 23^1$
$3^1 5^3 7^2 29^1$	$3^3 5^1 11^2 37^1$	$3^5 5^1 11^2 73^1$	$3^3 5^1 13^1 23^2$
$3^2 5^1 7^1 29^1$	$3^4 5^1 11^1 37^1$	$3^2 5^3 11^2 79^1$	$3^3 5^1 13^2 23^1$
$3^1 5^3 7^2 31^2$	$3^2 5^2 11^1 41^1$	$3^5 5^1 11^2 79^1$	$3^4 5^1 13^1 23^1$
$3^1 5^3 7^3 31^1$	$3^3 5^1 11^2 41^1$	$3^2 5^3 11^2 83^1$	$3^2 5^2 13^2 29^1$
$3^1 5^4 7^2 31^1$	$3^4 5^1 11^1 41^1$	$3^5 5^1 11^3 83^1$	$3^2 5^3 13^1 29^1$

$3^3 5^2 13^1 29^1$	$3^4 5^2 13^1 101^1$	$3^4 5^2 13^1 163^1$	$3^4 5^2 13^1 239^1$
$3^4 5^1 13^2 29^1$	$3^3 5^2 13^2 103^1$	$3^3 5^2 13^2 167^1$	$3^3 5^4 13^1 241^1$
$3^5 5^1 13^1 29^1$	$3^3 5^3 13^1 103^1$	$3^3 5^3 13^1 167^1$	$3^4 5^2 13^1 241^1$
$3^2 5^2 13^2 31^1$	$3^4 5^2 13^1 103^1$	$3^4 5^2 13^1 167^1$	$3^3 5^4 13^1 251^1$
$3^2 5^3 13^1 31^1$	$3^3 5^2 13^2 107^1$	$3^3 5^2 13^2 173^1$	$3^4 5^2 13^1 251^1$
$3^3 5^2 13^1 31^1$	$3^3 5^3 13^1 107^1$	$3^3 5^3 13^1 173^1$	$3^3 5^4 13^1 257^1$
$3^4 5^1 13^2 31^1$	$3^4 5^2 13^1 107^1$	$3^4 5^2 13^1 173^1$	$3^4 5^2 13^1 257^1$
$3^5 5^1 13^1 31^2$	$3^3 5^2 13^2 109^1$	$3^3 5^2 13^2 179^1$	$3^3 5^4 13^1 263^1$
$3^6 5^1 13^1 31^1$	$3^3 5^3 13^1 109^1$	$3^3 5^3 13^1 179^1$	$3^4 5^2 13^1 263^1$
$3^2 5^3 13^2 37^1$	$3^4 5^2 13^1 109^1$	$3^4 5^2 13^1 179^1$	$3^3 5^4 13^1 269^1$
$3^3 5^2 13^1 37^1$	$3^3 5^2 13^2 113^1$	$3^3 5^2 13^2 181^1$	$3^4 5^2 13^1 269^1$
$3^5 5^1 13^2 37^2$	$3^3 5^3 13^1 113^1$	$3^3 5^3 13^1 181^1$	$3^3 5^4 13^1 271^1$
$3^6 5^1 13^2 37^1$	$3^4 5^2 13^1 113^1$	$3^4 5^2 13^1 181^1$	$3^4 5^2 13^1 271^1$
$3^2 5^3 13^2 41^2$	$3^3 5^2 13^2 127^1$	$3^3 5^2 13^3 191^1$	$3^3 5^4 13^1 277^1$
$3^2 5^3 13^3 41^1$	$3^3 5^3 13^1 127^1$	$3^3 5^3 13^1 191^1$	$3^4 5^2 13^1 277^1$
$3^2 5^4 13^2 41^1$	$3^4 5^2 13^1 127^1$	$3^4 5^2 13^1 191^1$	$3^3 5^4 13^1 281^1$
$3^3 5^2 13^1 41^1$	$3^3 5^2 13^2 131^1$	$3^3 5^2 13^3 193^1$	$3^4 5^2 13^1 281^1$
$3^2 5^4 13^2 43^2$	$3^3 5^3 13^1 131^1$	$3^3 5^3 13^1 193^1$	$3^3 5^4 13^1 283^1$
$3^2 5^4 13^3 43^1$	$3^4 5^2 13^1 131^1$	$3^4 5^2 13^1 193^1$	$3^4 5^2 13^1 283^1$
$3^2 5^5 13^2 43^1$	$3^3 5^2 13^2 137^1$	$3^3 5^2 13^3 197^1$	$3^3 5^4 13^1 293^1$
$3^3 5^2 13^1 43^1$	$3^3 5^3 13^1 137^1$	$3^3 5^3 13^1 197^1$	$3^4 5^2 13^1 293^1$
$3^3 5^2 13^1 47^1$	$3^4 5^2 13^1 137^1$	$3^4 5^2 13^1 197^1$	$3^3 5^4 13^1 307^1$
$3^3 5^2 13^1 53^1$	$3^3 5^2 13^2 139^1$	$3^3 5^2 13^3 199^1$	$3^4 5^2 13^1 307^1$
$3^3 5^2 13^1 59^1$	$3^3 5^3 13^1 139^1$	$3^3 5^3 13^1 199^1$	$3^3 5^4 13^1 311^1$
$3^3 5^2 13^1 61^1$	$3^4 5^2 13^1 139^1$	$3^4 5^2 13^1 199^1$	$3^4 5^2 13^1 311^1$
$3^3 5^2 13^1 67^1$	$3^3 5^2 13^2 149^1$	$3^3 5^3 13^1 211^1$	$3^3 5^4 13^1 313^1$
$3^3 5^2 13^1 71^1$	$3^3 5^3 13^1 149^1$	$3^4 5^2 13^1 211^1$	$3^4 5^2 13^1 313^1$
$3^3 5^2 13^1 73^1$	$3^4 5^2 13^1 149^1$	$3^3 5^3 13^1 223^1$	$3^3 5^5 13^1 317^1$
$3^3 5^2 13^1 79^1$	$3^3 5^2 13^2 151^1$	$3^4 5^2 13^1 223^1$	$3^4 5^2 13^1 317^1$
$3^3 5^2 13^1 83^1$	$3^3 5^3 13^1 151^1$	$3^3 5^4 13^1 227^1$	$3^3 5^5 13^1 331^1$
$3^3 5^2 13^1 89^1$	$3^4 5^2 13^1 151^1$	$3^4 5^2 13^1 227^1$	$3^4 5^2 13^1 331^1$
$3^3 5^2 13^2 97^1$	$3^3 5^2 13^2 157^1$	$3^3 5^4 13^1 229^1$	$3^3 5^5 13^1 337^1$
$3^3 5^3 13^1 97^1$	$3^3 5^3 13^1 157^1$	$3^4 5^2 13^1 229^1$	$3^4 5^2 13^1 337^1$
$3^4 5^2 13^1 97^1$	$3^4 5^2 13^1 157^1$	$3^3 5^4 13^1 233^1$	$3^3 5^6 13^1 347^1$
$3^3 5^2 13^2 101^1$	$3^3 5^2 13^2 163^1$	$3^4 5^2 13^1 233^1$	$3^4 5^2 13^1 347^1$
$3^3 5^3 13^1 101^1$	$3^3 5^3 13^1 163^1$	$3^3 5^4 13^1 239^1$	$3^3 5^6 13^1 349^2$

$3^3 5^7 13^1 349^1$	$3^4 5^3 17^1 53^1$	$3^6 5^3 17^1 97^1$	$3^6 5^3 17^2 151^1$
$3^4 5^2 13^1 349^1$	$3^5 5^2 17^1 53^1$	$3^4 5^3 17^2 101^1$	$3^5 5^4 17^2 157^1$
$3^4 5^2 13^1 353^1$	$3^3 5^4 17^2 59^2$	$3^5 5^4 17^1 101^1$	$3^6 5^3 17^2 157^1$
$3^4 5^2 13^1 359^1$	$3^3 5^4 17^3 59^1$	$3^6 5^3 17^1 101^1$	$3^5 5^4 17^2 163^1$
$3^4 5^2 13^1 367^1$	$3^3 5^5 17^2 59^1$	$3^4 5^3 17^3 103^1$	$3^6 5^3 17^3 163^1$
$3^4 5^2 13^1 373^1$	$3^4 5^2 17^2 59^1$	$3^4 5^4 17^2 103^1$	$3^7 5^3 17^2 163^1$
$3^4 5^2 13^1 379^1$	$3^4 5^3 17^1 59^1$	$3^5 5^3 17^2 103^1$	$3^5 5^4 17^2 167^1$
$3^4 5^2 13^1 383^1$	$3^5 5^2 17^1 59^1$	$3^5 5^4 17^1 103^1$	$3^6 5^3 17^3 167^2$
	$3^3 5^5 17^3 61^2$	$3^6 5^3 17^1 103^1$	$3^7 5^3 17^2 167^1$
$3, 5, 17, [19, 251]:$	$3^4 5^2 17^2 61^2$	$3^4 5^4 17^2 107^1$	$3^5 5^4 17^3 173^1$
$3^2 5^2 17^1 19^2$	$3^4 5^2 17^3 61^1$	$3^5 5^3 17^2 107^1$	$3^5 5^5 17^2 173^1$
$3^2 5^2 17^2 19^1$	$3^4 5^3 17^1 61^1$	$3^5 5^4 17^1 107^1$	$3^6 5^4 17^2 173^1$
$3^2 5^3 17^1 19^1$	$3^5 5^2 17^2 61^1$	$3^7 5^3 17^1 107^1$	$3^7 5^3 17^3 173^1$
$3^3 5^2 17^1 19^1$	$3^6 5^2 17^1 61^1$	$3^4 5^4 17^2 109^1$	$3^8 5^3 17^2 173^2$
$3^4 5^1 17^1 19^2$	$3^4 5^3 17^1 67^1$	$3^5 5^3 17^2 109^1$	$3^9 5^3 17^2 173^1$
$3^4 5^1 17^2 19^1$	$3^5 5^2 17^2 67^1$	$3^5 5^4 17^1 109^1$	$3^5 5^5 17^2 179^1$
$3^5 5^1 17^1 19^1$	$3^4 5^3 17^1 71^1$	$3^7 5^3 17^1 109^2$	$3^6 5^4 17^2 179^1$
$3^2 5^3 17^2 23^2$	$3^5 5^2 17^2 71^1$	$3^8 5^3 17^1 109^1$	$3^8 5^3 17^3 179^2$
$3^2 5^4 17^2 23^1$	$3^4 5^3 17^1 73^1$	$3^4 5^4 17^2 113^1$	$3^8 5^3 17^4 179^1$
$3^3 5^2 17^1 23^1$	$3^5 5^2 17^2 73^1$	$3^5 5^3 17^2 113^1$	$3^9 5^3 17^3 179^1$
$3^3 5^2 17^1 29^1$	$3^4 5^3 17^2 79^1$	$3^5 5^5 17^1 113^2$	$3^5 5^5 17^3 181^1$
$3^3 5^2 17^1 31^1$	$3^4 5^4 17^1 79^1$	$3^5 5^6 17^1 113^1$	$3^5 5^6 17^2 181^1$
$3^3 5^2 17^2 37^1$	$3^5 5^3 17^1 79^1$	$3^6 5^4 17^1 113^1$	$3^6 5^4 17^2 181^1$
$3^3 5^3 17^1 37^1$	$3^6 5^2 17^2 79^2$	$3^5 5^3 17^2 127^1$	$3^9 5^3 17^4 181^2$
$3^4 5^2 17^1 37^1$	$3^6 5^2 17^3 79^1$	$3^6 5^5 17^1 127^2$	$3^{10} 5^3 17^3 181^2$
$3^3 5^2 17^2 41^2$	$3^7 5^2 17^2 79^1$	$3^6 5^7 17^1 127^1$	$3^6 5^4 17^2 191^1$
$3^3 5^3 17^1 41^1$	$3^4 5^3 17^2 83^1$	$3^7 5^4 17^1 127^2$	$3^6 5^4 17^2 193^1$
$3^4 5^2 17^1 41^1$	$3^4 5^4 17^1 83^1$	$3^7 5^5 17^1 127^1$	$3^6 5^4 17^2 197^1$
$3^3 5^3 17^1 43^1$	$3^5 5^3 17^1 83^1$	$3^8 5^4 17^1 127^1$	$3^6 5^4 17^2 199^1$
$3^4 5^2 17^1 43^1$	$3^7 5^2 17^3 83^2$	$3^5 5^3 17^2 131^1$	$3^6 5^4 17^3 211^1$
$3^3 5^3 17^1 47^2$	$3^9 5^2 17^3 83^1$	$3^7 5^5 17^1 131^1$	$3^6 5^5 17^2 211^1$
$3^3 5^3 17^2 47^1$	$3^4 5^3 17^2 89^1$	$3^5 5^3 17^2 137^1$	$3^7 5^4 17^2 211^1$
$3^3 5^4 17^1 47^1$	$3^5 5^3 17^1 89^1$	$3^5 5^3 17^2 139^1$	$3^6 5^5 17^3 223^1$
$3^4 5^2 17^1 47^1$	$3^4 5^3 17^2 97^1$	$3^5 5^4 17^2 149^1$	$3^7 5^4 17^3 223^1$
$3^3 5^3 17^2 53^1$	$3^5 5^3 17^1 97^2$	$3^6 5^3 17^2 149^1$	$3^7 5^5 17^2 223^1$
$3^4 5^2 17^2 53^1$	$3^5 5^4 17^1 97^1$	$3^5 5^4 17^2 151^1$	$3^8 5^4 17^2 223^2$

$3^{9}5^{4}17^{2}223^{1}$	$3, 5, 19, [23, 89]:$	$3^{4}5^{3}19^{3}61^{1}$	$3^{3}5^{4}23^{2}29^{1}$
$3^{6}5^{6}17^{3}227^{2}$	$3^{3}5^{2}19^{1}23^{1}$	$3^{4}5^{4}19^{2}61^{1}$	$3^{4}5^{2}23^{1}29^{2}$
$3^{6}5^{6}17^{4}227^{1}$	$3^{3}5^{2}19^{1}29^{1}$	$3^{5}5^{3}19^{1}61^{1}$	$3^{4}5^{2}23^{2}29^{1}$
$3^{6}5^{7}17^{3}227^{1}$	$3^{3}5^{2}19^{2}31^{1}$	$3^{4}5^{4}19^{3}67^{2}$	$3^{4}5^{3}23^{1}29^{1}$
$3^{7}5^{4}17^{3}227^{2}$	$3^{3}5^{3}19^{1}31^{1}$	$3^{4}5^{5}19^{2}67^{1}$	$3^{5}5^{2}23^{1}29^{1}$
$3^{7}5^{4}17^{4}227^{1}$	$3^{4}5^{2}19^{1}31^{1}$	$3^{5}5^{3}19^{2}67^{1}$	$3^{3}5^{5}23^{2}31^{2}$
$3^{7}5^{5}17^{2}227^{1}$	$3^{3}5^{3}19^{1}37^{2}$	$3^{5}5^{4}19^{1}67^{2}$	$3^{4}5^{2}23^{2}31^{2}$
$3^{8}5^{4}17^{3}227^{1}$	$3^{3}5^{3}19^{2}37^{1}$	$3^{5}5^{5}19^{1}67^{1}$	$3^{4}5^{3}23^{1}31^{1}$
$3^{6}5^{7}17^{4}229^{2}$	$3^{3}5^{4}19^{1}37^{1}$	$3^{6}5^{4}19^{1}67^{1}$	$3^{5}5^{2}23^{1}31^{1}$
$3^{7}5^{5}17^{2}229^{1}$	$3^{4}5^{2}19^{1}37^{1}$	$3^{7}5^{3}19^{1}67^{2}$	$3^{4}5^{3}23^{2}37^{1}$
$3^{8}5^{4}17^{3}229^{1}$	$3^{3}5^{3}19^{2}41^{2}$	$3^{5}5^{3}19^{2}71^{1}$	$3^{4}5^{4}23^{1}37^{1}$
$3^{7}5^{5}17^{3}233^{1}$	$3^{3}5^{4}19^{2}41^{1}$	$3^{6}5^{4}19^{1}71^{2}$	$3^{5}5^{3}23^{1}37^{1}$
$3^{7}5^{6}17^{2}233^{2}$	$3^{4}5^{2}19^{2}41^{1}$	$3^{6}5^{5}19^{1}71^{1}$	$3^{4}5^{4}23^{2}41^{2}$
$3^{7}5^{7}17^{2}233^{1}$	$3^{4}5^{3}19^{1}41^{1}$	$3^{7}5^{4}19^{1}71^{1}$	$3^{5}5^{3}23^{2}41^{1}$
$3^{8}5^{4}17^{3}233^{2}$	$3^{5}5^{2}19^{1}41^{1}$	$3^{5}5^{3}19^{2}73^{1}$	$3^{5}5^{4}23^{1}41^{1}$
$3^{8}5^{5}17^{2}233^{1}$	$3^{3}5^{4}19^{2}43^{2}$	$3^{6}5^{5}19^{1}73^{2}$	$3^{6}5^{3}23^{1}41^{1}$
$3^{9}5^{4}17^{3}233^{1}$	$3^{3}5^{5}19^{3}43^{1}$	$3^{7}5^{4}19^{1}73^{2}$	$3^{5}5^{3}23^{2}43^{2}$
$3^{7}5^{5}17^{3}239^{1}$	$3^{4}5^{2}19^{2}43^{1}$	$3^{7}5^{5}19^{1}73^{1}$	$3^{5}5^{4}23^{2}43^{1}$
$3^{8}5^{6}17^{2}239^{2}$	$3^{4}5^{3}19^{1}43^{1}$	$3^{9}5^{4}19^{1}73^{1}$	$3^{5}5^{5}23^{1}43^{2}$
$3^{8}5^{7}17^{2}239^{1}$	$3^{5}5^{2}19^{1}43^{1}$	$3^{5}5^{4}19^{2}79^{1}$	$3^{6}5^{3}23^{2}43^{1}$
$3^{9}5^{6}17^{2}239^{1}$	$3^{4}5^{3}19^{1}47^{1}$	$3^{6}5^{3}19^{2}79^{2}$	$3^{6}5^{4}23^{1}43^{1}$
$3^{10}5^{5}17^{2}239^{2}$	$3^{5}5^{2}19^{2}47^{1}$	$3^{6}5^{3}19^{3}79^{1}$	$3^{5}5^{6}23^{3}47^{2}$
$3^{7}5^{5}17^{3}241^{2}$	$3^{7}5^{2}19^{1}47^{2}$	$3^{7}5^{3}19^{2}79^{1}$	$3^{6}5^{4}23^{2}47^{1}$
$3^{7}5^{5}17^{4}241^{1}$	$3^{4}5^{3}19^{1}53^{2}$	$3^{5}5^{5}19^{2}83^{2}$	
$3^{7}5^{6}17^{3}241^{1}$	$3^{4}5^{3}19^{2}53^{1}$	$3^{5}5^{5}19^{3}83^{1}$	$3, 5, 29, [31, 31]:$
$3^{8}5^{5}17^{3}241^{1}$	$3^{4}5^{4}19^{1}53^{1}$	$3^{6}5^{4}19^{2}83^{1}$	$3^{5}5^{3}29^{2}31^{2}$
$3^{9}5^{6}17^{2}241^{2}$	$3^{5}5^{3}19^{1}53^{1}$	$3^{9}5^{3}19^{3}83^{2}$	$3^{5}5^{4}29^{1}31^{2}$
$3^{9}5^{7}17^{2}241^{1}$	$3^{6}5^{2}19^{3}53^{2}$	$3^{6}5^{4}19^{3}89^{2}$	$3^{5}5^{4}29^{2}31^{1}$
$3^{10}5^{6}17^{2}241^{1}$	$3^{7}5^{2}19^{2}53^{2}$	$3^{6}5^{5}19^{2}89^{1}$	$3^{6}5^{3}29^{1}31^{2}$
$3^{8}5^{6}17^{3}251^{2}$	$3^{7}5^{2}19^{3}53^{1}$	$3^{7}5^{4}19^{2}89^{1}$	$3^{6}5^{3}29^{2}31^{1}$
$3^{8}5^{7}17^{4}251^{1}$	$3^{9}5^{2}19^{2}53^{1}$		$3^{6}5^{4}29^{1}31^{1}$
$3^{9}5^{6}17^{3}251^{1}$	$3^{4}5^{3}19^{2}59^{1}$	$3, 5, 23, [29, 47]:$	
$3^{10}5^{5}17^{4}251^{2}$	$3^{5}5^{3}19^{1}59^{1}$	$3^{3}5^{3}23^{2}29^{2}$	
$3^{11}5^{5}17^{3}251^{2}$	$3^{4}5^{3}19^{2}61^{2}$	$3^{3}5^{4}23^{1}29^{2}$	

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References

[1] H. L. Abbott, C. E. Aull, E. Brown and D. Suryanarayana, Quasiperfect numbers, *Acta Arith.* **29** (1976), 439–447.

[2] H. L. Abbott, C. E. Aull, E. Brown and D. Suryanarayana, Corrections to the paper "Quasiperfect numbers", *Acta Artith.* **29** (1976), 427–428.

[3] Cattaneo, Sui numeri quasiperfetti, *Boll. Un. Mat. Ital.* (3) **6** (1951), 59–62.

[4] L. E. Dickson, Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors, *Amer. J. Math.* **22** (1913), 413–422.

[5] A. Ferrier, Table errata 176, *MTAC* **4** (1950), 222.

[6] Richard K. Guy, Unsolved Problems in Number Theory, *Springer-Verlag*, 2004, third edtion.

[7] P. Hagis and G. L. Cohen, Some results concerning quasiperfect numbers, *J. Austral. Math. Soc. Ser. A* **33** (1982), 275–286.

[8] F. Herzog, Table errata 571, *Math. Comp.* **34** (1980), 652.

[9] M. Kishore, Odd integers N with five distinct prime factors for which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{12}$, *Math. Comp.* **32**, no. 141 (1978), 303–309.

[10] P. Nielsen, Odd perfect numbers have at least nine distinct prime factors, *Math. Comp.* **76**, no. 260 (2007), 2109–2126.

[11] A. Tripathi, A note on products of primes that differ by a fixed integer, *Fibonacci Quart.* **48**, no. 2 (2010), 144–149.

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