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The convergence of the sequences coding the ground model reals

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Abstract. We investigate the convergence λ_1 on a complete Boolean algebra $\mathbb{B}$ defined in the following way: a sequence $x = \langle x_n : n \in \omega\rangle$ in $\mathbb{B}$ converges to the point $\limsup x$ of $\mathbb{B}$, if in each generic extension $V_{\mathbb{B}}[G]$ the real coded by the name $\tau_x = \{\langle \check{n}, x_n\rangle : n \in \omega\}$ belongs to the ground model V; otherwise, x has no limit points. It is shown that λ_1 generates the same topology as the convergence $\bar{\lambda}_4$, generalizing the sequential convergence on the Aleksandrov cube and that for a c.B.a. $\mathbb{B}$ the following conditions are equivalent: (1) The algebra $\mathbb{B}$ is $(\omega, 2)$-distributive; (2) The (L2)-closure of λ_1, $\bar{\lambda}_1$, is a topological convergence; (3) $\bar{\lambda}_1 = \bar{\lambda}_4$; (4) $\lambda_1 = \lambda_4$; and, for the algebras satisfying $\mathrm{hcc}(\mathbb{B}) > \mathfrak{c}$, (5) $\bar{\lambda}_1$ is a weakly topological convergence. Also, it is shown that the convergence $\bar{\lambda}_1$ is not weakly topological, if forcing by $\mathbb{B}$ produces splitting reals.

1. Preliminaries

Topologies and convergence structures on Boolean algebras as well as the interplay between the topological, algebraic and forcing-related properties of Boolean algebras are extensively investigated. The results concerning this interplay are useful because, for example, they enable us to attack algebraic problems by topological methods (see e.g. [3]) or topological problems using the techniques of forcing [7].

In this paper we investigate the convergence λ_1 on an arbitrary complete

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Boolean algebra $\mathbb{B}$ defined in the following way: a sequence $x = \langle x_n : n \in \omega \rangle$ converges to $\limsup x$, if $1 \Vdash \tau_x \in V$, where $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ and, otherwise, x has no limit points. In addition, we compare this convergence with some convergences considered in [8]. One of them is the algebraic convergence [11], [1] related to the von Neumann and the Maharam problem and generalizing the convergence on the Cantor cube; another one is a generalization of the convergence on the Aleksandrov cube considered in [9].

Our notation is mainly standard. So, ω denotes the set of natural numbers, Y^X the set of all functions $f : X \to Y$ and $\omega^{\uparrow\omega}$ the set of all strictly increasing functions from ω into ω. A *sequence* in a set X is each function $x : \omega \to X$; instead of $x(n)$ we usually write x_n and also $x = \langle x_n : n \in \omega \rangle$. The *constant sequence* $\langle a, a, a, \dots \rangle$ is denoted by $\langle a \rangle$. If $f \in \omega^{\uparrow\omega}$, the sequence $y = x \circ f$ is said to be a *subsequence* of the sequence x and we write $y \prec x$.

If $\langle X, \mathcal{O} \rangle$ is a topological space, a point $a \in X$ is said to be a *limit point of a sequence* $x \in X^\omega$ (we will write: $x \to_{\mathcal{O}} a$) iff each neighborhood U of a contains all but finitely many members of the sequence. A space $\langle X, \mathcal{O} \rangle$ is called *sequential* iff a set $A \subset X$ is closed whenever it contains each limit of each sequence in A.

If X is a non-empty set, each mapping $\lambda : X^\omega \to P(X)$ is a *convergence* on X and the mapping $u_\lambda : P(X) \to P(X)$, defined by $u_\lambda(A) = \bigcup_{x \in A^\omega} \lambda(x)$, the *operator of sequential closure* determined by λ. If λ_1 is another convergence on X, then we will write $\lambda \leq \lambda_1$ iff $\lambda(x) \subset \lambda_1(x)$, for each sequence $x \in X^\omega$. Clearly, $\leq$ is a partial order on the set $\mathrm{Conv}(X) = \{\lambda : \lambda \text{ is a convergence on } X\}$.

If $\langle X, \mathcal{O} \rangle$ is a topological space, then the mapping $\lim_{\mathcal{O}} : X^\omega \to P(X)$ defined by $\lim_{\mathcal{O}}(x) = \{a \in X : x \to_{\mathcal{O}} a\}$ is *the convergence on X determined by the topology $\mathcal{O}$* and for the operator $\lambda = \lim_{\mathcal{O}}$ we have (see [2])

(L1) $\forall a \in X \;\; a \in \lambda(\langle a \rangle)$;

(L2) $\forall x \in X^\omega \; \forall y \prec x \; \lambda(x) \subset \lambda(y)$;

(L3) $\forall x \in X^\omega \; \forall a \in X \; ((\forall y \prec x \; \exists z \prec y \; a \in \lambda(z)) \Rightarrow a \in \lambda(x))$.

A convergence $\lambda : X^\omega \to P(X)$ is called a *topological convergence* iff there is a topology $\mathcal{O}$ on X such that $\lambda = \lim_{\mathcal{O}}$. The following fact (see, for example, [8]) shows that each convergence has a minimal topological extension and connects topological and convergence structures.

Fact 1.1. *Let $\lambda : X^\omega \to P(X)$ be a convergence on a non-empty set X. Then*

(a) *There is the maximal topology $\mathcal{O}_\lambda$ on X satisfying $\lambda \leq \lim_{\mathcal{O}}$;*

(b) $\mathcal{O}_\lambda = \{O \subset X : \forall x \in X^\omega \, (O \cap \lambda(x) \neq \emptyset \Rightarrow \exists n_0 \in \omega \; \forall n \geq n_0 \; x_n \in O)\}$;

(c) $\langle X, \mathcal{O}_\lambda \rangle$ *is a sequential space;*

(d) $\mathcal{O}_\lambda = \{X \setminus F : F \subset X \wedge u_\lambda(F) = F\}$, *if* λ *satisfies* (L1) *and* (L2)*;*

(e) $\lim_{\mathcal{O}_\lambda} = \min\{\lambda' \in \mathrm{Conv}(X) : \lambda'$ *is topological and* $\lambda \leq \lambda'\}$*;*

(f) $\mathcal{O}_{\lim_{\mathcal{O}_\lambda}} = \mathcal{O}_\lambda$*;*

(g) *If* $\lambda_1 : X^\omega \to P(X)$ *and* $\lambda_1 \leq \lambda$, *then* $\mathcal{O}_\lambda \subset \mathcal{O}_{\lambda_1}$*;*

(h) λ *is a topological convergence iff* $\lambda = \lim_{\mathcal{O}_\lambda}$.

In our proofs we will mainly use the technique of *forcing* (see [4]). So, if $\mathbb{B}$ is a complete Boolean algebra belonging to the ground model V of ZFC, $V^{\mathbb{B}}$ will be the class of $\mathbb{B}$-*names*. For a formula $\varphi(v_0, \ldots, v_n)$ and $\tau_0, \ldots, \tau_n \in V^{\mathbb{B}}$ the corresponding *Boolean value* will be denoted by $\|\varphi(\tau_0, \ldots, \tau_n)\|$. If G is a $\mathbb{B}$-*generic filter over* V and $\tau \in V^{\mathbb{B}}$, the G-*evaluation* of τ will be denoted by τ_G. For $A \in V$, the corresponding $\mathbb{B}$-name will be $\check{A} = \{\langle a, 1\rangle : a \in A\}$.

Subsets of ω are called *reals* and can be coded by convenient names. Namely, each real belonging to a generic extension has a nice name of the form $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$, where $x_n = \|\check{n} \in \tau\|$, for each $n \in \omega$.

A real $r \in [\omega]^\omega \cap V_{\mathbb{B}}[G]$ will be called: *new* iff $r \notin V$; *old* iff $r \in V$; *dependent* iff there is $A \in [\omega]^\omega \cap V$ such that $A \subset r$ or $A \subset \omega \setminus r$; *independent* or a *splitting real* iff it is not dependent [6]; *supported* iff there is $A \in [\omega]^\omega \cap V$ such that $A \subset r$; *unsupported* iff it is not supported [5]. Using the elementary properties of forcing it is easy to prove the following two facts (see [9])

Fact 1.2. *Let* $x = \langle x_n : n \in \omega \rangle$ *be a sequence in a complete Boolean algebra* $\mathbb{B}$ *and* $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ *the corresponding* $\mathbb{B}$*-name for a subset of* ω*. Then*

(a) $\|\tau_x = \check{\omega}\| = \bigwedge_{n\in\omega} x_n$*;*

(b) $\|\tau_x$ *is cofinite* $\| = \bigvee_{k\in\omega} \bigwedge_{n\geq k} x_n$ $(= \liminf x)$*;*

(c) $\|\tau_x$ *is old infinite* $\| = \bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in\omega} x_n{}^{\chi_A(n)}$*; where* $x_n^1 = x_n$, $x_n^0 = x_n'$.

(d) $\|\tau_x$ *is supported* $\| = \bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in A} x_n$*;*

(e) $\|\tau_x$ *is dependent* $\| = \bigvee_{A\in[\omega]^\omega} (\bigwedge_{n\in A} x_n \vee \bigwedge_{n\in A} x_n')$*;*

(f) $\|\tau_x$ *is infinite* $\| = \bigwedge_{k\in\omega} \bigvee_{n\geq k} x_n$ $(= \limsup x)$*;*

(g) $\|\tau_x = \check{\omega}\| \leq \|\tau_x$ *is cofinite* $\| \leq \|\tau_x$ *is old infinite* $\| \leq \|\tau_x$ *is supported* $\| \leq$ $\|\tau_x$ *is infinite dependent* $\| \leq \|\tau_x$ *is infinite* $\|$.

Proof. We prove (c) and the rest of the proof is similar.
$\|\tau_x \text{ is old infinite}\| = \|\exists A \in ([\omega]^\omega)^{V\check{}} (\forall n \in A\ (n \in \tau_x) \wedge \forall n \in \check{\omega} \setminus A\ (n \notin \tau_x))\| = \bigvee_{A\in[\omega]^\omega} (\bigwedge_{n\in A} x_n \wedge \bigwedge_{n\in\omega\setminus A} x_n') = \bigvee_{A\in[\omega]^\omega} \bigwedge_{n\in\omega} x_n^{\chi_A(n)}$. □

Fact 1.3. *If* $x = \langle x_n : n \in \omega \rangle$ *is a sequence in a c.B.a.* $\mathbb{B}$ *and* $f \in \omega^{\uparrow\omega}$*, then* $y = x \circ f$ *is a subsequence of* x *and for the* $\mathbb{B}$*-names* τ_x *and* τ_y *we have*

(a) $1 \Vdash \tau_y = f^{-1}[\tau_x]$;

(b) $\limsup y = \| |f[\omega]\check{} \cap \tau_x| = \check{\omega} \|$;

(c) $\liminf y = \| f[\omega]\check{} \subset^* \tau_x \|$;

(d) $\liminf x \leq \liminf y \leq \limsup y \leq \limsup x$.

2. The convergence $\boldsymbol{\lambda_1}$

First, choosing a convenient notation, we present this research in the context of some previous results. Let $\mathbb{B}$ be a complete Boolean algebra and let the convergences $\lambda_i : \mathbb{B}^\omega \to P(\mathbb{B})$, for $i \in \{0,1,2,3,4\}$, be defined by

$$\lambda_i(x) = \begin{cases} \{b_4(x)\} & \text{if } b_i(x) = b_4(x), \\ \emptyset & \text{if } b_i(x) < b_4(x), \end{cases} \tag{1}$$

where

$$b_0(x) = \|\tau_x \text{ is cofinite}\| = \liminf x,$$

$$b_1(x) = \|\tau_x \text{ is old infinite}\|,$$

$$b_2(x) = \|\tau_x \text{ is supported}\|,$$

$$b_3(x) = \|\tau_x \text{ is infinite dependent}\|,$$

$$b_4(x) = \|\tau_x \text{ is infinite}\| = \limsup x.$$

Then λ_0 is the well known *algebraic convergence* [11] generating the *sequential topology* $\mathcal{O}_{\lambda_0}$ on $\mathbb{B}$ [1] related to the von-Neumann and the Maharam problem, λ_1 will be considered in this paper and the convergences λ_2, λ_3 and λ_4 were investigated in [8] and [9] and are related in the following way (see [8]).

Fact 2.1. *Let* $\mathbb{B}$ *be a complete Boolean algebra. Then*

(a) $\lambda_2 \leq \lambda_3 \leq \lambda_4$;

(b) λ_2, λ_3 *and* λ_4 *satisfy condition* (L1)*, but do not satisfy* (L2)*;*

(c) $\lambda_2 = \lambda_3$ *iff* $\lambda_2 = \lambda_4$ *iff the algebra* $\mathbb{B}$ *is* $(\omega, 2)$*-distributive;*

(d) $\lambda_3 = \lambda_4$ *iff forcing by* $\mathbb{B}$ *does not produce splitting reals.*

Thus $\lambda_1(x) = \{\limsup x\}$, if $\|\tau_x \text{ is old infinite}\| = \|\tau_x \text{ is infinite}\|$ and $\lambda_1(x) = \emptyset$, otherwise. Preliminarily we have

Theorem 2.2. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) *For each sequence x in $\mathbb{B}$ we have*

$$\lambda_1(x) = \begin{cases} \{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is old,} \\ \emptyset & \text{otherwise;} \end{cases}$$

(b) $\lambda_0 \leq \lambda_1 \leq \lambda_2$;

(c) *The convergence λ_1 satisfies condition* (L1), *but does not satisfy* (L2);

(d) $\lambda_1 = \lambda_2$ *iff the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive;*

(e) $\lambda_1 = \lambda_4$ *iff the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive.*

Proof. (a)

$$\begin{aligned} \lambda_1(x) \neq \emptyset &\Leftrightarrow \|\tau_x \text{ is infinite}\| \wedge \|\tau_x \text{ is old}\| = \|\tau_x \text{ is infinite}\| \\ &\Leftrightarrow \|\tau_x \text{ is infinite}\| \leq \|\tau_x \text{ is old}\| \\ &\Leftrightarrow 1 \Vdash \tau_x \text{ is infinite} \Rightarrow \tau_x \text{ is old} \\ &\Leftrightarrow 1 \Vdash \tau_x \text{ is finite} \vee \tau_x \text{ is old} \\ &\Leftrightarrow 1 \Vdash \tau_x \text{ is old.} \end{aligned}$$

(b) follows from Fact 1.2(g).

(c) For a constant sequence $x = \langle a \rangle$ we have $a \Vdash \tau_x = \check{\omega}$ and $a' \Vdash \tau_x = \check{\emptyset}$, which implies $1 \Vdash$ "τ_x is old". Since $\limsup x = a$, by (a) we have $a \in \lambda_1(\langle a \rangle)$ and (L1) holds. For the sequence $x = \langle 1, 0, 1, 0, \dots \rangle$ we have $1 \Vdash \tau_x = \{0, 2, 4, \dots\}\check{} \in V$ and, by (a), $\lambda_1(x) = \{\limsup x\} = \{1\}$. But $y = \langle 0, 0, 0, \dots \rangle \prec x$ and $1 \Vdash \tau_y = \check{\emptyset}$, which, by (a), implies $\lambda_1(y) = \{0\} \not\supset \lambda_1(x)$.

(d) By Theorem 7.5 of [8] for each sequence x in $\mathbb{B}$ we have

$$\lambda_2(x) = \begin{cases} \{\limsup x\} & \text{if } 1 \Vdash \tau_x \text{ is finite or supported,} \\ \emptyset & \text{otherwise.} \end{cases} \tag{2}$$

($\Leftarrow$) If $\mathbb{B}$ is $(\omega, 2)$-distributive, it does not produce new reals and, hence, for each sequence x in $\mathbb{B}$ we have $1 \Vdash$ "τ_x is old" and, clearly, $1 \Vdash$ "τ_x is finite or supported". So, by (a) and (2), $\lambda_1(x) = \{\limsup x\} = \lambda_2(x)$.

($\Rightarrow$) Suppose that the algebra $\mathbb{B}$ is not $(\omega, 2)$-distributive. Then there is an extension $V_{\mathbb{B}}[G]$ containing a new set $X \subset \omega$. Let σ be a $\mathbb{B}$-name such that $X = \sigma_G$ and $1 \Vdash \sigma \subset \check{\omega}$ and let $b \in G$, where

$$b \Vdash \sigma \text{ is new.} \tag{3}$$

If $y = \langle y_n : n \in \omega \rangle$, where $y_n = \|\check{n} \in \sigma\|$, $n \in \omega$, for $\tau_y = \{\langle \check{n}, y_n \rangle : n \in \omega\}$ we have

$$1 \Vdash \sigma = \tau_y. \tag{4}$$

For $x = \langle y_0, 1, y_1, 1, y_2, 1 \dots \rangle$ we have $1 \Vdash \{1, 3, 5, \dots\}\check{} \subset \tau_x$ and, hence, $1 \Vdash \tau_x$ is supported, which, by (2) implies $\lambda_2(x) \neq \emptyset$.

On the other hand $y = x \circ f$, where $f : \omega \to \omega$ is defined by $f(k) = 2k$, so, by Fact 1.3(a), $1 \Vdash \tau_y = \check{f}^{-1}[\tau_x]$, which, together with (3) and (4), implies $b \Vdash$ "$\check{f}^{-1}[\tau_x]$ is new". Now, since $f \in V$, we have $b \Vdash$ "τ_x is new" and, by (a), $\lambda_1(x) = \emptyset$. So $\lambda_1 \neq \lambda_2$.

(e) follows from (d) and Fact 2.1(c). □

Remark 2.3. Imitating the proof of the part (a) of the previous theorem one can easily show that, if $\mathbb{B}$ is a complete Boolean algebra, x a sequence in $\mathbb{B}$ and τ_x the corresponding name for a real, then the real determined by τ_x is

- always old iff $\lambda_1(x) \neq \emptyset$;
- sometimes new, but always supported iff $\lambda_1(x) = \emptyset$ and $\lambda_2(x) \neq \emptyset$;
- sometimes unsupported, but always unsplitting iff $\lambda_2(x) = \emptyset$ and $\lambda_3(x) \neq \emptyset$;
- sometimes splitting iff $\lambda_3 = \emptyset$ and $\lambda_4(x) \neq \emptyset$.

(Here "always" means in each and "sometimes" in some generic extension.)

By Fact 2.1(a) and Theorem 2.2(b) we have $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$; by Fact 2.1(c), $\lambda_2 = \lambda_3 < \lambda_4$ is impossible and, by Fact 2.1(c) and Theorem 2.2(d), $\lambda_1 = \lambda_2$ iff $\lambda_2 = \lambda_3$. Now, using Fact 2.1(c), (d) and Theorem 2.2(d), we show that, up to these restrictions, everything is possible.

Example 2.4. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ holds in each $(\omega, 2)$-distributive and, in particular, each atomic complete Boolean algebra.

$\lambda_1 < \lambda_2 < \lambda_3 = \lambda_4$ holds in each complete Boolean algebra which produces new reals, but does not produce splitting reals, for example in r.o.($\mathbb{P}$), where $\mathbb{P}$ is the Sacks or the Miller forcing.

$\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ holds in each complete Boolean algebra which produces splitting reals, for example in r.o.($\mathbb{P}$), where $\mathbb{P}$ is the Cohen or the random forcing.

3. The closure of $\boldsymbol{\lambda_1}$ under (L2)

By Theorem 2.2(c), the convergence λ_1 does not satisfy (L2) and, hence, it is never a topological convergence. The minimal closures of a convergence under (L2) and (L3) are described in the following general fact (see [8]).

Fact 3.1. *Let $\lambda : X^\omega \to P(X)$ be a convergence satisfying condition* (L1). *Then*

(a) *The mapping $\bar\lambda : X^\omega \to P(X)$ defined by $\bar\lambda(y) = \bigcup_{x\in X^\omega, f\in\omega^{\uparrow\omega}, y=x\circ f} \lambda(x)$ is the minimal convergence bigger than λ and satisfying* (L1) *and* (L2)*;*

(b) *$\bar\lambda^* : X^\omega \to P(X)$ defined by $\bar\lambda^*(y) = \bigcap_{f\in\omega^{\uparrow\omega}} \bigcup_{g\in\omega^{\uparrow\omega}} \bar\lambda(y \circ f \circ g)$ is the minimal convergence bigger than $\bar\lambda$ and satisfying* (L1)–(L3)*;*

(c) $\lambda \le \bar\lambda \le \bar\lambda^* \le \lim_{\mathcal{O}_\lambda}$*;*

(d) $\mathcal{O}_\lambda = \mathcal{O}_{\bar\lambda} = \mathcal{O}_{\bar\lambda^*}$.

For a subset A of a complete Boolean algebra $\mathbb{B}$ let $A\uparrow = \{b \in \mathbb{B} : \exists a \in A\, a \le b\}$. The (L2)-closures of the convergences λ_2, λ_3 and λ_4 are described in the following fact (see [8] and [9]).

Fact 3.2. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) *$\bar\lambda_4(y) = \{\limsup y\}\uparrow$, for each sequence y in $\mathbb{B}$;*

(b) *$\bar\lambda_2 = \bar\lambda_3 = \bar\lambda_4$;*

(c) *The convergence $\bar\lambda_4$ generalizes the convergence on the Aleksandrov cube.*

Now, concerning the convergence λ_1 we have

Theorem 3.3. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) *The closure of λ_1 under* (L2) *is given by*

$$\bar\lambda_1(y) = \begin{cases} \{\limsup y\}\uparrow & \textit{if } 1 \Vdash \tau_y \textit{ is old}, \\ \emptyset & \textit{otherwise}; \end{cases} \tag{5}$$

(b) *$\bar\lambda_1 = \bar\lambda_4$ iff the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive.*

Proof. (a)

Claim 1. $\bar\lambda_1(y) = \{\limsup y\}\uparrow$ if and only if $1 \Vdash \tau_y$ is old.

Proof of Claim 1. ($\Rightarrow$) Let $\bar\lambda_1(y) = \{\limsup y\}\uparrow$. Then, by Fact 3.1(a) the set $\bar\lambda_1(y) = \bigcup_{x\in\mathbb{B}^\omega, f\in\omega^{\uparrow\omega}, y=x\circ f} \lambda_1(x)$ is nonempty and, hence there are $x \in \mathbb{B}^\omega$ and $f \in \omega^{\uparrow\omega}$ such that $y = x \circ f$ and $\lambda_1(x) \ne \emptyset$. By Theorem 2.2(a), $1 \Vdash$ "τ_x is old" and by Fact 1.3(a), $1 \Vdash \tau_y = f^{-1}[\tau_x]$, which implies $1 \Vdash$ "τ_y is old".

($\Leftarrow$) Let $1 \Vdash$ "τ_y is old". According to Fact 3.1(a) we show that

$$\bigcup_{x\in\mathbb{B}^\omega, f\in\omega^{\uparrow\omega}, y=x\circ f} \lambda_1(x) = \{\limsup y\}\uparrow .$$

($\subset$) Suppose that $x \in \mathbb{B}^\omega$, $f \in \omega^{\uparrow\omega}$, $y = x \circ f$ and $b \in \lambda_1(x)$. Then $b = \limsup x$ and, since $y \prec x$, by Fact 1.3(d) we have $\limsup y \le \limsup x = b$.

($\supset$) Let $b \geq \limsup y$. Let $x = \langle y_0, b, y_1, b, y_2, \dots \rangle$ and $f, g \in \omega^{\uparrow\omega}$, where $f(k) = 2k$ and $g(k) = 2k+1$. Then $y = x \circ f$ and, if $z = x \circ g$, using Facts 1.2(f) and 1.3(b) we have

$$\begin{aligned}\limsup x &= \| |\tau_x| = \check{\omega} \| = \| |\tau_x \cap \check{f}[\omega]| = \check{\omega} \| \vee \| |\tau_x \cap \check{g}[\omega]| = \check{\omega} \| \\ &= \| |\tau_y| = \check{\omega} \| \vee \| |\tau_z| = \check{\omega} \| = \| |\tau_y| = \check{\omega} \| \vee b = b.\end{aligned}$$

So, by Theorem 2.2(a), for a proof that $b \in \lambda_1(x)$ it remains to be shown that $1 \Vdash$ "τ_x is old", which follows from $1 \Vdash$ "τ_y is old" and the following subclaim.

Subclaim 1. (i) $b' \Vdash \tau_x = \check{f}[\tau_y]$; (ii) $b \Vdash \tau_x = \check{f}[\tau_y] \cup \{1, 3, 5, \dots\}\check{}$.

Proof of Subclaim 1. By Fact 1.3(a) we have $1 \Vdash \tau_y = \check{f}^{-1}[\tau_x]$ and, hence,

$$1 \Vdash \check{f}[\tau_y] \subset \tau_x. \tag{6}$$

Let G be a $\mathbb{B}$-generic filter over V.

(i) If $b' \in G$, then for $n \in (\tau_x)_G$ we have $x_n \in G$ and, since $b \notin G$, there is $k \in \omega$ such that $x_n = x_{2k} = y_k$. Hence $k \in (\tau_y)_G$ and $n = f(k) \in f[(\tau_y)_G]$. So $b' \Vdash \tau_x \subset \check{f}[\tau_y]$ and, by (6), $b' \Vdash \tau_x = \check{f}[\tau_y]$.

(ii) Clearly, $b \Vdash \{1, 3, 5, \dots\}\check{} \subset \tau_x$ and, by (6), $b \Vdash \check{f}[\tau_y] \subset \tau_x$. On the other hand, let $b \in G$ and $n \in (\tau_x)_G$, that is $x_n \in G$. If n is odd, we are done. Otherwise, as in (a) we show that $n \in f[(\tau_y)_G]$. Claim 1 is proved.

Claim 2. $\bar{\lambda}_1(y) \neq \emptyset \Leftrightarrow 1 \Vdash \tau_y$ is old.

Proof of Claim 2. ($\Rightarrow$) Suppose that $a \in \bar{\lambda}_1(y)$. Then, by Fact 3.1(a), there are $x \in \mathbb{B}^\omega$ and $f \in \omega^{\uparrow\omega}$ such that $y = x \circ f$ and $a \in \lambda_1(x)$, which, by Theorem 2.2(a), implies $1 \Vdash$ "τ_x is old". By Fact 1.3(a) we have $1 \Vdash \tau_y = f^{-1}[\tau_x]$ and, consequently, $1 \Vdash$ "τ_y is old".

($\Leftarrow$) If $\bar{\lambda}_1(y) = \emptyset$, then, since $\lambda_1 \leq \bar{\lambda}_1$, we have $\lambda_1(y) = \emptyset$ and, by Theorem 2.2(a), $\neg 1 \Vdash \tau_y$ is old.

(b) It is well known [4] that $\mathbb{B}$ is $(\omega, 2)$-distributive iff forcing by $\mathbb{B}$ does not produce new reals, that is $1 \Vdash \tau_y$ is old, for each sequence y in $\mathbb{B}$. So we apply (a) and Fact 3.2(a). □

4. The topology generated by λ_1

By Theorem 3.3(b) and Fact 3.1(d), if $\mathbb{B}$ is an $(\omega, 2)$-distributive algebra, then $\mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_4}$. In this section we show more, that on each complete Boolean algebra the convergences λ_1, λ_2, λ_3 and λ_4 generate the same topology, investigated in [9]. Concerning the convergences λ_0, λ_2, λ_3 and λ_4 we have (see [8] and [9])

Fact 4.1. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) $\mathcal{O}_{\lambda_2} = \mathcal{O}_{\lambda_3} = \mathcal{O}_{\lambda_4}$;

(b) $\mathcal{O}_{\lambda_4}$ *is a sequential T_0 connected compact topology on $\mathbb{B}$;*

(c) $\mathcal{O}_{\lambda_4}$ *and its dual generate the sequential topology, $\mathcal{O}_{\lambda_0}$, when $\mathbb{B}$ is a Maharam algebra.*

We will use the following general fact (see [8]).

Fact 4.2. *Let $\lambda : X^\omega \to P(X)$ be a convergence satisfying* (L1) *and* (L2) *and let the mappings $u^\alpha_\lambda : P(X) \to P(X)$, $\alpha \leq \omega_1$, be defined by recursion in the following way: for $A \subset X$*

$u^0_\lambda(A) = A$,

$u^{\alpha+1}_\lambda(A) = u_\lambda(u^\alpha_\lambda(A))$ *and*

$u^\gamma_\lambda(A) = \bigcup_{\alpha<\gamma} u^\alpha_\lambda(A)$, *for a limit $\gamma \leq \omega_1$.*

Then $u^{\omega_1}_\lambda$ is the closure operator in the space $\langle X, \mathcal{O}_\lambda\rangle$.

We will say that a subset A of a c.B.a. $\mathbb{B}$ is *upward closed* iff $A = A\uparrow$. A sequence x in $\mathbb{B}$ will be called *decreasing* if $x_0 \geq x_1 \geq x_2 \geq \dots$.

Lemma 4.3. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) *The set $\bar\lambda_1(x)$ is upward closed, for each sequence x in $\mathbb{B}$;*

(b) *If x is a decreasing sequence in $\mathbb{B}$, then $\lambda_1(x) = \{\bigwedge_{n\in\omega} x_n\}$;*

(c) *If $A \subset \mathbb{B}$ is an upward closed set, then $u_{\bar\lambda_1}(A) = u_{\bar\lambda_2}(A)$;*

(d) *The set $u_{\bar\lambda_1}(A)$ is upward closed, for each $A \subset \mathbb{B}$.*

Proof. (a) follows from Theorem 3.3.

(b) If $x = \langle x_n : n \in \omega\rangle$ is decreasing, then $\limsup x = \bigwedge_{n\in\omega}\bigvee_{k\geq n} x_k = \bigwedge_{n\in\omega} x_n$ and, by Theorem 2.2(a), it remains to be shown that $1 \Vdash \tau_x$ is old. If G is a $\mathbb{B}$-generic filter over V, then $(\tau_x)_G = \{n : x_n \in G\}$, so if $m < n \in (\tau_x)_G$, then $x_m \geq x_n \in G$, which implies $x_m \in G$ and, hence, $m \in (\tau_x)_G$. Thus $(\tau_x)_G$ is either a finite set or equal to ω and, consequently, belongs to V.

(c) Let $A \subset \mathbb{B}$ be an upward closed set.

($\subset$) Since $\lambda_1 \leq \lambda_2 \leq \bar\lambda_2$, by the minimality of $\bar\lambda_1$ (see Fact 3.1(a)) we have $\bar\lambda_1 \leq \bar\lambda_2$ and, hence, $u_{\bar\lambda_1}(A) = \bigcup_{x\in A^\omega} \bar\lambda_1(x) \subset \bigcup_{x\in A^\omega} \bar\lambda_2(x) = u_{\bar\lambda_2}(A)$.

($\supset$) By Fact 3.2 we have $\bar\lambda_2(x) = \{\limsup x\}\uparrow$. So, for $x \in A^\omega$ we show that $\{\limsup x\}\uparrow \subset u_{\bar\lambda_1}(A)$. Let $\limsup x = b$. Then the sequence $t = \langle t_n : n \in \omega\rangle$ defined by

$$t_n = b \vee \bigvee_{k\geq n} x_k$$

is decreasing and, since $t_n \geq x_n \in A$, we have $t \in A^\omega$. Since

$$\bigwedge_{n\in\omega} t_n = b \vee \bigwedge_{n\in\omega} \bigvee_{k\geq n} x_k = b \vee \limsup x = b$$

by (b) we have $b \in \lambda_1(t) \subset \bar\lambda_1(t)$ and, by (a), $\{b\}\uparrow \subset \bar\lambda_1(t)\uparrow = \bar\lambda_1(t) \subset u_{\bar\lambda_1}(A)$.

(d) We prove that $u_{\bar\lambda_1}(A)\uparrow \subset u_{\bar\lambda_1}(A)$. If $b \geq a \in u_{\bar\lambda_1}(A)$, then there is $x \in A^\omega$ such that $a \in \bar\lambda_1(x)$. By (a) we have $b \in \bar\lambda_1(x)$, which implies $b \in u_{\bar\lambda_1}(A)$. □

Theorem 4.4. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) $u^{\omega_1}_{\bar\lambda_1}(A) = u^{\omega_1}_{\bar\lambda_2}(A)$, *for each* $A \subset \mathbb{B}$;

(b) $\mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_2} = \mathcal{O}_{\lambda_3} = \mathcal{O}_{\lambda_4}$.

Proof. (a) ($\subset$) Since $\bar\lambda_1 \leq \bar\lambda_2$, we have $u^{\omega_1}_{\bar\lambda_1}(A) \subset u^{\omega_1}_{\bar\lambda_2}(A)$, for each $A \subset \mathbb{B}$.

($\supset$) First, for $A \subset \mathbb{B}$ using induction we show that for each $\alpha \leq \omega_1$

$$u^{\alpha}_{\bar\lambda_2}(u_{\bar\lambda_1}(A)) = u^{\alpha}_{\bar\lambda_1}(u_{\bar\lambda_1}(A)) \text{ and this set is upward closed.} \tag{7}$$

By Lemma 4.3(d), (7) is true for $\alpha = 0$.

Let $\beta \leq \omega_1$ and suppose that (7) holds for each $\alpha < \beta$.

If β is a limit ordinal, then, by the induction hypothesis, we have

$$u^{\beta}_{\bar\lambda_2}(u_{\bar\lambda_1}(A)) = \bigcup_{\alpha<\beta} u^{\alpha}_{\bar\lambda_2}(u_{\bar\lambda_1}(A)) = \bigcup_{\alpha<\beta} u^{\alpha}_{\bar\lambda_1}(u_{\bar\lambda_1}(A)) = u^{\beta}_{\bar\lambda_1}(u_{\bar\lambda_1}(A))$$

and, since the union of upward closed sets is upward closed, (7) is true for β.

If $\beta = \alpha + 1$, then, by the induction hypothesis we have

$$u^{\alpha+1}_{\bar\lambda_2}(u_{\bar\lambda_1}(A)) = u_{\bar\lambda_2}(u^{\alpha}_{\bar\lambda_2}(u_{\bar\lambda_1}(A))) = u_{\bar\lambda_2}(u^{\alpha}_{\bar\lambda_1}(u_{\bar\lambda_1}(A))). \tag{8}$$

By the hypothesis the set $u^{\alpha}_{\bar\lambda_1}(u_{\bar\lambda_1}(A))$ is upward closed and, by Lemma 4.3(c),

$$u_{\bar\lambda_2}(u^{\alpha}_{\bar\lambda_1}(u_{\bar\lambda_1}(A))) = u_{\bar\lambda_1}(u^{\alpha}_{\bar\lambda_1}(u_{\bar\lambda_1}(A))) = u^{\alpha+1}_{\bar\lambda_1}(u_{\bar\lambda_1}(A)) \tag{9}$$

and $u^{\beta}_{\bar\lambda_2}(u_{\bar\lambda_1}(A)) = u^{\beta}_{\bar\lambda_1}(u_{\bar\lambda_1}(A))$ follows from (8) and (9). By Lemma 4.3(d) and (9) this set is upward closed and the proof of (7) is over.

Since $A \subset u_{\bar\lambda_1}(A) \subset u^{\omega_1}_{\bar\lambda_1}(A)$, by Fact 4.2 we have $u^{\omega_1}_{\bar\lambda_1}(u_{\bar\lambda_1}(A)) = u^{\omega_1}_{\bar\lambda_1}(A)$. Using (7) we obtain $u^{\omega_1}_{\bar\lambda_2}(A) \subset u^{\omega_1}_{\bar\lambda_2}(u_{\bar\lambda_1}(A)) = u^{\omega_1}_{\bar\lambda_1}(u_{\bar\lambda_1}(A)) = u^{\omega_1}_{\bar\lambda_1}(A)$.

(b) By (a) and Fact 4.2 we have $\mathcal{O}_{\bar\lambda_1} = \mathcal{O}_{\bar\lambda_2}$ and, by Fact 3.1(d), $\mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_2}$. By Fact 4.1(a), the other two equalities hold as well. □

Thus the topology $\mathcal{O}_{\lambda_1}$, generated by the convergence λ_1, has the properties given in Fact 4.1(b) and (c).

5. Topological and weakly topological convergences

In this section we investigate the classes of complete Boolean algebras on which the convergence $\bar\lambda_1$ (satisfying conditions (L1) and (L2)) is topological or weakly topological. According to [8], a convergence $\lambda : X^\omega \to P(X)$ will be called *weakly topological* iff it satisfies conditions (L1) and (L2) and its (L3)-closure, λ^*, is a topological convergence. The following general fact can be found in [8].

Fact 5.1. *A convergence $\lambda : X^\omega \to P(X)$ satisfying* (L1) *and* (L2) *is weakly topological iff $\lambda^* = \lim_{\mathcal{O}_\lambda}$, that is for each $x \in X^\omega$ and $a \in X$*

$$a \in \lim_{\mathcal{O}_\lambda}(x) \Leftrightarrow \forall y \prec x \;\; \exists z \prec y \;\; a \in \lambda(z).$$

By [9], for the convergence $\bar\lambda_4$ we have

Fact 5.2. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) *$\bar\lambda_4$ is a topological convergence iff the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive;*

(b) *If the algebra $\mathbb{B}$ satisfies $(\hbar)$, then $\bar\lambda_4$ is a weakly topological convergence.*

We note that, according to [7], a complete Boolean algebra satisfies condition $(\hbar)$ iff $\forall x \in \mathbb{B}^\omega \; \exists y \prec x \; \forall z \prec y \;\; \limsup z = \limsup y$. More about condition $(\hbar)$ (implied by the ccc) can be found in [10].

For the convergence $\bar\lambda_1$ we have the following analogue of Fact 5.2(a).

Theorem 5.3. *$\bar\lambda_1$ is a topological convergence iff the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive.*

Proof. ($\Rightarrow$) Let $\bar\lambda_1$ be a topological convergence. Then, by Fact 1.1(h), $\bar\lambda_1 = \lim_{\mathcal{O}_{\bar\lambda_1}}$. By Fact 3.1(d) and Theorem 4.4(b) we have $\mathcal{O}_{\bar\lambda_1} = \mathcal{O}_{\lambda_1} = \mathcal{O}_{\lambda_2}$ thus $\bar\lambda_1 = \lim_{\mathcal{O}_{\lambda_2}} \geq \lambda_2$. Since $\lambda_1 \leq \lambda_2 \leq \bar\lambda_1$, by Fact 3.1(a) we have $\bar\lambda_1 = \bar\lambda_2$ and, by Fact 3.2(b) and Theorem 3.3(b) the algebra $\mathbb{B}$ is $(\omega, 2)$-distributive.

($\Leftarrow$) follows from Theorem 3.3(b) and Fact 5.2(a). □

Now we deal with the question on which algebras the convergence $\bar\lambda_1$ is weakly topological. First we describe its (L3)-closure, $\bar\lambda_1^*$, in terms of forcing.

Theorem 5.4. *Let $\mathbb{B}$ be a complete Boolean algebra. Then for $y \in \mathbb{B}^\omega$ we have*

(a) $\bar\lambda_1^*(y) = \bigcap_{A \in [\omega]^\omega} \bigcup_{B \in [A]^\omega \wedge \| \tau_y \cap \check{B} \text{ is old}\| = 1} \| |\tau_y \cap \check{B}| = \check{\omega} \| \uparrow$;

(b) *$\bar\lambda_1^*(y) \neq \emptyset$ iff $\mathcal{D}_y = \{B \in [\omega]^\omega : \|\tau_y \cap \check{B}$ is old$\| = 1\}$ is a dense set in the poset $\langle [\omega]^\omega, \subset \rangle$.*

Proof. (a) By Fact 3.1(b), for $y \in \mathbb{B}^\omega$ we prove that

$$\bigcap_{f\in\omega^{\uparrow\omega}} \bigcup_{g\in\omega^{\uparrow\omega}} \bar\lambda_1(y\circ f\circ g) = \bigcap_{A\in[\omega]^\omega} \bigcup_{B\in[A]^\omega \wedge \,\|\tau_y\cap\check B \text{ is old}\|=1} \||\tau_y \cap \check B| = \check\omega\|\!\uparrow$$

($\subset$) Suppose that for each $f \in \omega^{\uparrow\omega}$ there is $g \in \omega^{\uparrow\omega}$ such that $a \in \bar\lambda_1(y\circ f\circ g)$, which, by Theorem 3.3 and Fact 1.3, means that

$$\|\tau_{y\circ f\circ g} \text{ is old}\| = 1 \quad \text{and} \quad a \in \||\tau_y \cap (f\circ g)[\omega]\check{}\,| = \check\omega\|\!\uparrow . \tag{10}$$

Let $A \in [\omega]^\omega$ and let $f \in \omega^{\uparrow\omega}$, where $A = f[\omega]$. By the assumption, there is $g \in \omega^{\uparrow\omega}$ such that (10) holds. Then $B = f[g[\omega]] \subset A$ and, since f and g are injections, $B \in [A]^\omega$. By (10), $a \in \||\tau_y \cap \check B| = \check\omega\|\!\uparrow$. By Fact 1.3, in each generic extension $V_{\mathbb{B}}[G]$ we have $(\tau_{y\circ f\circ g})_G = (f\circ g)^{-1}[(\tau_y)_G] = (f\circ g)^{-1}[(\tau_y)_G \cap B]$ and, hence, $(\tau_y)_G \cap B = f[g[(\tau_{y\circ f\circ g})_G]]$. Thus

$$\|\tau_{y\circ f\circ g} \text{ is old} \Leftrightarrow \tau_y \cap \check B \text{ is old}\| = 1, \tag{11}$$

which together with (10) implies $\|\tau_y \cap \check B \text{ is old}\| = 1$.

($\supset$) Suppose that for each $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that

$$\|\tau_y \cap \check B \text{ is old}\| = 1 \text{ and } a \in \||\tau_y \cap \check B| = \check\omega\|\!\uparrow . \tag{12}$$

Let $f \in \omega^{\uparrow\omega}$ and $A = f[\omega]$. By the assumption, there is $B \in [A]^\omega$ such that (12) holds. If $g \in \omega^{\uparrow\omega}$ where $g[\omega] = f^{-1}[B]$, then $B = (f\circ g)[\omega]$ and, by (12), we have $a \in \||\tau_y \cap (f\circ g)[\omega]\check{}\,| = \check\omega\|\!\uparrow$.By (11), $\|\tau_{y\circ f\circ g}$ is old$\| = 1$, thus $a \in \bar\lambda_1(y\circ f\circ g)$.

(b) ($\Rightarrow$) Let $a \in \bar\lambda_1^*(y)$ and $A \in [\omega]^\omega$. By (a) there is $B \in [A]^\omega$ such that $\|\tau_y \cap \check B \text{ is old}\| = 1$ and $a \geq \||\tau_y \cap \check B| = \check\omega\|$. Thus $B \subset A$ and $B \in \mathcal{D}_y$.

($\Leftarrow$) Let $\mathcal{D}_y$ be a dense set in $\langle[\omega]^\omega, \subset\rangle$ and $a = \||\tau_y| = \check\omega\|$. Since for each $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that $\|\tau_y \cap \check B \text{ is old}\| = 1$ and, clearly, $a \geq \||\tau_y \cap \check B| = \check\omega\|$, by (a) we have $a \in \bar\lambda_1^*(y)$. □

Theorem 5.5. *If there is a sequence y in $\mathbb{B}$ such that $\|\tau_y$ is splitting$\| > 0$, then $\bar\lambda_1^*(y) = \emptyset$ and the convergence $\bar\lambda_1$ is not weakly topological.*

Proof. Let $\|\tau_y$ is splitting $\| = b > 0$ and suppose that $\bar\lambda_1^*(y) \neq \emptyset$. Then, by (b), there is $B \in [\omega]^\omega$ such that $1 \Vdash \tau_y \cap \check B$ is old. But then $b \Vdash$ "$\tau_y \cap \check B$ is old $\wedge\, \tau_y$ is splitting", which is impossible. Thus $\bar\lambda_1^*(y) = \emptyset$. By Theorem 4.4, $\lim_{\lambda_1}(y) = \lim_{\lambda_4}(y) \supset \lambda_4(y) \ni \limsup y$ and, hence, $\bar\lambda_1^*(y) \neq \lim_{\lambda_1}(y)$ so $\bar\lambda_1$ is not a weakly topological convergence. □

Concerning the previous theorem we remark that it is possible that the convergence $\bar{\lambda}_1$ is not weakly topological, although forcing by $\mathbb{B}$ does not produce splitting reals (see Example 5.8). In contrast to Fact 5.2(b) we have

Example 5.6. The ccc (and, consequently, condition ($\hbar$)) does not imply that the convergence $\bar{\lambda}_1$ is weakly topological. The Cohen algebra is ccc, produces splitting reals and, by Theorem 5.5, $\bar{\lambda}_1$ is not a weakly topological convergence.

Now, inside a wide class of complete Boolean algebras, we characterize the algebras on which the convergence $\bar{\lambda}_1$ is weakly topological.

Theorem 5.7. *Let $\mathbb{B}$ be a Boolean algebra such that $hcc(\mathbb{B}) > \mathfrak{c}$ (i.e. below each $b \in \mathbb{B}^+$ there is an antichain of size $\mathfrak{c}$). Then*

$$\bar{\lambda}_1 \text{ is a weakly topological convergence } \Leftrightarrow \mathbb{B} \text{ is } (\omega, 2)\text{-distributive.} \tag{13}$$

Proof. ($\Leftarrow$) This implication follows from Theorem 5.3.

($\Rightarrow$) If $\mathbb{B}$ is not $(\omega, 2)$-distributive, then $b = \|\exists r \subset \check{\omega}\ (r \text{ is new})\| > 0$ and, by the Maximum Principle, there is a name π such that

$$b \Vdash \pi \subset \check{\omega} \wedge \pi \text{ is new.} \tag{14}$$

We choose an enumeration $[\omega]^\omega = \{S_\alpha : \alpha < \mathfrak{c}\}$, bijections $f_\alpha : S_\alpha \to \omega$, $\alpha < \mathfrak{c}$, and a maximal antichain under b, $\{b_\alpha : \alpha < \mathfrak{c}\}$. Now, for the $\mathbb{B}$-name σ defined by $\sigma = \{\langle \check{n}, \bigvee_{\alpha<\mathfrak{c}} (b_\alpha \wedge \|f_\alpha(n)\check{} \in \pi\|)\rangle : n \in \omega\}$ it is easy to prove that $b_\alpha \Vdash \sigma = f_\alpha^{-1}[\pi]$, for $\alpha < \mathfrak{c}$, (see [7, Th. 4, Cl. 1]) and, clearly, $1 \Vdash \sigma = \tau_x$, where $x = \langle x_n : n \in \omega\rangle$ and $x_n = \bigvee_{\alpha<\mathfrak{c}} b_\alpha \wedge \|f_\alpha(n)\check{} \in \pi\|$, $n \in \omega$. Thus

$$b_\alpha \Vdash \tau_x = f_\alpha^{-1}[\pi]. \tag{15}$$

Let us prove

$$\forall B \in [\omega]^\omega \quad \|\tau_x \cap \check{B} \text{ is new}\| > 0. \tag{16}$$

Let $B \in [\omega]^\omega$ and $\alpha < \mathfrak{c}$, where $B = S_\alpha$. Let G be a $\mathbb{B}$-generic filter over V containing b_α. Since $b_\alpha < b$ we have $b \in G$ and, by (14), $\pi_G \notin V$. By (15), $(\tau_x)_G = f_\alpha^{-1}[\pi_G] \subset B$. Now, $f_\alpha^{-1}[\pi_G] \in V$ would imply $f_\alpha[f_\alpha^{-1}[\pi_G]] = \pi_G \in V$, which is false. Thus $f_\alpha^{-1}[\pi_G] = (\tau_x)_G = (\tau_x)_G \cap B \notin V$ and (16) is proved.

By (16) we have $\mathcal{D}_x = \{B \in [\omega]^\omega : 1 \Vdash \tau_x \cap \check{B} \text{ is old}\} = \emptyset$ so, by Theorem 5.4(b), $\bar{\lambda}_1^*(x) = \emptyset$. But, by Theorem 4.4, $\limsup x \in \lim_{\mathcal{O}_{\lambda_1}}(x) = \lim_{\mathcal{O}_{\bar{\lambda}_1}}(x)$ and, by Fact 5.1, $\bar{\lambda}_1$ is not a weakly topological convergence. □

Example 5.8. $\bar{\lambda}_1$ is not a weakly topological convergence on the Sacks algebra. Namely, if $\mathbb{B}$ is the Boolean completion of the Sacks forcing, $\mathbb{B}$ is homogeneous, has antichains of size $\mathfrak{c}$, adds new reals and we apply Theorem 5.7.

Is the equivalence (13) a theorem of ZFC? In the following theorem, using a result of Veličković [12], we show that under the CH, a possible counterexample can not be nicely definable.

Theorem 5.9. (CH) *If $\mathbb{B} = r.o.(\mathbb{P})$, where $\mathbb{P}$ is a Suslin forcing notion, then*

$$\bar\lambda_1 \textit{ is a weakly topological convergence } \Leftrightarrow \mathbb{B} \textit{ is } (\omega,2)\textit{-distributive.}$$

Proof. ($\Leftarrow$) This implication follows from Theorem 5.3.

($\Rightarrow$) If the algebra $\mathbb{B}$ is not $(\omega,2)$-distributive, then $b = \|\exists r\subset\check\omega\ (r \text{ is new})\| > 0$. If there exists an uncountable antichain below b, then, as in the proof of Theorem 5.7, we show that $\bar\lambda_1$ is not a weakly topological convergence. Otherwise, $\mathbb{B}|b$ is a non-atomic ccc forcing, clearly, $\mathbb{P}\cap b\downarrow$ is a non-atomic ccc Suslin forcing and, by a result of Veličković [12], produces splitting reals. Now, by Theorem 5.5, $\bar\lambda_1$ is not a weakly topological convergence again. □

6. A diagram

Here we describe the relations between the convergence structures considered in this paper.

Theorem 6.1. *Let $\mathbb{B}$ be a complete Boolean algebra. Then*

(a) *If $\mathcal{A}\subset[\omega]^\omega$ is a mad family, y a sequence in $\mathbb{B}$ and $1\Vdash \tau_y$ kills $\check{\mathcal{A}}$, then $\bar\lambda_1^*(y) = \mathbb{B}$ and $\bar\lambda_4(y) = \{1\}$;*

(b) *If forcing by $\mathbb{B}$ produces a splitting real in each extension, then the convergences $\bar\lambda_1^*$ and $\bar\lambda_4$ are not comparable;*

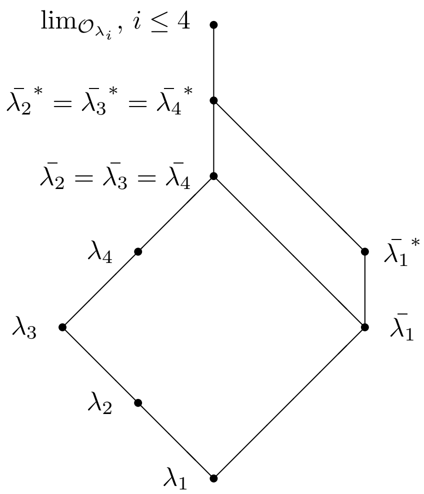
Proof. (a) Suppose that $1 \Vdash |\tau_y| = \check\omega \wedge \forall A\in\check{\mathcal{A}}\ |\tau_y\cap A| < \check\omega$. Then $\|\tau_y \text{ is infinite}\| = 1$ and, by Facts 1.2 and 3.2(a), we have $\bar\lambda_4(y) = \{1\}\!\uparrow = \{1\}$.

Using Theorem 5.4(a) we prove that $0\in\bar\lambda_1^*(y)$ (which implies $\bar\lambda_1^*(y) = \mathbb{B}$). For $A\in[\omega]^\omega$, by the maximality of $\mathcal{A}$, there is $A_1\in\mathcal{A}$ such that $B = A\cap A_1 \in [A]^\omega$. Since $1\Vdash |\tau_y\cap A_1| < \check\omega$, we have $\||\tau_y\cap\check B| < \check\omega\| = 1$ which implies $\|\tau_y\cap\check B \text{ is old}\| = 1$ and $\||\tau_y\cap\check B| = \check\omega\| = 0$.

(b) ($\bar\lambda_4 \not\le \bar\lambda_1^*$). By the assumption, there is $y\in\mathbb{B}^\omega$ such that $\|\tau_y \text{ is splitting}\| > 0$ so, by Theorem 5.5, $\bar\lambda_1^*(y) = \emptyset$ and $\bar\lambda_4(y)\neq\emptyset$.

($\bar\lambda_1^* \not\le \bar\lambda_4$). It is known (see [7, Lemma 1]) that there is a mad family $\mathcal{A}\subset[\omega]^\omega$ which is killed in each generic extension of the ground model containing new reals. By the assumption, forcing by $\mathbb{B}$ produces new reals in each extension and, hence, we have $1 \Vdash \exists x \subset \check{\omega}$ (x kills $\check{\mathcal{A}}$) and, by the Maximum Principle, there is a $\mathbb{B}$-name σ such that $1 \Vdash \sigma \subset \check{\omega}$ and $1 \Vdash \sigma$ kills $\check{\mathcal{A}}$. If $y_n = \|\check{n} \in \sigma\|$, $n \in \omega$, then $1 \Vdash \tau_y = \sigma$ and $1 \Vdash \tau_y$ kills $\check{\mathcal{A}}$. By (a) we have $\bar{\lambda}_1^*(y) = \mathbb{B}$ and $\bar{\lambda}_4(y) = \{1\}$. $\square$

In the following diagram $\lambda' \leq \lambda''$ denotes that for each c.B.a. $\mathbb{B}$ and each sequence x in $\mathbb{B}$, $\lambda'(x) \subset \lambda''(x)$.



In the sequel we show that the diagram is correct. By Fact 2.1(a) and Theorem 2.2(b) we have $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ and, by Example 2.4, all the inequalities can be strict. By Fact 3.2(b) we have $\bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}_4$, which implies $\bar{\lambda}_2^* = \bar{\lambda}_3^* = \bar{\lambda}_4^*$ and $\lim_{\mathcal{O}_{\lambda_2}} = \lim_{\mathcal{O}_{\lambda_3}} = \lim_{\mathcal{O}_{\lambda_4}}$. By Theorem 4.4(b) we have $\lim_{\mathcal{O}_{\lambda_1}} = \lim_{\mathcal{O}_{\lambda_2}}$. By Fact 3.1, $\lambda_1 \leq \lambda_4$ implies $\bar{\lambda}_1 \leq \bar{\lambda}_4$ and $\bar{\lambda}_1^* \leq \bar{\lambda}_4^*$.

The convergence $\bar{\lambda}_1^*$ is not comparable with $\lambda_2, \lambda_3, \lambda_4$ and $\bar{\lambda}_4$. The relation $\bar{\lambda}_1^* \not\leq \bar{\lambda}_4$ is proved in (b) of Theorem 6.1. For a proof that $\bar{\lambda}_1^* \not\geq \lambda_2$ we follow Theorem 5.5. If $y \in \mathbb{B}^\omega$, where $\|\tau_y$ is splitting$\| > 0$, then $\bar{\lambda}_1^*(y) = \emptyset$. For the sequence $z = \langle y_0, 1, y_1, 1, \ldots \rangle$ we have $y \prec z$ and, since $\bar{\lambda}_1^*$ fulfills (L2), we have $\bar{\lambda}_1^*(z) = \emptyset$. But $1 \Vdash \{1, 3, 5, \ldots\}\check{} \subset \tau_z$, thus $1 \Vdash$ "τ_z is supported" which, by (2), implies $\lambda_2(z) \neq \emptyset$.

The convergence $\bar{\lambda}_1$ is not comparable with λ_2, λ_3, and λ_4. Namely, $\bar{\lambda}_1(\langle 0 \rangle) = \{0\}\uparrow$ and $\lambda_4(\langle 0 \rangle) = \{0\}$ implies $\bar{\lambda}_1 \not\leq \lambda_4$. The relation $\bar{\lambda}_1 \not\geq \lambda_2$ follows from $\bar{\lambda}_1^* \not\geq \lambda_2$, proved above. $\square$

Remark 6.2. For the $(\omega, 2)$-distributive algebras the diagram collapses to the diagram containing two elements, e.g. λ_1 and $\bar{\lambda}_1$ (see Example 2.4 and Fact 5.2(a)).

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