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On the diophantine equation $xy + yz + zx = m$

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Let m, n be arbitrary fixed positive integers. K. KOVÁCS [1] investigated the diophantine equation

$$\sum_{1\le i<j\le n} x_i x_j = m \tag{1}$$

for x_i positive integers. The case $n = 1$ or $n = 2$ is trivial. For $n > 3$, KOVÁCS proved that (1) has a solution if $m \ge 136n^2$. In the case $n = 3$, the problem is still open. KOVÁCS examined that the equation

$$xy + yz + zx = m \tag{2}$$

has a solution for all $m \le 10^7$ except $m = 1$, 2, 4, 18, 22, 30, 42, 58, 70, 78, 102, 130, 190, 210, 330 and 462.

In this note we prove the following theorem using the properties of quadratic residues and the Chinese remainder theorem.

Theorem. *Let $E(X)$ be the number of $m \le X$ for which* (2) *has no solutions. Then for any $\varepsilon > 0$*

$$E(X) = O\left(X 2^{-(1-\varepsilon)(\log X)/\log\log X}\right).$$

PROOF. It is easy to see that the equation (2) has a solution if and only if there exist $x, y \ge 1, xy < m$ such that

$$m \equiv xy \pmod{x+y}. \tag{3}$$

Replacing $x + y = t$ the congruence (3) goes over into

$$x^2 \equiv -m \pmod{t},\ 1 \le x < t,\ x(t-x) < m. \tag{4}$$

For any $Y \le \sqrt{x}$ and prime $p \le Y$ let $S_p(X,Y) = \{Y^2 \le m \le X \mid (4)$ has a solution for $p\}$.

Each reduced residue system mod p contains exactly $(p-1)/2$ quadratic residues which yields that

$$|S_p(X,Y)| = \frac{1}{2}(1-1/p)X + O(Y^2) + O((p-1)/2). \tag{5}$$

Using the Chinese remainder theorem for the primes $p, q, r, \dots \le Y$ and a simple sieve we have

$$\begin{aligned} X - E(X) \ge & \sum_{p \le Y} |S_p(X,Y)| - \sum_{p<q\le Y} |S_p(X,Y) \cap S_q(X,Y)| \\ & + \sum_{p<q<r\le Y} |S_p(X,Y) \cap S_p(X,Y) \cap S_r(X,Y)| - \dots \\ & = X\left(1 - \prod_{p\le Y}\left(1 - \frac{1}{2}(1-1/p)\right)\right) + O\left(Y^2 2^{\pi(Y)}\right) \\ & + O\left(\prod_{p\le Y}(1+(p-1)/2)\right) \\ & = X + O\left(X(\log Y)/2^{\pi(Y)}\right) + O\left(Y^2 2^{\pi(Y)}\right) \\ & + O\left(e^{\psi(Y)}(\log Y)/2^{\pi(Y)}\right). \end{aligned}$$

Using the well-known results $\prod_{p\le Y}(1+1/p) = O(\log Y)$, $\pi(Y) = \sum_{p\le Y} 1 = Y/\log Y(1+o(1))$ and $\psi(Y) = \sum_{p\le Y} \log p = Y(1+o(1))$ the choice $X = e^{\psi(Y)}$ implies

$$E(X) = O\left(X 2^{-(1-\varepsilon)(\log X)/\log\log X}\right).$$

References

[1] K. Kovács, About some positive solutions of the diophantine equation $\sum_{1\le i<j\le n} a_i a_j = m$, *Publ. Math. Debrecen* **40** (1992), 207–210.

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