Publ. Math. Debrecen
82/2 (2013), 309–324
DOI: 10.5486/PMD.2013.5220

On prime radical of submodules

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Abstract. Let R be a commutative ring with identity. A proper submodule N of an R-module M is called *P-prime* [resp. *P-primary*], if for each $r \in R$ and $a \in M$, $ra \in N$ implies that $a \in N$ or $r \in P = (N : M)$ [resp. $r \in P = \sqrt{(N : M)}$]. The intersection of all prime submodules of M containing a submodule B denoted by $\operatorname{rad}(B)$ is called the radical of B. We will try to formulate and find the forms of elements of $\operatorname{rad}(B)$, and we study when the radicals of primary submodules are prime.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider R to be a ring, M a unitary R-module, B a submodule of M, and $\mathbb{N}$ the set of positive integers. By $B \leq M$ [resp. $B < M$], we mean B is a submodule [resp. a proper submodule] of M.

It is said that $N < M$ is a *prime* submodule of M, if the condition $ra \in N$, $r \in R$ and $a \in M$ implies that $a \in N$ or $rM \subseteq N$ (see [1], [2], [4]–[6], [8], [12]–[15], [17]–[21]).

For any subset B of M, the *envelope* of B, $E(B)$ (or $E_M(B)$) is

$$E(B) = \{x \mid x = ra,\ r^n a \in B,\ \text{for some } r \in R,\ a \in M,\ n \in \mathbb{N}\}.$$

Recall that a ring R is called *absolutely flat* (or *Von Neumann regular*), in case $Rx = Rx^2$, for every $x \in R$.

In general $E(B)$ is not a submodule of M, indeed according to [8, Proposition 2.1], R is an absolutely flat ring if and only if $E(B)$ is a submodule of M

Mathematics Subject Classification: 13C99, 13A15, 13C13.
Key words and phrases: distributive module, primary submodule, prime submodule, radical formula, radical of a submodule, secondary module.

for any submodule B of every R-module M. The submodule of M generated by $E(B)$ is denoted by $\langle E(B)\rangle$ and it is a module version of the radical of ideals, and obviously $B \subseteq \langle E(B)\rangle$.

The intersection of all prime submodules of M containing B is denoted by $\mathrm{rad}(B)$ or $\mathrm{rad}_M(B)$. If there does not exist any prime submodule of M containing B, then we consider $\mathrm{rad}(B) = M$.

It is said that a module M *satisfies the radical formula (s.t.r.f.)*, if $\langle E(B)\rangle = \mathrm{rad}(B)$ for every submodule B of M. We say a ring R *s.t.r.f.*, if every R-module s.t.r.f. The s.t.r.f. concept has been studied in many papers recently, see for example [2], [4], [8], [13]–[15], [17]–[21].

Recall that an R-module M is multiplication if every submodule of M is of the form IM, where I is an ideal of R (see [10]).

An R-module $0 \neq S$ is said to be *P-secondary*, if for each $r \in R$, $rS = S$ or $r \in P = \sqrt{\mathrm{Ann}(S)}$. If S is a P-secondary module, then P is a prime ideal of R. Evidently every divisible module over an integral domain is 0-secondary.

According to [16, Section 6], a *secondary representation* of an R-module M, is an expression of M as a finite sum of P_i-secondary submodules S_i, that is $M = S_1 + S_2 + S_3 + \cdots + S_n$. If M has a secondary representation, then it is said that M is a *secondary representable module*.

Recall that a *serial module* is a module in which every two submodules are comparable. A module M [resp. ring R] is called *distributive* [resp. *arithmetical*], when $I + (J \cap K) = (I + J) \cap (I + K)$, for every three arbitrary submodules [resp. ideals] I, J and K of M [resp. R]. By [7, Theorem 2.16] M is distributive if and only if $M_{\mathfrak{M}}$ is a serial module for every maximal ideal $\mathfrak{M}$ of R.

In Section 2 of this paper we will find the forms of elements of radicals of submodules for some particular modules such as modules over rings of Krull dimension zero, multiplication modules and secondary representable modules. In Section 3 we will find the formulas of radical of primary submodules. It is proved that $\mathrm{rad}(Q) = PM$, if Q is a P-primary submodule of a distributive module M. Also we introduce some modules of which the radicals of primary submodules are prime submodules.

2. Some formulas for radicals of submodules

Throughout this section, we consider:

$$\mathcal{A} = \{(B, M) \mid M \text{ is a module and } B \leq M\},$$

and we suppose $\mathcal{B}$ to be the class of all modules.

Definition 1. Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a function such that $\phi(B, M) \leq M$ for any $(B, M) \in \mathcal{A}$. We will say that a module M is ϕ-radical, if $\mathrm{rad}_M(B) \subseteq \phi(B, M)$, for every $B \leq M$. If every R-module is ϕ-radical, then we will say the ring R is ϕ-radical.

We define $\phi_0 : \mathcal{A} \longrightarrow \mathcal{B}$ by $\phi_0(B, M) = \langle E_M(B)\rangle$. Then a module M is ϕ_0-radical if and only if M s.t.r.f.

We consider $N(R)$ to be the nilradical of R, which is the intersection of all prime ideals (or all nilpotent elements) of R. Evidently for every submodule B of any R-module M, we have:

$$B + N(R)M \subseteq B + \sqrt{(B : M)}M \subseteq \langle E(B)\rangle \subseteq \mathrm{rad}(B). \qquad (*)$$

In the following, we are trying to study the equalities $B + N(R)M = \mathrm{rad}(B)$ and $B + \sqrt{(B : M)}M = \mathrm{rad}(B)$. Hence throughout this paper, we consider:

$$\phi_1(B, M) = B + \sqrt{(B : M)}M, \quad \phi_2(B, M) = B + N(R)M,$$

$$\phi_3(B, M) = \sqrt{(B : M)}M.$$

From $(*)$ in the above, if a module M is ϕ_i-radical, for some $1 \leq i \leq 3$, then $\phi_i(B, M) = \langle E(B)\rangle = \mathrm{rad}(B)$ for every submodule B of M. In particular M s.t.r.f.

Definition 2. It will be said that ϕ commutes with the localization, if $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a function such that $\phi(B, M) \leq M$ for any $(B, M) \in \mathcal{A}$, and $\phi(B_{\mathfrak{M}}, M_{\mathfrak{M}}) \subseteq (\phi(B, M))_{\mathfrak{M}}$ for each $(B, M) \in \mathcal{A}$ and every maximal ideal $\mathfrak{M}$ of R.

According to [19, Lemma 1.5], ϕ_0 commutes with the localization.

Lemma 2.1. *Let M be an R-module and suppose ϕ commutes with the localization. If for any maximal ideal $\mathfrak{M}$ of R, the $R_{\mathfrak{M}}$-module $M_{\mathfrak{M}}$ is ϕ-radical, then M is ϕ-radical.*

Proof. Let B be a proper submodule of an R-module M and $\mathfrak{M}$ a maximal ideal of R. According to [19, Proposition 1.6], $(\mathrm{rad}_M(B))_{\mathfrak{M}} \subseteq \mathrm{rad}_{M_{\mathfrak{M}}}(B_{\mathfrak{M}})$. Hence by our assumption $(\mathrm{rad}_M(B))_{\mathfrak{M}} \subseteq \phi(B_{\mathfrak{M}}, M_{\mathfrak{M}}) \subseteq (\phi(B, M))_{\mathfrak{M}}$, for each maximal ideal $\mathfrak{M}$ of R. Therefore $\mathrm{rad}_M(B) \subseteq \phi(B, M)$. $\square$

Note that if for a submodule B of an R-module M, the ideal $(B : M)$ is a prime ideal, then B need not be a prime submodule. For example consider $M = \mathbb{Z} \oplus \mathbb{Z}$, $B = 0 \oplus 2\mathbb{Z}$, and $R = \mathbb{Z}$. Then $(B : M) = 0$ is a prime ideal, however B is not a prime submodule of M, because $2(0, 1) \in B$, but $(0, 1) \notin B$. Compare this note with the following lemma, which its proof is straightforward.

Lemma 2.2. *Let B be a submodule of an R-module M. If $(B : M)$ is a maximal ideal of R, then B is a prime submodule of M.*

Theorem 2.3. *Let R be a ring. Then the following are equivalent:*

(i) $\operatorname{rad}(B) = B + N(R)M$, *for every submodule B of any R-module M;*

(ii) $\operatorname{rad}(B) = B + N(R)M$, *for every submodule B of any finitely generated R-module M;*

(iii) $\operatorname{rad}(B) = B + \sqrt{(B : M)}M$, *for every submodule B of any R-module M;*

(iv) $\operatorname{rad}(B) = B + \sqrt{(B : M)}M$, *for every submodule B of any finitely generated R-module M;*

(v) $\dim R = 0$.

Proof. (i) $\Longrightarrow$ (iii) Note that $N(R) \subseteq \sqrt{(B : M)}$.

(iii) $\Longrightarrow$ (iv) The proof is clear.

(iv) $\Longrightarrow$ (v) We show that $R/N(R)$ is an absolutely flat ring, and so by [3, p. 44, Ex. 11], $\dim R = 0$. Let I be an ideal of R containing $N(R)$ and consider $M = R/I \oplus R$.

Evidently $(0 : M) = 0$, and by our assumption, M is ϕ_1-radical, so $\langle E_M(0)\rangle = \operatorname{rad}(0) = \sqrt{(0 : M)}M = N(R)M = (I/I) \oplus N(R)$.

Suppose that an R-module M' is a direct sum of two R-modules M_1 and M_2, i.e., $M' = M_1 \oplus M_2$, then according to [2, Lemma 2.3], $\langle E_{M'}(0)\rangle = \langle E_{M_1}(0)\rangle \oplus \langle E_{M_2}(0)\rangle$. Thus $\langle E_M(0)\rangle = \langle E_{R/I}(I/I)\rangle \oplus \langle E_R(0)\rangle = (\sqrt{I}/I) \oplus N(R)$. Consequently $I/I = \sqrt{I}/I$, that is $\sqrt{I} = I$, for every ideal I of R containing $N(R)$. Therefore $R/N(R)$ is an absolutely flat ring.

(v) $\Longrightarrow$ (i) According to [3, Proposition 3.11], for any maximal ideal $\mathfrak{M}$ of R, $(N(R))_{\mathfrak{M}} = N(R_{\mathfrak{M}})$, hence the function ϕ_2 commutes with the localization, so by (2.1), we can assume that $(R, \mathfrak{M})$ is a local ring of dimension zero, and thus $N(R) = \mathfrak{M}$.

Let B be a proper submodule of an R-module M. Now since $\mathfrak{M} \subseteq (B + \mathfrak{M}M : M)$, we have $(B + \mathfrak{M}M : M) = \mathfrak{M}$ or $B + \mathfrak{M}M = M$. So by (2.2), $\operatorname{rad}(B) \subseteq B + \mathfrak{M}M = B + N(R)M$.

(i) $\Longrightarrow$ (ii) The proof is clear.

(ii) $\Longrightarrow$ (iv) The proof is similar to that of (i) $\Longrightarrow$ (iii). □

Recall that an R-module M is called *weak multiplication* if every prime submodule of M is of the form IM, where I is an ideal of R (see [1], [5]). According to [5, Theorem 2.7], every finitely generated weak multiplication module is a multiplication module.

According to P. Gabriel [11], a module M is *finitely annihilated* if there exists a finite subset T of M with $\mathrm{Ann}(T) = \mathrm{Ann}(M)$. Evidently every finitely generated module is finitely annihilated.

In [10], the ideal $\theta(M)$ is introduced as $\theta(M) = \sum_{a\in M}(Ra : M)$.

Lemma 2.4 ([9, Proposition 4]). *Every multiplication module over a local ring is cyclic.*

The following is a generalization of [12, Theorem 5.6(II)].

Proposition 2.5. *Let M be an R-module. Then the following are equivalent:*

(i) *M is a finitely annihilated locally cyclic module;*

(ii) *M is a finitely generated multiplication module;*

(iii) *M is a finitely generated module and $\mathrm{rad}(B) = \sqrt{(B : M)}M$, for every submodule B of M;*

(iv) *M is a finitely generated module and $N = (N : M)M$, for every prime submodule N of M;*

(v) *$\theta(M)M = M$, and $PM \neq M$ for every maximal ideal P of R containing $\mathrm{Ann}(M)$, and $\mathrm{rad}(B) = \sqrt{(B : M)}M$ for every submodule B of M.*

Proof. (i) $\Longrightarrow$ (ii) By [10, Theorem 3.1], if $M \neq PM$, for any maximal ideal P of R containing $\mathrm{Ann}(M)$, then M is finitely generated. On the contrary let $M = \mathfrak{M}M$, for some maximal ideal $\mathfrak{M}$ of R containing $\mathrm{Ann}(M)$. By our assumption $M_{\mathfrak{M}}$ is a cyclic $R_{\mathfrak{M}}$-module, and $M_{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}}M_{\mathfrak{M}}$, then by Nakayama's lemma there exist $r \in R$ and $s \in R \setminus \mathfrak{M}$ such that $(r/s)M_{\mathfrak{M}} = 0$ and $1 - (r/s) \in \mathfrak{M}_{\mathfrak{M}}$.

Suppose that $T = \{t_1, t_2, t_3, \dots, t_n\}$ is a finite subset of M with $\mathrm{Ann}(T) = \mathrm{Ann}(M)$. Then $(r/s)(t_i/1) = 0$, for each $1 \leq i \leq n$. So there exists $s_i \in R \setminus \mathfrak{M}$ with $s_i r t_i = 0$, for each $1 \leq i \leq n$. Put $\bar{s} = s_1 s_2 s_3 \dots s_n$. Then $\bar{s}r \in \mathrm{Ann}(T) = \mathrm{Ann}(M) \subseteq \mathfrak{M}$. Hence $r \in \mathfrak{M}$, and so $r/s \in \mathfrak{M}_{\mathfrak{M}}$, which is a contradiction, since $1 - (r/s) \in \mathfrak{M}_{\mathfrak{M}}$. According to [9, Proposition 5] every finitely generated locally cyclic module is multiplication.

(ii) $\Longrightarrow$ (iii) Let $\mathfrak{M}$ be a maximal ideal of R. According to [3, Proposition 3.11], $(\sqrt{(B : M)})_{\mathfrak{M}} = \sqrt{(B : M)_{\mathfrak{M}}}$, and since M is finitely generated, $(B : M)_{\mathfrak{M}} = (B_{\mathfrak{M}} : M_{\mathfrak{M}})$.

So $(\phi_3(B, M))_{\mathfrak{M}} = (\sqrt{(B : M)}M)_{\mathfrak{M}} = \sqrt{(B_{\mathfrak{M}} : M_{\mathfrak{M}})}M_{\mathfrak{M}} = \phi_3(B_{\mathfrak{M}}, M_{\mathfrak{M}})$, Thus $\phi_3|_{\mathcal{A}'}$ commutes with the localization, where

$$\mathcal{A}' = \{(B, M) \in \mathcal{A} \mid M \text{ is finitely generated}\}.$$

Therefore by (2.1), we may assume that R is a local ring and so by (2.4), M is cyclic.

Let $M = R/I$, where I is an ideal of R. Suppose B is a submodule of R/I. Then $B = J/I$, where J is an ideal of R containing I. One can easily check that $\mathrm{rad}(B) = (\cap P)/I$, where P runs over all prime ideals of R containing J. Hence $\mathrm{rad}(J/I) = \sqrt{J}/I = \sqrt{(J/I : R/I)}R/I$.

(iii) $\Longrightarrow$ (iv) The proof is clear.

(iv) $\Longrightarrow$ (i) Since M is a finitely generated weak multiplication R-module, by [5, Theorem 2.7], M is multiplication. Now by (2.4), M is locally cyclic.

(ii) $\Longrightarrow$ (v) We have $Ra = (Ra : M)M$, for each $a \in M$. So $\theta(M)M = (\sum_{a\in M}(Ra : M))M = \sum_{a\in M}((Ra : M)M) = \sum_{a\in M} Ra = M$. If $PM = M$, for a maximal ideal P of R containing $\mathrm{Ann}(M)$, then by Nakayama's Lemma, there exists $r \in \mathrm{Ann}(M)$ with $1 - r \in P$, which is impossible.

(v) $\Longrightarrow$ (iii) We have $\theta(M) = R$, otherwise there exists a prime ideal P of R containing $\theta(M)$. So $M = \theta(M)M \subseteq PM$, which is a contradiction. Now $1 \in \theta(M)$ implies that there exist $n \in \mathbb{N}$ and $a_1, a_2, a_3, \dots, a_n \in M$ such that $1 = r_1 + r_2 + \cdots + r_n$, where $r_i \in (Ra_i : M)$, $\forall\, 1 \le i \le n$. So $M \subseteq r_1M + r_2M + \cdots + r_nM \subseteq Ra_1 + Ra_2 + \cdots + Ra_n$. □

Note 1. There is no a ring R such that $\mathrm{rad}(N) = (N : M)M$ for every prime submodule N of every (finitely generated) R-module M.

Proof. Consider $M = R \oplus R$ and suppose that P is a prime ideal of R. It is easy to see that $P \oplus R$ is a P-prime submodule of M. Then $P \oplus R = \mathrm{rad}(P \oplus R) = \sqrt{(P \oplus R : M)}M = PM = P \oplus P$, which is a contradiction. □

If A_1, A_2 are two subsets of M, then we consider $A_1 + A_2$ as the subset $\{a_1 + a_2 \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$ of M.

Theorem 2.6. *Let $M = \sum_{i=1}^{n} M_i$ such that M_i is a P_i-secondary submodule of M for each $1 \le i \le n$. Then $\mathrm{rad}(B) = \sum_{i=1}^{n} P_iM_i + E(B)$, for each $B \le M$. In particular $\mathrm{rad}(B) = \sqrt{(B : M)}M + E(B)$, if M is a secondary module.*

Proof. By [18, Corollary 3.11], every secondary representable module s.t.r.f., and since $E(B) \subseteq \sum_{i=1}^{n} P_iM_i + E(B) \subseteq \langle E(B)\rangle$, it is enough to show that $\sum_{i=1}^{n} P_iM_i + E(B)$ is a submodule of M.

It is easy to see that $\sum_{i=1}^{n} P_iM_i + E(B)$ is closed under the multiplication. Therefore it is sufficient to prove that $\alpha + \beta \in \sum_{i=1}^{n} P_iM_i + E(B)$, for each $\alpha, \beta \in \sum_{i=1}^{n} P_iM_i + E(B)$.

Suppose $\alpha = x_1 + x$, $\beta = y_1 + y$, where $x_1, y_1 \in \sum_{i=1}^{n} P_iM_i$ and $x, y \in E(B)$. We prove that $x + y \in \sum_{i=1}^{n} P_iM_i + E(B)$.

There exist $r, s \in R$, $a, b \in M$ and a positive integer k such that $x = ra$, $y = sb$, and $r^k a, s^k b \in B$. If for some $1 \leq i \leq n$, $r \in P_i$ or $s \in P_i$, then obviously $ra + sb \in \sum_{i=1}^n P_i M_i + E(B)$. Now suppose $r, s \notin P_i$, for all $1 \leq i \leq n$. Since $rs \notin P_i$ for all $1 \leq i \leq n$, $rsM = M$. Thus there exists $c \in M$ with $rsc = ra + sb$. Consequently $(rs)^k c = (rs)^{k-1}(ra + sb) = s^{k-1} r^k a + r^{k-1} s^k b \in B$. So $x + y = (rs)c \in E(B)$.

Now let $x + y = z_1 + z$, where $z_1 \in \sum_{i=1}^n P_i M_i$ and $z \in E(B)$. Thus $\alpha + \beta = x_1 + y_1 + z_1 + z \in \sum_{i=1}^n P_i M_i + E(B)$, which completes the proof.

If M is a P-secondary module, then it is enough to show that $\sqrt{(B : M)} = P$.

First note that $(B : M) \subseteq P$. To see the proof, let $r \in (B : M) \setminus P$. Since M is P-secondary, $M = rM \subseteq B$, which is a contradiction.

Evidently $\mathrm{Ann}(M) \subseteq (B : M)$, then $P = \sqrt{\mathrm{Ann}(M)} \subseteq \sqrt{(B : M)} \subseteq P$. □

Lemma 2.7. *Let M_1 be a P_1-secondary submodule of a module M, and Q a P-primary submodule of M. Then $P = P_1$, if $M_1 \nsubseteq Q$.*

Proof. Consider $r \in P_1 = \sqrt{\mathrm{Ann}(M_1)}$. Then $r^m M_1 = 0 \subseteq Q$, for some $m \in \mathbb{N}$ and $M_1 \nsubseteq Q$, hence $r \in P$.

Now suppose $t \in P$. Then $t^k M_1 \subseteq Q$, for some $k \in \mathbb{N}$, and $M_1 \nsubseteq Q$, then $t^k M_1 \neq M_1$. Therefore $tM_1 \neq M_1$, and as M_1 is a secondary module, $t \in \sqrt{\mathrm{Ann}(M_1)} = P_1$. Hence $P_1 = P$. □

Let S and S' be two P-secondary submodules of M. Then $S + S'$ is itself a P-secondary submodule. To see the proof, note that evidently $\sqrt{\mathrm{Ann}(S + S')} = \sqrt{\mathrm{Ann}(S) \cap \mathrm{Ann}(S')} = \sqrt{P} = P$. Now assume $r \in R \setminus P$. Then $rS = S$ and $rS' = S'$, and so $r(S + S') = S + S'$.

Corollary 2.8. *Let $M = \sum_{i \in I} M_i$ such that M_i is a P_i-secondary submodule of M for each $i \in I$.*

(i) $\langle E(B) \rangle = \sum_{i \in I} P_i M_i + E(B)$, *for each $B \leq M$.*

(ii) $\mathrm{rad}(B) = \sum_{i \in I} P_i M_i + E(B)$, *for each finitely generated submodule B of M. Particularly* $\mathrm{rad}(B) = \langle E(B) \rangle$.

Proof. (i) To see the assertion, follow the proof of (2.6).

(ii) Note that the sum of two P_i-secondary submodules is itself a P_i-secondary submodule. Hence we may suppose $P_i \neq P_j$, for each $i, j \in I$, $i \neq j$.

Since B is finitely generated, it is contained in a finite number of M_i's, let $B \leq M' = \sum_{j=1}^n M_{i_j}$. We show that:

$$\mathrm{rad}_M(B) \subseteq \mathrm{rad}_{M'}(B). \tag{$*$}$$

To prove this, it is enough to show that if N is a prime submodule of M' containing B, then N is a prime submodule of M.

Since $N \subset M'$, then $M_{i_k} \not\subseteq N$, for some $1 \leq k \leq n$. Then (2.7) implies that $(N : M) = P_{i_k}$. We prove that $M_i \subseteq N$, for each $i_k \neq i \in I$.

If for some $i_k \neq i \in I$, $M_i \not\subseteq N$, then again by (2.7), $P_i = (N : M) = P_{i_k}$, which is a contradiction. Therefore for each $i_k \neq i \in I$, $M_i \subseteq N$.

To prove that N is a prime submodule of M, let $ra \in N$, where $a \in M$ and $r \in R$. Suppose $a = a_{i_k} + \sum_{i=1, i \neq i_k}^{m} a_i$, where $a_{i_k} \in M_{i_k}$ and $a_i \in M_i$, for each $1 \leq i \leq m, i \neq i_k$. Then $ra_{i_k} \in N$ and since N is a prime submodule of M', we have $a_{i_k} \in N$ or $rM_{i_k} \subseteq N$. This implies that $a \in N$ or $rM \subseteq N$.

Now from $(*)$ and (2.6), we get:

$$\mathrm{rad}_M(B) \subseteq \sum_{j=1}^{n} P_{i_j} M_{i_j} + E_{M'}(B) \subseteq \sum_{i \in I} P_i M_i + E_M(B) \subseteq \langle E_M(B) \rangle \subseteq \mathrm{rad}_M(B).$$

□

3. Radicals of primary submodules

In this section, we will try to find some formulas for primary submodules of some particular modules. Also we establish the conditions under which the radical of a primary submodule Q of a module M is a prime submodule, if $\mathrm{rad}(Q) \neq M$. This subject has been noticed in [6], [17], [20].

Definition 3. Let M be an R-module. If for any primary submodule Q of M, $\mathrm{rad}(Q) = M$ or $\mathrm{rad}(Q)$ is a prime submodule of M, then we say that for M radical of primary submodules are prime submodules (for M r.p.a.p.).

If for every R-module r.p.a.p., then we say that for the ring R r.p.a.p.

According to [21] an R-module M is called *special*, if for each maximal ideal $\mathfrak{M}$ of R, each $a \in \mathfrak{M}$ and each $m \in M$, there exist $c \in R \setminus \mathfrak{M}$ and $k \in \mathbb{N}$ such that $ca^k m = 0$. *Semi-simple modules* (direct sum of simple modules), *locally Artinian modules* (modules in which every cyclic submodule is Artinian) and *semi-Artinian modules* (modules of which every homomorphic image has a nonzero simple submodule) are special (see [21, Section 3]).

The following lemma is the main result of [6].

Lemma 3.1.

(1) *If one of the following conditions is satisfied for a ring R, then for the ring R r.p.a.p.*

R is a ZPI-ring, $\dim R = 0$, R is an almost multiplication ring, an arithmetical ring with locally ACC on principal ideals, or a ring with DCC on principal ideals.

(2) *If M is a special module, a secondary representable module, a module with DCC on cyclic submodules, or a module with DCC on the submodules of the form $\{r^k M \mid k \in \mathbb{N}\}$ for each $r \in R$, then for the module M r.p.a.p.*

In this section, we will generalize the results of [6]. In (3.8) of this paper, we study when for $M = M_1 \oplus M_2$ r.p.a.p., where for M_1 r.p.a.p. Indeed the modules introduced in (3.1)(2) are some particular cases of (3.8), where $M_1 = 0$.

Compare the following result with (2.5)(iii) and Note 1.

Theorem 3.2. *Let Q be a P-primary submodule of an R-module M. Then $\operatorname{rad}(Q) = PM$, if M is a distributive or a multiplication module.*

Proof. Let $\mathfrak{M}$ be a maximal ideal of R. First we prove that $P_{\mathfrak{M}} = \sqrt{(Q_{\mathfrak{M}} : M_{\mathfrak{M}})}$.

If $P \subseteq \mathfrak{M}$, then by [3, Proposition 4.8(ii)], $Q_{\mathfrak{M}}$ is a $P_{\mathfrak{M}}$-primary submodule of $M_{\mathfrak{M}}$, and particularly $P_{\mathfrak{M}} = \sqrt{(Q_{\mathfrak{M}} : M_{\mathfrak{M}})}$. If $P \not\subseteq \mathfrak{M}$, then [3, Proposition 4.8(i)] implies that $Q_{\mathfrak{M}} = M_{\mathfrak{M}}$, and so $P_{\mathfrak{M}} = R_{\mathfrak{M}} = \sqrt{(Q_{\mathfrak{M}} : M_{\mathfrak{M}})}$.

Therefore $(\phi_3(Q, M))_{\mathfrak{M}} = (PM)_{\mathfrak{M}} = \sqrt{(Q_{\mathfrak{M}} : M_{\mathfrak{M}})} M_{\mathfrak{M}} = \phi_3(Q_{\mathfrak{M}}, M_{\mathfrak{M}})$. Thus $\phi_3|_{\mathcal{A}''}$ commutes with the localization, where

$$\mathcal{A}'' = \{(Q, M) \in \mathcal{A} \mid Q \text{ is primary }\}.$$

So by (2.1), we may suppose that R is a local ring.

Now assume that M is a distributive module. By [7, Theorem 2.16], every distributive module over a local ring is a serial module, then we can suppose that M is a serial module.

Consider $m \in M \setminus Q$. Then $Q \subseteq Rm$. Let $q \in Q$. Then $q = tm$ for some $t \in R$, and as Q is a P-primary submodule, $t \in P$. Thus $q \in PM$. So $Q \subseteq PM$, which implies that $Q \subseteq PM \subseteq \operatorname{rad}(Q)$, hence it is enough to show that PM is a prime submodule of M.

Let $s \in R$ and $x \in M \setminus PM$ such that $sx \in PM$. Then $sx = \sum_{i=1}^n a_i y_i$, where $a_i \in P$, $y_i \in M$ for each $1 \leq i \leq n$. Since every two submodules of M are comparable, we may suppose that $sx = ay$, where $a \in P$ and $y \in M$.

Let z be an arbitrary element of M and take $M_1 = Rx + Ry + Rz$.

Note that $Q \subseteq PM \subseteq Rx \subseteq M_1$. If $\operatorname{rad}_{M_1}(Q) \neq M_1$, then $\operatorname{rad}_{M_1}(Q)$ is a prime submodule of M_1, since $\operatorname{rad}_{M_1}(Q)$ is an intersection of a chain of prime submodules. Thus since $sx = ay \in PM_1 \subseteq \sqrt{(Q : M_1)} M_1 \subseteq \operatorname{rad}_{M_1}(Q)$, consequently $x \in \operatorname{rad}_{M_1}(Q)$ or $sM_1 \subseteq \operatorname{rad}_{M_1}(Q)$.

We show that:

$$\sqrt{(Q:M_1)}M_1 \subseteq PM. \tag{$*$}$$

For the proof, let $\alpha \in \sqrt{(Q:M_1)}M_1$. Then $\alpha = \sum_{i=1}^{k} t_i w_i$, where $t_i \in \sqrt{(Q:M_1)}$, $w_i \in M_1$ for each $1 \leq i \leq k$. Since every two submodules of M_1 are comparable, one can assume that $\alpha = tw$, where $t \in \sqrt{(Q:M_1)}$ and $w \in M_1$. Suppose that $t^\ell M_1 \subseteq Q$, where $\ell \in \mathbb{N}$. Then as $t^\ell w \in Q$, we have $w \in Q \subseteq PM$ or $t \in P$. Thus $\alpha = tw \in PM$.

If $x \in \operatorname{rad}_{M_1}(Q)$. As every two submodules of M are comparable, M_1 is cyclic, and so by (2.5) and $(*)$, $x \in \operatorname{rad}_{M_1}(Q) = \sqrt{(Q:M_1)}M_1 \subseteq PM$, which is a contradiction. Consequently $sM_1 \subseteq \operatorname{rad}_{M_1}(Q)$, then $sz \in \operatorname{rad}_{M_1}(Q) = \sqrt{(Q:M_1)}M_1 \subseteq PM$. Thus $sM \subseteq PM$. This completes the assertion, when M is a distributive module.

Now assume that M is a multiplication module. By (2.4), M is cyclic, and the proof is given by (2.5)(iii). □

Finitely generated distributive modules are characterized in the following corollary.

Corollary 3.3. *Let M be a finitely generated R-module. Then the following are equivalent:*

(i) *M is a distributive module;*

(ii) *M is a multiplication module and $R/\operatorname{Ann}(M)$ is an arithmetical ring.*

Proof. According to [7, Theorem 2.16], M is a distributive R-module if and only if $M_{\mathfrak{M}}$ is a serial module for every maximal ideal $\mathfrak{M}$ of R. $(*)$

If M is a multiplication module, then by (2.4), M is locally cyclic. Put $R' = R/\operatorname{Ann}(M)$. Then as M is a faithful multiplication R'-module, it follows that $M_{\mathfrak{M}} \cong R'_{\mathfrak{M}}$, for each maximal ideal $\mathfrak{M}$ of R' $(**)$

(i) $\Longrightarrow$ (ii) By (3.2) and (2.5)(iv), M is a multiplication R-module, and evidently a multiplication and distributive R'-module. Thus by $(*)$, $M_{\mathfrak{M}}$ is a serial module for every maximal ideal $\mathfrak{M}$ of R' and by $(**)$, $M_{\mathfrak{M}} \cong R'_{\mathfrak{M}}$. Hence R' is an arithmetical ring, by $(*)$.

(ii) $\Longrightarrow$ (i) By $(**)$, $M_{\mathfrak{M}} \cong R'_{\mathfrak{M}}$, for each maximal ideal $\mathfrak{M}$ of R. According to $(*)$, $R'_{\mathfrak{M}}$, is a valuation ring. Hence $M_{\mathfrak{M}}$ is a serial module, which implies that M is a distributive module, by $(*)$. □

Lemma 3.4. *Let M be an R-module and Q a P-primary submodule of M. If P is a maximal ideal, then $\operatorname{rad}(Q)$ is a prime submodule, if $\operatorname{rad}(Q) \neq M$.*

Proof. Note that $P \subseteq (\operatorname{rad}(Q):M)$. Now apply (2.2). □

The following is a generalization of some parts of (3.1)(2).

Proposition 3.5. *Let M be an R-module and Q a P-primary submodule of M. If one of the following holds, then P is a maximal ideal.*

(i) *M has DCC on the cyclic submodules of the form $\{Rr^k m \mid k \in \mathbb{N}\}$, for each $r \in R$ and $m \in M$.*

(ii) *R has DCC on the ideals of the form $\{Rr^k \mid k \in \mathbb{N}\}$, for each $r \in R$.*

(iii) *M is finitely generated and it has DCC on the submodules of the form $\{r^k M \mid k \in \mathbb{N}\}$, for each $r \in R$.*

Proof. Let $r \in R \setminus P$ and $m \in M \setminus Q$. We prove that $r^n(1-rs)m = 0$, for some $s \in R$ and $n \in \mathbb{N}$. Hence $r^n(1-rs)m \in Q$, which implies that $1 - rs \in P$. Thus $P + Rr = R$.

(i) Since the chain $\cdots \subseteq Rr^3 m \subseteq Rr^2 m \subseteq Rrm$ stops, there exists $n \in \mathbb{N}$ with $Rr^n m = Rr^{n+1} m$. Then $r^n(1-rs)m = 0$, for some $s \in R$.

(ii) Note that the chain $\cdots \subseteq Rr^3 \subseteq Rr^2 \subseteq Rr$ stops, then there exists a positive integer n with $Rr^n = Rr^{n+1}$. So $r^n(1-rs) = 0$, for some $s \in R$.

(iii) First suppose that $Q + rM = M$. Then as $r\frac{M}{Q} = \frac{M}{Q}$, Nakayama's lemma implies that there exists $t \in (Q : M)$ such that $t - 1 \in Rr$. Therefore $(Q : M) + Rr = R$, and so $P + Rr = R$.

Now assume $m \in M \setminus Q + rM$. The chain $\cdots \subseteq r^3 M \subseteq r^2 M \subseteq rM$ stops, so there exists a positive integer n with $r^n M = r^{n+1} M$, that is $r^n m = r^{n+1} m'$, for some $m' \in M$. Now since $r^n(m - rm') \in Q$ and $r \notin P$, $m - rm' \in Q$, which is impossible. $\square$

Corollary 3.6. *Let M be an R-module and Q a P-primary submodule of M. Then $\mathrm{rad}(Q) = Q + PM$, if one of the following holds:*

(i) *M has DCC on the cyclic submodules of the form $\{Rr^k m \mid k \in \mathbb{N}\}$, for each $r \in R$ and $m \in M$.*

(ii) *R has DCC on the ideals of the form $\{Rr^k \mid k \in \mathbb{N}\}$, for each $r \in R$.*

(iii) *M is finitely generated and it has DCC on the submodules of the form $\{r^k M \mid k \in \mathbb{N}\}$, for each $r \in R$.*

Proof. Evidently $Q + PM \subseteq \mathrm{rad}(Q)$ and hence if $Q + PM = M$, then $\mathrm{rad}(Q) = M$, which completes the assertion.

Now suppose that $Q + PM \neq M$. Clearly $P \subseteq (Q + PM : M)$, and by (3.5), P is a maximal ideal, then $(Q + PM : M)$ is a maximal ideal, which implies that $Q + PM$ is a prime submodule of M containing Q, by (2.2). Therefore $\mathrm{rad}(Q) = Q + PM$. $\square$

Proposition 3.7. *Let $M = M_1 \oplus M_2$, where M_1, M_2 are submodules of M such that for M_1 r.p.a.p. If Q is a primary submodule of M containing M_2, then $\mathrm{rad}_M(Q) = M$ or $\mathrm{rad}_M(Q)$ is a prime submodule of M.*

Proof. Let Q' be a proper submodule of M containing M_2. One can easily see that:

(i) Q' is a primary submodule of M if and only if $M_1 \cap Q'$ is a primary submodule of M_1.

(ii) Q' is a prime submodule of M if and only if $M_1 \cap Q'$ is a prime submodule of M_1. Now we prove that:

(iii) $\mathrm{rad}_{M_1}(M_1 \cap Q') = M_1 \cap (\mathrm{rad}_M(Q'))$.

By (ii) in above, $\mathrm{rad}_{M_1}(M_1 \cap Q') \subseteq M_1 \cap (\mathrm{rad}_M(Q'))$. Conversely assume that T_1 is an arbitrary prime submodule of M_1 containing $(M_1 \cap Q')$. Since $T_1 \subseteq M_1$, from the modular law we get $M_1 \cap (T_1 + Q') = T_1 + M_1 \cap Q' = T_1$. As $T_1 = M_1 \cap (T_1 + Q')$ is a prime submodule of M_1, by part (ii), $(T_1 + Q')$ is a prime submodule of M containing Q', and hence $M_1 \cap (\mathrm{rad}_M(Q')) \subseteq M_1 \cap (T_1 + Q') = T_1$. Consequently $\mathrm{rad}_{M_1}(M_1 \cap Q') = M_1 \cap (\mathrm{rad}_M(Q'))$.

Now for the proof of this proposition, let $r(m_1 + m_2) \in \mathrm{rad}_M(Q)$, where $r \in R$, $m_1 \in M_1$ and $m_2 \in M_2$. Note that $rm_2 \in M_2 \subseteq Q \subseteq \mathrm{rad}_M(Q)$. Then by (iii), $rm_1 \in \mathrm{rad}_{M_1}(M_1 \cap Q)$. According to (i), $(M_1 \cap Q)$ is a primary submodule of M_1 and for M_1 r.p.a.p, then $m_1 \in \mathrm{rad}_{M_1}(M_1 \cap Q)$, or $rM_1 \subseteq \mathrm{rad}_{M_1}(M_1 \cap Q)$. Thus $m_1 \in \mathrm{rad}_M(Q)$ or $rM_1 \subseteq \mathrm{rad}_M(Q)$, according to (iii). Now from $M_2 \subseteq \mathrm{rad}_M(Q)$, we get $m_1 + m_2 \in \mathrm{rad}_M(Q)$ or $rM \subseteq \mathrm{rad}_M(Q)$. □

In [21] a module M was called *generalized torsion divisible*, when $M = \sum_{i \in I} M_i$ for submodules M_i, such that for each $i \in I$, there exists a prime ideal $\mathfrak{P}_i$ of R such that $\mathfrak{P}_i M_i = 0$ and M_i is a torsion divisible $\frac{R}{\mathfrak{P}_i}$-module.

Theorem 3.8. *Let $M = M_1 \oplus M_2$, where M_1, M_2 are two R-modules such that for M_1 r.p.a.p. Then for M r.p.a.p., if one of the following holds:*

(i) *R is an integral domain and M_2 is a divisible R-module.*

(ii) *M_2 is a generalized torsion divisible R-module.*

(iii) *M_2 is a special R-module.*

(iv) *$M_2 = \sum_{i \in I} S_i$, where for each $i \in I$, S_i is a P_i-secondary submodule of M and $\sqrt{\mathrm{Ann}(M_1)} \nsubseteq P_i$ for each $i \in I$.*

(v) *M_2 has DCC on the cyclic submodules of the form $\{Rr^n m \mid n \in \mathbb{N}\}$, for each $r \in R$ and $m \in M_2$.*

(vi) *M_2 is finitely generated and it has DCC on the submodules of the form $\{r^n M_2 \mid n \in \mathbb{N}\}$, for each $r \in R$.*

(vii) *$M_2 = \sum_{i \in I} S_i$, where for each $i \in I$, S_i is a submodule of M_2 such that $\sqrt{\mathrm{Ann}(S_i)}$ is a maximal ideal of R, in particular when M_2 is semi-simple.*

(viii) *$\sqrt{\mathrm{Ann}(M_2)}$ is a finite intersection of maximal ideals.*

Proof. Let Q be a P-primary submodule of M such that $\mathrm{rad}(Q) \neq M$. If $0 \oplus M_2 \subseteq Q$, then according to (3.7), $\mathrm{rad}(Q)$ is a primary submodule of M. Also note that if P is a maximal ideal, then by (3.4), $\mathrm{rad}(Q)$ is a prime submodule.

(i) If $(Q : M) = 0$, then Q is a prime submodule of M, and obviously $\mathrm{rad}(Q) = Q$ is a prime submodule of M.

Now suppose that $0 \neq r \in (Q : M)$. Then $0 \oplus M_2 = 0 \oplus rM_2 \subseteq rM \subseteq Q$.

(ii) Assume that $M_2 = \sum_{i \in I} M_i$ is a sum of submodules M_i such that for each $i \in I$, there exists a prime ideal $P_i \subseteq \mathrm{Ann}(M_i)$ of R and M_i is a torsion divisible $\frac{R}{P_i}$-module.

We show that $0 \oplus M_2 \subseteq Q$. Let $i \in I$ and $x_i \in M_i$. Since M_i is a torsion $\frac{R}{P_i}$-module, there exists $r \in R \setminus P_i$ such that $r(0, x_i) = (0, 0) \in Q$. Hence $(0, x_i) \in Q$, or there exists $n \in \mathbb{N}$ with $r^n M \subseteq Q$. Therefore $(0, x_i) \in Q$, or $(0, x_i) \in 0 \oplus M_i = 0 \oplus r^n M_i \subseteq r^n M \subseteq Q$.

(iii) Let $\mathfrak{M}$ be a maximal ideal of R containing P. If $P = \mathfrak{M}$, then by (3.4), $\mathrm{rad}(Q)$ is a prime submodule of M.

Now suppose that $a \in \mathfrak{M} \setminus P$ and consider $m_2 \in M_2$. Then there exist a positive integer n, and an element $c \in R \setminus \mathfrak{M}$ such that $ca^n m_2 = 0$. So $ca^n(0, m_2) \in Q$, and so $(0, m_2) \in Q$, hence $0 \oplus M_2 \subseteq Q$.

(iv) Since the sum of two P_i-secondary submodules is P_i-secondary, we may assume $P_i \neq P_j$, for each $i \neq j \in I$. Evidently one of the following two cases is satisfied:

Case 1. For each $i \in I$, $0 \oplus S_i \subseteq Q$.

Case 2. For some $i \in I$, $0 \oplus S_i \nsubseteq Q$.

If Case 1 holds, then since $0 \oplus M_2 \subseteq Q$, the result is given by (3.7).

Now suppose that Case 2 is satisfied. Assume that $0 \oplus S_1 \nsubseteq Q$. In this case we show that for each prime submodule N of M containing Q, $(N : M) = P_1$, and consequently $\mathrm{rad}(Q)$ is a prime submodule of M.

According to (2.7), $P_1 = P$. If for some $1 \neq j \in I$, $0 \oplus S_j \nsubseteq Q$, then again by (2.7), $P_j = P = P_1$, which is a contradiction.

Therefore for each $1 \neq j \in I$, $0 \oplus S_j \subseteq Q$. $(*)$

Now if $0 \oplus S_1 \nsubseteq N$ for all prime submodules N of M containing Q, then

(2.7) implies that $P_1 = P = (N : M)$. Thus $\mathrm{rad}(Q)$ is a prime submodule of M. Otherwise let $0 \oplus S_1 \subseteq N_0$ for some prime submodule N_0 of M containing Q.

Now from $(*)$ we get $0 \oplus M_2 \subseteq N_0$. $(**)$

According to our hypothesis, $\sqrt{\mathrm{Ann}(M_1)} \nsubseteq P_1$, then let $a \in \sqrt{\mathrm{Ann}(M_1)} \setminus P_1$. Thus $a^\ell(M_1 \oplus 0) \in Q$, for some $\ell \in \mathbb{N}$, and since $a^\ell \notin P_1 = P$, $M_1 \oplus 0 \subseteq Q \subseteq N_0$. Now from $(**)$ we have $M \subseteq N_0$, which is a contradiction.

(v) Let $(0, m_2) \in 0 \oplus M_2 \setminus Q$ and suppose $r \in R \setminus P$. Then there exists $k \in \mathbb{N}$ with $Rr^k m_2 = Rr^{k+1} m_2$. So there exists $s \in R$ with $r^k(1 - st)m_2 = 0$. Then $r^k(1 - st)(0, m_2) \in Q$, which implies that $P + Rr = R$, that is P is a maximal ideal of R.

(vi) Let $r \in R \setminus P$. There exists $k \in \mathbb{N}$ with $r^k M_2 = r^{n+1} M_2$. By Nakayama's lemma there exists $s \in Rr$ such that $(s-1)r^k M_2 = 0$. Then $(s-1)r^k(0 \oplus M_2) \in Q$, which implies that $s - 1 \in P$. Therefore $P + Rr = R$, that is P is a maximal ideal.

(vii) If $0 \oplus M_2 \nsubseteq Q$, then there exists $j \in I$ such that $0 \oplus S_j \nsubseteq Q$. Now let $r \in \sqrt{\mathrm{Ann}(S_j)}$. Then evidently for some $n \in \mathbb{N}$, $r^n(0 \oplus S_j) \subseteq Q$, and this implies that $r \in P$. Hence $\sqrt{\mathrm{Ann}(S_j)} \subseteq P$, and so P is a maximal ideal.

(viii) Let $\cap_{i=1}^n \mathfrak{M}_i = \sqrt{\mathrm{Ann}(M_2)}$, where $\mathfrak{M}_i$ is a maximal ideal for each $1 \leq i \leq n$. If for some $1 \leq j \leq n$, $\mathfrak{M}_j \subseteq P$, then P is a maximal ideal.

Now assume $r_i \in \mathfrak{M}_i \setminus P$, for each $1 \leq i \leq n$. Then $r = r_1 r_2 r_3 \dots r_n \in \sqrt{\mathrm{Ann}(M_2)} \setminus P$, and so for some $m \in \mathbb{N}$, $r^m(0 \oplus M_2) \subseteq Q$. This implies that $0 \oplus M_2 \subseteq Q$. □

Corollary 3.9. *Let $n \in \mathbb{N}$ and $M = \oplus_{i=1}^n M_i$, where for each $1 \leq i \leq n$, M_i is an R-module. Then for M r.p.a.p., if for each i, M_i is a quotient of a module introduced in one of* (i) *to* (viii) *of the previous theorem.*

Proof. Observe that, if for a module r.p.a.p., then for any quotient of that module r.p.a.p. □

Note 2. Let $M = \oplus_{i \in I} M_i$, where for each i, M_i is an R-module. If for M r.p.a.p., then for each M_i r.p.a.p., because each M_i is a quotient of M. However the converse is not true. For example let $M = R \oplus R$, where $R = \mathbb{Z}[x]$. Then by [6, p. 3, Note(d)] for the R-module M, the radical of primary submodules are not necessarily prime.

Proposition 3.10. *Let $R = \oplus_{i=1}^n R_i$, where each R_i is a ring. Then for the ring R r.p.a.p., if and only for each ring R_i, r.p.a.p.*

Proof. ($\Longleftarrow$) Let M be an R-module. For each $i \in I$, consider $M_i = e_i M$, where $e_i = \{\delta_{ij}\}_{j \in I}$. The proof follows from the following simple observations:

(1) M_i is an R_i-module, for each $1 \leq i \leq n$.

(2) For each submodule B of M, $B = \oplus_{i=1}^{n} B_i$, where $B_i = e_i B$, and B_i is a submodule of M_i. Also, B is a prime [resp. primary] submodule of the R-module M if and only if each B_i is a prime [resp. primary] submodule of the R_i-module M_i. Furthermore, $\mathrm{rad}(B) = \oplus_{i=1}^{n} \mathrm{rad}(B_i)$.

($\Longrightarrow$) Consider $i \in I$. Then $R_i \cong R/K$, for some ideal K of R. Now let M be an R/K-module. Then obviously M is an R-module by considering the natural epimorphism $R \longrightarrow R/K$. The proof is completed by the following evident facts:

(1) Prime [resp. primary] submodules of M as an R/K-module are exactly the prime [resp. primary] submodules of M as an R-module.

(2) $\mathrm{rad}_{R/K}(B) = \mathrm{rad}_R(B)$, for each submodule B of M. $\square$

Corollary 3.11. *Let $R = \oplus_{i=1}^{n} R_i$, where for each i, R_i is a ring. Then for the ring R r.p.a.p., if for each i, R_i is a quotient of a ring introduced in* (3.1)(1).

Acknowledgment. The authors would like to thank the referees of this paper for their valuable comments and suggestions.

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(Received April 21, 2011; revised December 19, 2011)