

## Variation problems and $E$ -valued horizontal harmonic forms on Finsler manifolds

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**Abstract.** This paper is mainly to find the variation backgrounds of strongly harmonic maps and strongly minimal immersions between Finsler manifolds, and obtain an equivalent statement of strongly harmonic map. First, an explicit example of non-trivial strongly minimal immersions is given. By using the vertical Laplacian, we introduce the notions of vertical mean value operator and vertical mean value section. We define the generalized energy functionals and the volume functionals, and prove that they are critical points for appropriate variations. Finally, we give the definition of horizontal harmonic  $p$ -forms with values in a vector bundle  $E$  via the horizontal Laplacian and derive the relation between an  $E$ -valued  $h$ -harmonic 1-form and a strongly harmonic map.

### 1. Preliminaries

In recent decades, Finsler geometry has developed rapidly. Studies on harmonic maps and minimal submanifolds have also made some progress ([1]–[8]). By using the Holmes-Thompson volume form, harmonic maps and minimal immersions between Finsler manifolds were introduced in [3], [5] and [6] respectively. A harmonic map between Finsler manifolds is also defined as the critical point of the energy functional. It is well known that a map between Riemannian manifolds is harmonic if and only if either its tension field vanishes or its differential is a harmonic 1-form with value in the pull-back tangent bundle. Generally, this

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result does not hold for Finsler manifolds. Therefore, a map between Finsler manifolds is called strongly harmonic if its tension field vanishes in [8]. Analogously, a minimal immersion in Finsler manifold is defined as the critical point of the volume functional. An isometric immersion is called strongly minimal if its mean curvature vector field vanishes.

There indeed exist some examples of non-trivial strongly minimal immersions and many examples of minimal immersions that are not strongly minimal (see Remark 2.4 and Example 2.6). So it is significant to study strongly harmonic maps and strongly minimal immersions. A natural and interesting problem is whether there are similar variation backgrounds for strongly harmonic maps and strongly minimal immersions.

The main purpose of the present paper is to find the variation backgrounds of strongly harmonic maps and strongly minimal immersions, and to derive the relation between a harmonic  $E$ -valued 1-form and a strongly harmonic map. It is well known that various kinds of Laplace operators play a very important role in differential geometry and physics, especially in the theory of harmonic integral and Böchner technique. The key point is to find a proper way to define the Laplace operator. Therefore, we first generalize and define some suitable differential operators on Finsler projective sphere bundle  $SM$ . By using the vertical Laplacian for functions on  $SM$ , notions of vertical mean value operator and vertical mean value section of a pull-back vector bundle over  $SM$  are introduced. Next we define the generalized energy functionals and the volume functionals such that strongly harmonic maps and strongly minimal immersions, including totally geodesic maps, are their critical points for appropriate variation vector fields respectively. The first variation formulae are calculated in a more straightforward way than former ones. Finally, we define a horizontal Laplacian for  $p$ -forms with values in the vector bundle  $E$  over  $SM$ . Using the horizontal Laplacian, we give the definition of an  $E$ -valued  $h$ -harmonic  $p$ -form, which is a horizontal harmonic  $p$ -form with value in the vector bundle  $E$ , and prove that a smooth map  $\phi$  from a Finsler manifold  $(M, F)$  to a Riemannian manifold  $(\widetilde{M}, \widetilde{F})$  is harmonic if and only if  $d\phi$  is an  $h$ -harmonic 1-form with value in  $\pi^*(\phi^*T\widetilde{M})$ .

## 2. Harmonic maps and minimal immersions

Let  $(M, F)$  be an  $n$ -dimensional smooth Finsler manifold. The natural projection  $\pi : TM \rightarrow M$  gives rise to the pull-back bundle  $\pi^*TM$  and its dual  $\pi^*T^*M$ . Let  $(x, y)$  be a point of  $TM$  with  $x \in M$ ,  $y \in T_xM$ , and let  $(x^i, y^i)$  be

the local coordinates on  $TM$  with  $y = y^i \partial / \partial x^i$ . We shall work on  $\tilde{TM} = TM \setminus \{0\}$  and rigidly use only objects that are invariant under positive rescaling in  $y$ , so that one may view them as objects on the projective sphere bundle  $SM$  using homogeneous coordinates. Denote by  $H$  the orthogonal complement of the vertical bundle  $V = \{X \in \mathcal{C}(T^*(\tilde{TM})) \mid \pi(X) = 0\}$  in  $T(\tilde{TM})$ , which is the horizontal bundle, and denote by  $H^* = \pi^* T^* M$  the horizontal subbundle of  $T^*(\tilde{TM})$ . We shall use the following convention of index ranges unless otherwise stated:

$$1 \leq i, j, \dots \leq n; \quad 1 \leq a, b, \dots \leq n-1; \quad \bar{i} = n+i;$$

$$1 \leq A, B, \dots \leq 2n-1; \quad 1 \leq \alpha, \beta, \dots \leq m.$$

The following quantities

$$g_{ij} := \frac{1}{2} [F^2]_{y^i y^j}, \quad A_{ijk} = \frac{F}{2} \left[ \frac{1}{2} F^2 \right]_{y^i y^j y^k}, \quad \eta_i = g^{jk} A_{ijk}, \quad (2.1)$$

are called the *fundamental tensor*, the *Cartan tensor* and the *Cartan form* respectively.

Set

$$\partial_i := \frac{\partial}{\partial x^i}, \quad \dot{\partial}_i := \frac{\partial}{\partial y^i}, \quad \delta_i := \partial_i - N_i^j \dot{\partial}_j, \quad \delta y^i = \frac{1}{F} (dy^i + N_j^i dx^j), \quad (2.2)$$

where  $N_j^i = \gamma_{jk}^i y^k - \frac{1}{F} A_{jk}^i \gamma_{pq}^k y^p y^q$  and  $\gamma_{jk}^i$  are the formal Christoffel symbols of the second kind for  $g_{ij}$ .  $\{\delta_i\}$  and  $\{F \dot{\partial}_i\}$  are the local adapted bases of  $H$  and  $V$  respectively, whose dual are  $\{dx^i\}$  and  $\{\delta y^i\}$ .

The *Hilbert form*  $\omega = [F]_{y^i} dx^i$  is a global section of the covector bundle  $\pi^* T^* M$  and its dual  $l = l^i \partial_i$ , with  $l^i = \frac{y^i}{F}$ , can be viewed as a global section of  $\pi^* TM$ .

Express  $X \in \mathcal{C}(T(\tilde{TM}))$ ,  $\psi \in \mathcal{C}(T^*(\tilde{TM}))$  as  $X = X^i \delta_i + X^{\bar{i}} \dot{\partial}_i$ ,  $\psi = \psi_i dx^i + \psi_{\bar{i}} \delta y^i$ . Then  $X \in \mathcal{C}(TSM)$ ,  $\psi \in \mathcal{C}(T^* SM)$  if and only if  $X^{\bar{i}} F_{y^i} = 0$ ,  $\psi_{\bar{i}} y^i = 0$ . Denote by  $X^H = X^i \delta_i$  and  $\psi^H = \psi_i dx^i$  the horizontal parts of  $X$  and  $\psi$  respectively, and by  $X^\perp = X^{\bar{i}} \dot{\partial}_i$  and  $\psi^\perp = \psi_{\bar{i}} \delta y^i$  the vertical parts.

Each fibre of  $\pi^* T^* M$  has a positively oriented orthonormal coframe  $\{\omega^i\}$  with  $\omega^n = \omega$ . Expand  $\omega^i$  as  $v_j^i dx^j$ , whereby the stipulated orientation implies that  $\det(v_j^i) = \sqrt{\det(g_{ij})}$ .

The pull-back of the Sasaki metric from  $TM \setminus \{0\}$  to  $SM$  is a Riemannian metric

$$\hat{g} := g_{ij} dx^i \otimes dx^j + \delta_{ab} \omega^{\bar{a}} \otimes \omega^{\bar{b}} = \delta_{AB} \omega^A \otimes \omega^B,$$

where  $\omega^{n+a} = v_j^a \delta y^j$ . The collection  $\{\omega^A\}$  is an ordered orthonormal coframe on  $SM$ . Thus, the volume element  $dV_{SM}$  of  $SM$  can be defined as

$$dV_{SM} = \omega^1 \wedge \cdots \wedge \omega^n \wedge \omega^{n+1} \wedge \cdots \wedge \omega^{2n-1} = \Omega dx \wedge d\tau,$$

where

$$\begin{aligned} \Omega &:= \det \left( \frac{g_{ij}}{F} \right), \quad dx = dx^1 \wedge \cdots \wedge dx^n, \\ d\tau &:= \sum_{i=1}^n (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n. \end{aligned} \tag{2.3}$$

The *volume form* of a Finsler  $n$ -manifold  $(M, F)$  is defined by

$$dV_M := \sigma(x) dx, \quad \sigma(x) := \frac{1}{c_{n-1}} \int_{S_x M} \Omega d\tau, \tag{2.4}$$

where  $c_{n-1}$  denotes the volume of the unit Euclidean  $(n - 1)$ -sphere  $S^{n-1}$ ,  $S_x M$  is the fibre of  $SM$  at point  $x$ .

It is well known that there exists uniquely the Chern connection  $\nabla$  on  $\pi^*TM$  with  $\nabla \frac{\partial}{\partial x^j} = \omega_j^i \frac{\partial}{\partial x^i}$  and  $\omega_j^i = \Gamma_{jk}^i dx^k$ . Another torsion-free Berwald connection  ${}^b\nabla$  is defined by

$${}^b\nabla = \nabla + \dot{A}, \quad B_{jk}^i = \Gamma_{jk}^i + \dot{A}_{jk}^i, \quad {}^b\omega_j^i = B_{jk}^i dx^k, \tag{2.5}$$

where “ $\cdot$ ” denotes the covariant derivative along the Hilbert form.

**Lemma 2.1** ([3], [7]). *For  $\Psi = \Psi_i dx^i + \Psi_{\bar{i}} \delta y^i \in \mathcal{C}(T^*SM)$ ,  $X = X^i \delta_i + X^{\bar{i}} \dot{\delta}_i \in \mathcal{C}(TSM)$  and  $f \in C^\infty(SM)$ , we have the following*

$$\operatorname{div}_{\hat{g}} \Psi = g^{ij} \left[ \Psi_{i|j} - \Psi_k \dot{A}_{ij}^k + \Psi_{\bar{i};j} \right] = g^{ij} \left[ \Psi_{i,j} + \Psi_{\bar{i};j} \right], \tag{2.6}$$

$$\operatorname{div}_{\hat{g}} X = X_{|i}^i - X^i \eta_i + X_{\bar{i}}^{\bar{i}} + 2X^{\bar{i}} \eta_i, \tag{2.7}$$

$$\Delta_{\hat{g}} f = \operatorname{div}_{\hat{g}}(df) = g^{ij} \left[ \delta_i \delta_j f - \delta_k f B_{ij}^k + F^2 \dot{\delta}_i \dot{\delta}_j f \right], \tag{2.8}$$

where “ $|$ ” and “ $|$ ” denote the horizontal covariant differentials with respect to the Berwald connection  ${}^b\nabla$  and the Chern connection  $\nabla$  respectively, and “ $;$ ” denotes the vertical covariant differential, i.e.  $(\cdot)_{;i} = F \dot{\delta}_i(\cdot)$ .

**Lemma 2.2** ([7], [9]). *Let  $(M, F)$  be a Finsler manifold. Then*

$$\int_{S_x M} g^{ij} \dot{\delta}_i \dot{\delta}_j [F^2 f] h \Omega d\tau = \int_{S_x M} f g^{ij} \dot{\delta}_i \dot{\delta}_j [F^2 h] \Omega d\tau, \tag{2.9}$$

for all smooth functions  $f, h$  on  $SM$ . In particular,

$$\int_{S_x M} g^{ij} \dot{\delta}_i \dot{\delta}_j [F^2 f] \Omega d\tau = 2n \int_{S_x M} f \Omega d\tau. \tag{2.10}$$

**Lemma 2.3** ([7]). *Let  $(M, F)$  be a Finsler manifold. Then*

$$\int_{SM} l^H(f) dV_{SM} = 0, \quad (2.11)$$

for any compactly supported function  $f \in C_0^\infty(SM)$ .

Let  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  be a non-degenerate smooth map, i.e.  $\ker d\phi = 0$ , and let  $\phi_t : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$ ,  $t \in (-\varepsilon, \varepsilon)$ , be a smooth variation of  $\phi$  with  $\phi_0 = \phi$  and  $\phi_t|_{\partial M} = \phi|_{\partial M}$ . Then  $V := \frac{\partial \phi_t}{\partial t}|_{t=0} \in \mathcal{C}(\phi^{-1}T\widetilde{M})$  is the variation vector field of  $\phi$ . The energy functional for  $\phi$  can be written as

$$\begin{aligned} E(\phi) &= \frac{1}{c_{n-1}} \int_{SM} \frac{1}{2} g^{ij} \tilde{g}_{\alpha\beta} \phi_i^\alpha \phi_j^\beta dV_{SM} \\ &= \frac{n}{2c_{n-1}} \int_{SM} \frac{\tilde{F}^2(\phi(x), d\phi y)}{F^2} dV_{SM} = \frac{n}{2c_{n-1}} \int_{SM} \|d\phi l\|_{\tilde{g}}^2 dV_{SM}. \end{aligned} \quad (2.12)$$

The first variation of the energy functional for  $\phi$  is given by [5], [7]

$$\frac{d}{dt} E(\phi_t)|_{t=0} = - \int_M \mu_\phi(V) dV_M, \quad (2.13)$$

where

$$\mu_\phi(V) = \frac{1}{c_{n-1}\sigma} \int_{S_x M} \langle \tau(\phi), V \rangle_{\tilde{g}} \Omega d\tau = \frac{n}{c_{n-1}\sigma} \int_{S_x M} \langle \tilde{\tau}(\phi), V \rangle_{\tilde{g}} \Omega d\tau, \quad (2.14)$$

$$\tilde{\tau}(\phi) := (\tilde{\nabla}_l d\phi)(l) = \frac{1}{F^2} \tilde{\tau}^\alpha \tilde{\partial}_\alpha, \quad \tilde{\tau}^\alpha = \phi_{ij}^\alpha y^i y^j - \phi_k^\alpha G^k + \tilde{G}^\alpha,$$

$$\tau(\phi) := \tau^\alpha \tilde{\partial}_\alpha, \quad \tau^\alpha = \frac{1}{2} g^{ij} \partial_i \partial_j [\tilde{\tau}_\beta] \tilde{g}^{\alpha\beta}, \quad (2.15)$$

here we have set  $\tilde{\partial}_\alpha = \frac{\partial}{\partial \tilde{x}^\alpha}$  for convenience,  $G^k$  and  $\tilde{G}^\alpha$  are the geodesic coefficients for  $(M, F)$  and  $(\widetilde{M}, \widetilde{F})$  respectively.  $\tau(\phi)$  is called the *tension field* and  $\mu_\phi$  is called the *tension form* of  $\phi$ . A harmonic map is naturally defined as the critical point of the energy functional. By (2.13),  $\phi$  is harmonic if and only if  $\mu_\phi = 0$ . In particular,  $\phi$  is called *strongly harmonic* if  $\tau(\phi) = 0$ .

Let  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  be an isometric immersion, that is,  $F(x, y) = \widetilde{F}(\phi(x), d\phi(y))$  for all  $(x, y) \in TM \setminus \{0\}$ . Then  $g_{ij}(x, y) = \tilde{g}_{\alpha\beta}(\tilde{x}, \tilde{y}) \phi_i^\alpha \phi_j^\beta$ . Denote by  $(\pi^*TM)^\perp$  the orthogonal complement of  $\pi^*TM$  in  $\pi^*(\phi^*T\widetilde{M})$  with respect to  $\tilde{g}$  and denote

$$\mathcal{V}^* = \{\xi \in \mathcal{C}(\pi^*(\phi^*T\widetilde{M})) \mid \xi(d\phi X) = 0, \forall X \in \mathcal{C}(TM)\},$$

which is called the *normalbundle* of  $\phi$  [1]. Setting

$$h = \tilde{\tau}(\phi), \quad H = \frac{1}{n}\tau(\phi), \quad \mu = \mu_\phi, \tag{2.16}$$

we see that  $h, H \in (\pi^*TM)^\perp$  and  $\mu \in \mathcal{V}^*$ . An isometric immersion  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  is called *minimal* if on any compact domain of  $M$ ,  $\phi$  is the critical point of its volume functional with respect to any variation vector field in  $\pi^*(\phi^*T\widetilde{M})$ . It has been shown in [6] that  $\phi$  is minimal if and only if  $\mu = 0$ .  $h, H$  and  $\mu$  are called the *normal curvature*, the *mean curvature normal vector field* and the *mean curvature form* of  $\phi$  respectively [1], [6]. Analogously,  $\phi$  is called *strongly minimal* if  $H = 0$ .

*Remark 2.4.* By (2.14), one can easily see that a strongly harmonic map (resp. strongly minimal immersion) must be harmonic (resp. minimal). In general, a harmonic map (resp. minimal immersion) is not necessarily strongly harmonic (resp. strongly minimal). It has been shown that in Randers space  $(\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$ , a submanifold  $(M, \alpha + \beta)$  is minimal if and only if  $(M, \alpha)$  is minimal in Riemannian manifold  $(\widetilde{M}, \tilde{\alpha})$ , [10] but it is not necessarily strongly minimal.

From (2.15), (2.16) and [15], a straightforward calculation gives the following

**Lemma 2.5.** *Let  $f : (M, \alpha + \beta) \rightarrow (\widetilde{M}, \tilde{\alpha} + \tilde{\beta})$  be an isometric immersion into a Randers  $(n + p)$ -space. If  $\tilde{\beta}$  is a closed 1-form, then*

$$h = \frac{\alpha^2}{F^2}[\bar{h} - \tilde{\beta}(\bar{h})\tilde{l}], \quad H = \frac{\alpha}{F}(\bar{H} - \tilde{\beta}(\bar{H})\tilde{l}) - \frac{(n + 1)\alpha\beta}{2nF^2}(\bar{h} - \tilde{\beta}(\bar{h})\tilde{l}), \tag{2.17}$$

where  $\bar{h}$  and  $\bar{H}$  are the normal curvature and the mean curvature normal vector field with respect to the Riemannian metric  $\tilde{\alpha}$ , respectively.

*Example 2.6.* Let  $(V^3, \tilde{F})$  be a Randers space with  $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ , where

$$\tilde{\alpha} = \sqrt{\sum_{\alpha} (\tilde{y}^\alpha)^2}, \quad \tilde{\beta} = \frac{\tilde{x}^2 d\tilde{x}^1 - \tilde{x}^1 d\tilde{x}^2}{(\tilde{x}^1)^2 + (\tilde{x}^2)^2} + d\tilde{x}^3. \tag{2.18}$$

Then  $d\tilde{\beta} = 0$ .

Let  $M$  be a helicoid defined by

$$f(u, v) = \{u \cos v, u \sin v, av\}, \tag{2.19}$$

where  $a \neq 0$  is a constant, and let  $F = f^*\tilde{F} = \alpha + \beta$  be the Randers metric on  $M$ . Then

$$\bar{H} = 0, \quad \bar{h} = \frac{-2ay^1y^2}{\alpha^2(u^2 + a^2)}(a \sin v, -a \cos v, u).$$

From (2.17), we have

$$H = \frac{(n+1)\alpha(1-a)y^2}{2nF^2}(\bar{h} - \tilde{\beta}(\bar{h})\tilde{\ell}).$$

It is obvious that  $H = 0$  if and only if  $a = 1$ . That is,  $M$  is a strongly minimal surface of  $(V^3, \tilde{F})$  if and only if  $a = 1$ . So  $M$  is minimal but not strongly minimal when  $a \neq 1$ .

### 3. The vertical mean value operator and the vertical mean value section

In Lemma 2.1, we set

$$\Delta^H f = \operatorname{div}_{\hat{g}}(df)^H = g^{ij} f_{|i,j}, \quad \Delta^\perp f = \operatorname{div}_{\hat{g}}(df)^\perp = F^2 g^{ij} \dot{\partial}_i \dot{\partial}_j f, \quad (3.1)$$

which are called the *horizontal Laplacian* and the *vertical Laplacian* for functions on  $SM$  respectively [11]. Obviously, we have  $\Delta_{\hat{g}} = \Delta^H + \Delta^\perp$  and

$$\int_{SM} \Delta^H f dV_{SM} = \int_{SM} \Delta^\perp f dV_{SM} = 0, \quad (3.2)$$

for any compactly supported function  $f \in C_0^\infty(SM)$ . In fact, we assert from Lemma 2.2 that the condition that  $f$  for  $\Delta^\perp$  is compactly supported is unnecessary, i.e.,

**Lemma 3.1.** *Let  $(M, F)$  be a Finsler manifold,  $f \in C^\infty(SM)$ . Then*

$$\int_{SM} \Delta^\perp f dV_{SM} = 0. \quad (3.3)$$

*Definition 3.2.* The map  $\mu^\perp : C^\infty(SM) \rightarrow C^\infty(SM)$  defined by  $\mu^\perp f = \frac{1}{2n} g^{ij} [F^2 f]_{y^i y^j}$  for any  $f \in C^\infty(SM)$  is called the *vertical mean value operator*.

A straightforward calculation gives that

$$g^{ij} \dot{\partial}_i \dot{\partial}_j [F^2 f] = 2nf + F^2 g^{ij} \dot{\partial}_i \dot{\partial}_j f.$$

**Lemma 3.3.**

$$\mu^\perp = \frac{1}{2n} \Delta^\perp + \operatorname{id}_{C^\infty(SM)}. \quad (3.4)$$

Set

$$\mathcal{M} = \mu^\perp[C^\infty(SM)], \quad \mathcal{M}_0 = \{f \in C^\infty(SM) \mid \mu^\perp f = f\} = \ker \Delta^\perp. \quad (3.5)$$

It follows from Lemma 3.1 that  $\Delta^\perp f = 0$  if and only if  $f$  is a vertical constant, i.e.,

$$\mathcal{M}_0 = \pi^*C^\infty(M) \subset \mathcal{M}. \quad (3.6)$$

Assume that  $M$  is compact. Then from Lemma 2.2, one obtains

$$(\mu^\perp f, h) = \int_{SM} \mu^\perp(f)h dV_{SM} = \int_{SM} f\mu^\perp(h) dV_{SM} = (f, \mu^\perp h). \quad (3.7)$$

**Lemma 3.4.** *The operator  $\mu^\perp$  is selfadjoint on  $C^\infty(SM)$ .*

From the above lemma, we see that  $(f, \mu^\perp h) = (\mu^\perp f, h) = 0$  always holds for any  $f \in \ker \mu^\perp$  and  $h \in \mathcal{C}(SM)$ . Thus, we have the following proposition by (3.4):  $\ker \mu^\perp$  is just the eigenspace of  $\Delta^\perp$  for eigenvalue  $-2n$ , and orthogonal to  $\mathcal{M}$  with respect to the global inner product. Thus  $\mathcal{M} \neq C^\infty(SM)$  if  $\ker \mu^\perp \neq \{0\}$ .

*Remark 3.5.* Generally,  $\ker \mu^\perp \neq \{0\}$ . For example, set  $f = a_{ij}l^i l^j$ , where  $a_{ij}$  are independent of  $y$ , satisfying  $g^{ij}a_{ij} = 0$ . Then  $f \in C^\infty(SM)$  and  $\mu^\perp f = 0$ .

Let  $\xi : E \rightarrow M$  be a smooth vector bundle over  $M$ ,  $\pi^*E$  be the pull-back bundle over  $SM$  and  $\{E_\alpha\}$  be a local frame field of  $E$ . We still denote by  $\{E_\alpha\}$  the lifts of  $\{E_\alpha\}$  to  $SM$ . Moreover, we can also define the vertical mean value operator  $\mu^\perp : \mathcal{C}(\pi^*E) \rightarrow \mathcal{C}(\pi^*E)$  by

$$\mu^\perp \tilde{X} = \mu^\perp(\tilde{X}^\alpha)E_\alpha, \quad \text{for } \tilde{X} = \tilde{X}^\alpha E_\alpha \in \mathcal{C}(\pi^*E). \quad (3.8)$$

It is easily to check that the definition above is independent of our choice of  $\{E_\alpha\}$ .

From Lemma 2.2, we have the following proposition: *Let  $(M, F)$  be a Finsler manifold and  $\tilde{X} \in \mathcal{C}(\pi^*E), \tilde{\psi} \in \mathcal{C}(\pi^*E^*)$ . Then*

$$\int_{S\mathcal{D}} (\mu^\perp \tilde{\psi})(\tilde{X}) dV_{S\mathcal{D}} = \int_{S\mathcal{D}} \tilde{\psi}(\mu^\perp \tilde{X}) dV_{S\mathcal{D}}, \quad (3.9)$$

for any compact domain  $\mathcal{D}$  on  $M$ .

Similarly, we denote the image set and the set of fixed points of  $\mu^\perp$  on  $\mathcal{C}(\pi^*E)$  by

$$\mathcal{M}(E) = \mu^\perp[\mathcal{C}(\pi^*E)], \quad \mathcal{M}_0(E) = \{\tilde{X} \in \mathcal{C}(\pi^*E) \mid \mu^\perp \tilde{X} = \tilde{X}\} = \pi^*\mathcal{C}(E), \quad (3.10)$$

respectively. The vector fields in  $\mathcal{M}(E)$  are called the *vertical mean value sections* of  $\pi^*E$ . Clearly,

$$\mathcal{M}_0(E) \subset \mathcal{M}(E) \subset \mathcal{C}(\pi^*E). \quad (3.11)$$

Denote by  $\tilde{\tau}^*(\phi)$  and  $\tau^*(\phi)$  the dual forms of  $\tilde{\tau}(\phi)$  and  $\tau(\phi)$  respectively. Then (2.15) and (2.16) show that  $\mu^\perp \tilde{\tau}^*(\phi) = \frac{1}{n} \tau^*(\phi)$ ,  $\mu^\perp h^* = H^*$ , i.e.  $\tilde{\tau}^*(\phi)$  and  $H^*$  are vertical mean value sections of  $\pi^*(\phi^*T^*M)$ .



**4. Variation backgrounds of strongly harmonic maps and totally geodesic maps**

Let  $(M, F)$  be a compact Finsler manifold and  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  be a non-degenerate smooth map. Denote  $\psi = \pi^* \phi : SM \rightarrow \widetilde{SM}, (x, y) \mapsto \phi(x)$  and consider the variation  $\psi_t$  of  $\psi$ , defined by

$$\psi_0(x, y) = \phi(x), \quad \psi_t(x, y)|_{x \in \partial M} = \phi|_{\partial M}, \tag{4.1}$$

where  $\psi_t(x, y)$  are homogeneous of degree zero with respect to  $y$ . Then  $\psi_t(x, y)$  induces a variation vector field  $V(x, y)$  as follows

$$V(x, y) := \frac{\partial \psi_t}{\partial t} \Big|_{t=0} = V^\alpha \tilde{\partial}_\alpha, \quad V(x, y)|_{x \in \partial M} = 0. \tag{4.2}$$

which is well defined. For example,  $\psi_t(x, y)$  can be defined as

$$\psi_t^\alpha(x, y) = \tilde{x}^\alpha(x, y, t) = \phi^\alpha(x) + tV^\alpha(x, y)$$

under the local coordinates  $(x^i, y^j), (\tilde{x}^\alpha, \tilde{y}^\alpha)$ . Defining  $\tilde{\phi}_t : SM \rightarrow \widetilde{SM}$  by

$$\tilde{\phi}_t(x, y) = (\psi_t(x, y), d\psi_t(y^H)), \tag{4.3}$$

and setting  $\tilde{F}_t = \tilde{\phi}_t^* \tilde{F}$ , we have

$$\tilde{F}_t(x, y) = \tilde{F}(\psi_t(x, y), d\psi_t(y^H)) = \|d\psi_t(y^H)\|_{\tilde{g}}.$$

The *generalized energy functional* for  $\psi_t$  is defined by

$$E(\psi_t) = \frac{n}{2c_{n-1}} \int_{SM} \frac{\tilde{F}_t^2}{F^2} dV_{SM} = \frac{n}{2c_{n-1}} \int_{SM} \|d\psi_t l^H\|_{\tilde{g}}^2 dV_{SM}. \tag{4.4}$$

Using (2.12), one obtains  $E(\psi_t)|_{t=0} = E(\phi)$  and

$$\frac{d}{dt} E(\psi_t) \Big|_{t=0} = \frac{n}{2c_{n-1}} \int_{SM} \frac{\partial}{\partial t} \frac{\tilde{F}_t^2}{F^2} \Big|_{t=0} dV_{SM}. \tag{4.5}$$

Denoting  $\tilde{F}_0 = \phi^* \tilde{F}$  by  $\tilde{F}$ , one obtains from (4.2) that

$$\begin{aligned} \frac{d}{dt} \frac{\tilde{F}_t^2}{F^2} \Big|_{t=0} &= \frac{2\tilde{F}}{F^2} [\tilde{F}_{\tilde{x}^\alpha} V^\alpha + \tilde{F}_{\tilde{y}^\alpha} y^H(V^\alpha)] = \frac{2\tilde{F}}{F^2} [\tilde{N}_\beta^\alpha \tilde{F}_{\tilde{y}^\alpha} V^\beta + \tilde{F}_{\tilde{y}^\alpha} y^H(V^\alpha)] \\ &= 2\tilde{g}(d\phi l, \tilde{\nabla}_{l^H} V) = 2l^H(\tilde{g}(d\phi l, V)) - 2(\tilde{\nabla}_{l^H} \tilde{g})(d\phi l, V) - 2\tilde{g}(\tilde{\nabla}_{l^H} d\phi l, V) \\ &= 2l^H(\tilde{g}(d\phi l, V)) - 2\tilde{g}(\tilde{\tau}(\phi), V). \end{aligned} \tag{4.6}$$

Combining the last two statements with Lemma 2.3, we have the following

**Theorem 4.1.** *Let  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  be a non-degenerate smooth map and  $\psi_t$  be a smooth variation satisfying (4.1) and (4.2). Then the first variation formula of the generalized energy functional is given by*

$$\frac{d}{dt}E(\psi_t)|_{t=0} = -\frac{n}{c_{n-1}} \int_{SM} \tilde{g}(\tilde{\tau}(\phi), V) dV_{SM}. \tag{4.7}$$

Firstly, if  $\frac{d}{dt}E(\psi_t)|_{t=0} = 0$  for any variation vector field  $V(x, y) \in \mathcal{C}(\pi^*(\phi^*T\widetilde{M}))$ , then  $\tilde{\tau}(\phi) = 0$ , i.e.  $\phi$  is totally geodesic. Secondly, if  $\frac{d}{dt}E(\psi_t)|_{t=0} = 0$  for any variation vector field  $V(x, y) \in \mathcal{M}(\phi^*T\widetilde{M})$ , i.e. for any  $W(x, y) \in \mathcal{C}(\pi^*(\phi^*T\widetilde{M}))$  and  $V(x, y) = \mu^\perp W(x, y)$ , then from (4.7) and (2.15), we have

$$\begin{aligned} 0 &= \frac{d}{dt}E(\psi_t)|_{t=0} = -\frac{n}{c_{n-1}} \int_{SM} \tilde{\tau}^*(\phi)(\mu^\perp W) dV_{SM} \\ &= -\frac{n}{c_{n-1}} \int_{SM} \mu^\perp \tilde{\tau}^*(\phi)(W) dV_{SM} = -\frac{1}{c_{n-1}} \int_{SM} \tau^*(\phi)(W) dV_{SM}. \end{aligned}$$

Hence  $\tau(\phi) = 0$ , which implies that  $\phi$  is strongly harmonic. Moreover, if  $\frac{d}{dt}E(\psi_t)|_{t=0} = 0$  for any variation vector field  $V(x, y) \in \mathcal{M}_0(\phi^*T\widetilde{M})$ , i.e. for any  $V(x, y) = V(x) \in \mathcal{C}(\phi^*T\widetilde{M})$ , then from (2.14) we have  $\mu_\phi = 0$ , which implies that  $\phi$  is harmonic.

**Theorem 4.2.** *Let  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  be a non-degenerate smooth map and  $\psi_t$  be a smooth variation satisfying (4.1) and (4.2). Then  $\phi$  is strongly harmonic if and only if it is the critical point of the generalized energy functional with respect to any variation vector field  $V(x, y) \in \mathcal{M}(\phi^*T\widetilde{M})$ . Furthermore,  $\phi$  is totally geodesic if and only if it is the critical point of the generalized energy functional with respect to any variation vector field  $V(x, y) \in \mathcal{C}(\pi^*(\phi^*T\widetilde{M}))$ .*

### 5. Variation backgrounds of strongly minimal immersions

Let  $(M, F)$  be a compact Finsler manifold and  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  be an isometric immersion. Then  $\phi$  induces a map  $\tilde{\phi} : (SM, \hat{g}) \rightarrow (S\widetilde{M}, \hat{\tilde{g}})$  with  $(x, y) \mapsto (\phi(x), d\phi y)$ .

**Lemma 5.1.** *The immersion  $\tilde{\phi}$  is isometric if and only if  $\phi$  is totally geodesic.*

PROOF. Set  $\tilde{h}_{ij}^\alpha = \frac{1}{2}[h^\alpha]_{y^i y^j}$ . Since  $\phi^* \delta \tilde{y}^\alpha = \tilde{h}_{ik}^\alpha l^k dx^i + \phi_i^\alpha \delta y^i$ , we have

$$\begin{aligned} \tilde{\phi}^* \hat{\tilde{g}} &= \tilde{g}_{\alpha\beta}(\phi_i^\alpha dx^i) \otimes (\phi_j^\beta dx^j) \\ &\quad + (\tilde{g}_{\alpha\beta} - \tilde{F}_\alpha \tilde{F}_\beta)(h_{ik}^\alpha l^k dx^i + \phi_i^\alpha \delta y^i) \otimes (h_{js}^\beta l^s dx^j + \phi_j^\beta \delta y^j) \end{aligned}$$

$$\begin{aligned}
 &= g_{ij} dx^i \otimes dx^j + (\tilde{g}_{\alpha\beta} - \tilde{F}_\alpha \tilde{F}_\beta) [h_{ik}^\alpha l^k h_{js}^\beta l^s dx^i \otimes dx^j \\
 &\quad + h_{ik}^\alpha l^k \phi_j^\beta dx^i \otimes \delta y^j + h_{jk}^\beta l^k \phi_i^\alpha \delta y^i \otimes dx^i + \phi_i^\alpha \phi_j^\beta \delta y^i \otimes \delta y^j].
 \end{aligned}$$

It is easy to verify that  $\hat{g} = \tilde{\phi}^* \hat{g}$  if and only if  $h_{ik}^\alpha l^k = 0$ . Thus  $h = h_{ik}^\alpha l^i l^k = 0$ , i.e.  $\phi$  is totally geodesic.

Now assume that only  $\phi$  is isometric, while  $\tilde{\phi}$  is not necessarily so. We also consider the variation  $\psi_t$  of  $\psi = \pi^* \phi$ , which satisfies (4.1) and (4.2). Set

$$F_t(x, y) = \tilde{F}(\psi_t, d\psi_t(y^H)), \quad g_{t|ij} = \left[ \frac{1}{2} F_t^2 \right]_{y^i y^j}. \tag{5.1}$$

Note that  $g_t$  are positive definite for all  $t \in (-\varepsilon, \varepsilon)$ , since  $g_0 = g$  is positive definite. That is,  $\{F_t\}$  are all Finsler metrics. By (2.4), our volume functional for  $(M, F_t)$  takes the form

$$\tilde{V}(t) = \text{Vol}(M, F_t) = \frac{1}{c_{n-1}} \int_{SM} \Omega_t d\tau \wedge dx, \tag{5.2}$$

where  $\Omega_t = \det \left( \frac{1}{F_t} g_{t|ij} \right)$ ,  $\Omega_0 = \Omega$ . Hence

$$\tilde{V}'(0) = \frac{1}{c_{n-1}} \int_{SM} \left( \frac{\partial}{\partial t} \Omega_t \right) |_{t=0} d\tau \wedge dx. \tag{5.3}$$

(5.1) together with (2.3) and (2.4) yields

$$\begin{aligned}
 \left( \frac{\partial}{\partial t} \Omega_t \right) |_{t=0} &= \Omega_0 \left( g_t^{ij} \frac{\partial g_{t|ij}}{\partial t} - \frac{n}{F_t} \frac{\partial F_t}{\partial t} \right) |_{t=0} \\
 &= \Omega \left\{ g^{ij} \left[ \frac{1}{2} \frac{\partial}{\partial t} (F_t^2) \right]_{y^i y^j} - \frac{n}{F_t} \frac{\partial F_t}{\partial t} \right\} |_{t=0} \\
 &= n\Omega \left\{ 2\mu^\perp \left( \frac{1}{F} \frac{\partial F_t}{\partial t} \right) |_{t=0} - \frac{1}{F} \frac{\partial F_t}{\partial t} |_{t=0} \right\}. \tag{5.4}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{1}{F} \frac{\partial F_t}{\partial t} |_{t=0} &= \frac{1}{F} [\tilde{F}_{x^\alpha} V^\alpha + \tilde{F}_{y^\alpha} y^H(V^\alpha)] = \frac{1}{F} [\tilde{N}_\beta^\alpha \tilde{F}_{y^\alpha} V^\beta + \tilde{F}_{y^\alpha} y^H(V^\alpha)] \\
 &= \tilde{g}(d\phi l, \tilde{\nabla}_{l^H} V) = l^H(\tilde{g}(d\phi l, V)) - \tilde{g}(h, V). \tag{5.5}
 \end{aligned}$$

By (3.7) and the above formulas, one obtains

$$\begin{aligned}
 \tilde{V}'(0) &= \frac{2n}{c_{n-1}} \int_{SM} \mu^\perp \left( \frac{1}{F} \frac{\partial F_t}{\partial t} \right) |_{t=0} dV_{SM} - \frac{n}{c_{n-1}} \int_{SM} \frac{1}{F} \frac{\partial F_t}{\partial t} |_{t=0} dV_{SM} \\
 &= \frac{n}{c_{n-1}} \int_{SM} \{ l^H(\tilde{g}(d\phi l, V)) - \tilde{g}(h, V) \} dV_{SM}. \quad \square
 \end{aligned}$$

Therefore, by Lemma 2.3, we have the following

**Theorem 5.2.** *Let  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  be an isometric immersion and  $\psi_t$  be a smooth variation satisfying (4.1) and (4.2). Then the first variation formula of the volume functional is given by*

$$\tilde{V}'(0) = -\frac{n}{c_{n-1}} \int_{SM} \tilde{g}(h, V) dV_{SM}. \tag{5.6}$$

Firstly, if  $\tilde{V}'(0) = 0$  for any variation vector field  $V(x, y) \in \mathcal{C}(\pi^*(\phi^*T\widetilde{M}))$ , then the normal curvature  $h = 0$ , i.e.  $(M, F)$  is totally geodesic. Secondly, if  $\tilde{V}'(0) = 0$  for any variation vector field  $V(x, y) \in \mathcal{M}(\phi^*T\widetilde{M})$ , i.e. for any  $W(x, y) \in \mathcal{C}(\pi^*(\phi^*T\widetilde{M}))$  and  $V(x, y) = \mu^\perp W(x, y)$ , then from (3.9) and (2.16), we have

$$\begin{aligned} 0 = \tilde{V}'(0) &= -\frac{n}{c_{n-1}} \int_{SM} h^*(\mu^\perp W) dV_{SM} \\ &= -\frac{n}{c_{n-1}} \int_{SM} \mu^\perp h^*(W) dV_{SM} = -\frac{n}{c_{n-1}} \int_{SM} H^*(W) dV_{SM}. \end{aligned}$$

Hence  $H = 0$ , which implies that  $\phi$  is strongly minimal. Moreover, if  $\tilde{V}'(0) = 0$  for any variation vector field  $V(x, y) \in \mathcal{M}_0(\phi^*T\widetilde{M})$ , i.e. for any  $V(x, y) = V(x) \in \mathcal{C}(\phi^*T\widetilde{M})$ , then we have  $\mu_\phi = 0$  from (2.14), which implies that  $\phi$  is strongly minimal.

**Theorem 5.3.** *Let  $\phi : (M, F) \rightarrow (\widetilde{M}, \widetilde{F})$  be an isometric immersion and  $\psi_t$  be a smooth variation satisfying (4.1) and (4.2). Then  $\phi$  is strongly minimal if and only if it is the critical point of the volume functional with respect to any variation vector field  $V(x, y) \in \mathcal{M}(\phi^*T\widetilde{M})$ . Furthermore,  $\phi$  is totally geodesic if and only if it is the critical point of the volume functional with respect to any variation vector field  $V(x, y) \in \mathcal{C}(\pi^*(\phi^*T\widetilde{M}))$ .*

### 6. Horizontal Laplacian for $E$ -valued $p$ -forms on $SM$

In this section, we consider the differential operators acting on smooth  $p$ -forms with values in a vector bundle [12]. Let  $(M, F)$  be a compact Finsler manifold without boundary and  $\xi : E \rightarrow SM$  be a smooth vector bundle over  $SM$ . Set

$$\mathcal{A}^p(E) = \mathcal{C}\left(\bigwedge^p T^*SM \otimes E\right), \quad \mathcal{H}^p(E) = \mathcal{C}\left(\bigwedge^p H^* \otimes E\right),$$

which are the space of  $p$ -forms and the space of horizontal  $p$ -forms on  $SM$  with values in  $E$  respectively. Let  $\{E_\alpha\}$  be a local frame field of  $E$  and assume there

exists a Riemannian metric  $\tilde{g}$  and a linear connection  $\tilde{\nabla}$  with  $\tilde{\nabla}E_\alpha = \tilde{\omega}_\alpha^\beta E_\beta$  on  $E$ .

For any  $\phi, \psi \in \mathcal{H}^p(E)$ , set

$$\phi = \frac{1}{p!} \phi_{i_1 i_2 \dots i_p}^\alpha dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \otimes E_\alpha, \quad \phi^{j_1 j_2 \dots j_p \alpha} = \phi_{i_1 i_2 \dots i_p}^\alpha g^{i_1 j_1} \dots g^{i_p j_p},$$

and define the global inner product in  $\mathcal{H}^p(E)$  as follows

$$(\phi, \psi) = \int_{SM} \langle \phi, \psi \rangle_{\mathcal{H}^p(E)} dV_{SM} = \frac{1}{p!} \int_{SM} \phi_{i_1 i_2 \dots i_p}^\alpha \psi^{i_1 i_2 \dots i_p \beta} \tilde{g}_{\alpha\beta} dV_{SM}. \quad (6.1)$$

Since the Chern connection  $\nabla$  is torsion-free, we can define the horizontal differential operator  $d^H : \mathcal{H}^p(E) \rightarrow \mathcal{H}^{p+1}(E)$  by

$$\begin{aligned} d^H \phi &= \frac{1}{p!} \delta_i (\phi_{i_1 i_2 \dots i_p}^\alpha) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \otimes E_\alpha \\ &\quad + \frac{1}{p!} \phi_{i_1 i_2 \dots i_p}^\alpha \tilde{\omega}_\alpha^\beta (\delta_i) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \otimes E_\beta \\ &= \frac{1}{p!} \phi_{i_1 i_2 \dots i_p | i}^\alpha dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \otimes E_\alpha, \end{aligned} \quad (6.2)$$

for any  $\phi = \frac{1}{p!} \phi_{i_1 i_2 \dots i_p}^\alpha dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \otimes E_\alpha \in \mathcal{H}^p(E)$ .

Accordingly, the horizontal codifferential operator  $\delta^H : \mathcal{H}^{p+1}(E) \rightarrow \mathcal{H}^p(E)$  can be defined such that  $(d^H \phi, \psi) = (\phi, \delta^H \psi)$  holds for any  $\phi \in \mathcal{H}^p(E), \psi \in \mathcal{H}^{p+1}(E)$ . Taking into account that  $g$  is parallel along the horizontal directions, one obtains from Lemma 2.1 that

$$\begin{aligned} \langle d^H \phi, \psi \rangle_{\mathcal{H}^p(E)} &= \frac{1}{(p+1)!} \{ (p+1) \phi_{i_1 i_2 \dots i_p | i}^\alpha \psi^{i i_1 i_2 \dots i_p \beta} \tilde{g}_{\alpha\beta} \} \\ &= X_{|i}^i - \frac{1}{p!} g^{ij} \phi^{i_1 i_2 \dots i_p \alpha} (\psi_{i i_1 i_2 \dots i_p | j}^\beta \tilde{g}_{\alpha\beta} - \psi_{i i_1 i_2 \dots i_p}^\beta \tilde{g}_{\alpha\beta | j}) \\ &= \operatorname{div}_{\tilde{g}} X + X^i \dot{\eta}_i - \frac{1}{p!} g^{ij} \phi^{i_1 i_2 \dots i_p \alpha} (\psi_{i i_1 i_2 \dots i_p | j}^\beta \tilde{g}_{\alpha\beta} - \psi_{i i_1 i_2 \dots i_p}^\beta \tilde{g}_{\alpha\beta | j}) \end{aligned}$$

where  $X^i = \frac{1}{p!} \phi_{i_1 i_2 \dots i_p}^\alpha \psi^{i i_1 i_2 \dots i_p \beta} \tilde{g}_{\alpha\beta}$ . Hence

$$\begin{aligned} (\phi, \delta^H \psi) &= (d^H \phi, \psi) = \int_{SM} \langle d^H \phi, \psi \rangle_{\mathcal{H}^p(E)} dV_{SM} \\ &= \frac{1}{p!} \int_{SM} \phi^{i_1 i_2 \dots i_p \alpha} g^{ij} (\psi_{i i_1 i_2 \dots i_p}^\beta \tilde{g}_{\alpha\beta} \dot{\eta}_j - \psi_{i i_1 i_2 \dots i_p | j}^\beta \tilde{g}_{\alpha\beta} - \psi_{i i_1 i_2 \dots i_p}^\beta \tilde{g}_{\alpha\beta | j}) dV_{SM}. \end{aligned}$$

**Theorem 6.1.** *The horizontal codifferential of the horizontal  $E$ -value  $(p+1)$ -form  $\psi$  can be expressed as*

$$\delta^H \psi = -\frac{1}{p!} g^{ij} (\psi_{i_1 i_2 \dots i_p | j}^\alpha - \psi_{i_1 i_2 \dots i_p}^\alpha \dot{\eta}_j + \psi_{i_1 i_2 \dots i_p}^\beta \tilde{g}_{\beta \gamma | j} \tilde{g}^{\gamma \alpha}) dx^{i_1} \wedge \dots \wedge dx^{i_p} \otimes E_\alpha. \tag{6.3}$$

*Definition 6.2.* The **horizontal Laplacian**  $\tilde{\Delta}^H : \mathcal{H}^p(E) \rightarrow \mathcal{H}^p(E)$  is defined by

$$\tilde{\Delta}^H = d^H \circ \delta^H + \delta^H \circ d^H. \tag{6.4}$$

In particular, an  $E$ -value  $p$ -form  $\phi$  is called  **$h$ -harmonic** if  $\tilde{\Delta}^H \phi = 0$ .

It is obvious that

$$\begin{aligned} \langle \tilde{\Delta}^H \phi, \psi \rangle &= \int_{SM} \langle (d^H \circ \delta^H + \delta^H \circ d^H) \phi, \psi \rangle_{\mathcal{H}^p(E)} dV_{SM} \\ &= \int_{SM} (\langle d^H \phi, d^H \phi \rangle_{\mathcal{H}^p(E)} + \langle \delta^H \phi, \delta^H \phi \rangle_{\mathcal{H}^p(E)}) dV_{SM} = \langle \phi, \tilde{\Delta}^H \phi \rangle, \end{aligned} \tag{6.5}$$

for any  $\phi, \psi \in \mathcal{H}^p(E)$ .

**Lemma 6.3.** *The operator  $\tilde{\Delta}^H$  is selfadjoint and positive with respect to the global inner product. An  $E$ -valued  $p$ -form  $\psi$  is  $h$ -harmonic if and only if  $d^H \psi = 0, \delta^H \psi = 0$ .*

Let  $\phi : (M, F) \rightarrow (\tilde{M}, \tilde{F})$  be a non-degenerate smooth map and  $E = \pi^*(\phi^* T\tilde{M})$ , then  $d\phi \in \mathcal{H}^1(E)$ . It follows from (6.2) that

$$\begin{aligned} d^H(d\phi)(X, Y) &= (\tilde{\nabla}_{X^H} d\phi)Y - (\tilde{\nabla}_{Y^H} d\phi)X = 0, \\ \delta^H(d\phi) &= g^{ij} (\phi_{i|j}^\alpha - \phi_i^\alpha \dot{\eta}_j + \phi_i^\beta \tilde{g}_{\beta \gamma | j} \tilde{g}^{\gamma \alpha}) \tilde{\partial}_\alpha, \end{aligned}$$

for any  $X, Y \in \mathcal{C}(\pi^* TM)$ . In particular, when  $(\tilde{M}, \tilde{F})$  is Riemannian,

$$\delta^H(d\phi) = g^{ij} \phi_{i,j}^\alpha \tilde{\partial}_\alpha = \tau(\phi).$$

**Theorem 6.4.** *Let  $\phi$  be a smooth map from a Finsler manifold  $(M, F)$  to a Riemannian manifold  $(\tilde{M}, \tilde{F})$ . Then  $\phi$  is harmonic if and only if  $d\phi$  is an  $h$ -harmonic 1-form with value in  $\pi^*(\phi^* T\tilde{M})$ .*

In particular, when  $E = \mathbf{R}$ , the expression of  $\tilde{\Delta}^H$  accords with that in [13]. For a 0-form, i.e. a function  $f \in C^\infty(SM)$ , we have

$$\tilde{\Delta}^H f = -g^{ij} f_{|i,j}. \tag{6.6}$$

Therefore, the relation between the  $h$ -Laplacian  $\Delta^H$  for scalar fields and the  $h$ -Laplacian  $\tilde{\Delta}^H$  for 0-forms is given by  $\Delta^H = -\tilde{\Delta}^H$ .

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