Initial hulls and zero dimensional objects

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Abstract. In this paper we use initial hulls to define zero dimensional (and other) objects in topological categories. In the process we characterize some initial hulls in the category of topological spaces. Examples of zero dimensional objects in several topological categories are given. We relate zero dimensional objects to preT_2 objects via a Theorem which generalizes their relationship for topological spaces.

1. Introduction

Topological categories; i.e., the domain of a topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ ($\mathcal{U}$ is concrete, has small (set-indexed) fibers, and every $\mathcal{U}$-source has an initial lift) have appeared extensively in the literature since the early 1970's (comprehensive expository treatments of categorical topology can be found in [1] or [16]). The prototype example of a topological category is $\mathcal{TOP}$, the category of topological spaces and continuous functions, which is topological over $\mathcal{SET}$, the category of sets and functions, with $\mathcal{U} : \mathcal{TOP} \to \mathcal{SET}$ being the standard forgetful functor. Given a non-empty collection of objects E_0 in a topological category $\mathcal{E}$, it is natural to try and find the smallest full subcategory of $\mathcal{E}$ which contains all objects in E_0, and which is also topological with respect to the inherited initial lift formulations from $\mathcal{E}$. This subcategory is refered to as the initial hull of E_0 (see [10], [17]). Several examples of initial hulls in $\mathcal{TOP}$ are given in [11]; in

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particular, the initial hulls for the collections of all T_1 and all T_2-spaces are stated (see proposition 7 below).

In [8] a topological space is defined to be zero dimensional provided it has a basis consisting of clopen (both open and closed) sets. These zero dimensional spaces have been used in the construction of many useful classes of topological spaces including Lusin spaces ([4]), Stone spaces ([9]), and non-Archimedean spaces ([6]). To extend classical notions such as dimension zero in $\mathcal{TOP}$ to an arbitrary topological category, one must express these notions in terms which are available in topological categories; i.e., in terms of final or initial structures, (co)limits, (in)discreteness, etc. The main goal of this paper is to employ initial hulls as a vehicle to extend certain classical notions based on separation, particularly dimension zero, to the general setting of a topological category.

The paper is organized as follows. Section 2 states the definition of the initial hull of a collection of objects in a topological category and we state a characterization of the initial hull which will be used throughout the remainder of the paper. Also included in this section are several examples of initial hulls in $\mathcal{TOP}$. Section 3 contains the main results of the paper; namely the use of initial hulls to generalize the classical notion of a zero dimensional topological space to zero dimensional objects in a topological category. Several examples of zero dimensional objects in some well-known topological categories are calculated. We also develop some general theoretical results concerning zero dimensional objects. In particular, we relate zero dimensional objects to the so-called pre-Hausdorff objects ([2], [3], [13], [14], [19]) in a topological category by way of a theorem which generalizes their relationship in $\mathcal{TOP}$. In Section 4 we conclude the paper by defining other objects in topological categories in a fashion similar to the method in Section 3; namely, by using initial hulls. We also mention an open problem concerning initial hulls which is of interest to the theoretical development of pre-Hausdorff objects. Throughout the paper, $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ will be assumed to be a topological functor, E_0 will denote a non-empty collection of subobjects of $\mathcal{E}$, and $\mathcal{TOP}$ and $\mathcal{SET}$ will denote the categories as mentioned above.

2. Initial hulls and examples

In this section we define initial hulls and we characterize some initial hulls in $\mathcal{TOP}$ which will be used later in the paper.

Definition 1. The initial hull of E_0, denoted $IH(E_0)$, is the smallest full concrete initially closed subcategory of $\mathcal{E}$ whose object class contains E_0.

The following proposition gives a useful characterization of initial hulls. A proof can be found in [17].

Proposition 2. $IH(E_0)$ *is determined by the following object class.*

$$\{e \in \mathcal{E} \mid \exists \mathcal{U} - initial\ source\{f_i : e \to e_i\}_{i \in I}\ with\ e_i \in E_0, \forall i \in I\}$$

That is, $IH(E_0)$ is the full subcategory of $\mathcal{E}$ consisting of all objects which can be induced from a family of objects in E_0.

Example 3. In $\mathcal{TOP}$, suppose E_0 is a family consisting of a single space which consists of a single point. Then $IH(E_0) = \mathcal{I}(\mathcal{SET})$, the collection of all indiscrete spaces. More generally, recall that each topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ has a right adjoint $\mathcal{I} : \mathcal{B} \to \mathcal{E}$ called the indiscrete functor, and that $\mathcal{I}(b)$ is the maximum element in $\mathcal{U}^{-1}(b)$ for each $b \in \mathcal{B}$. We denote the collection of all these indiscrete objects by $\mathcal{I}(\mathcal{B})$. Suppose $\mathcal{B}$ is any category with a terminal object 1. If $* = \mathcal{I}(1)$ and E_0 consists solely of $*$, then $IH(E_0) = \mathcal{I}(\mathcal{B})$. This follows immediately from the fact that the initial lift of an indiscrete object yields an indiscrete object (see Lemma 2.1.2 in [19]).

In a topological category $\mathcal{E}$, the initial hull of a family of objects E_0 may be trivial in the sense that $IH(E_0) = \mathcal{E}$. In $\mathcal{TOP}$, we characterize the families of spaces whose initial hull is trivial in this sense.

Proposition 4. *A family of topological spaces F satisfies $IH(F) = \mathcal{TOP}$ iff F contains a space which contains a Sierpinski space (as a subspace).*

PROOF. Suppose F is a family of topological spaces containing the space (Y, σ) which contains the Sierpinski space (S, σ_S), where $S = \{p, q\}$ and let (X, τ) be any topological space. Suppose the point p has no proper neighborhood in σ_S, and there exists a proper open set $O_S \in \sigma$ such that $O_S \cap S = \{q\}$. For each $U \in \tau$, define a function $f_U : X \to Y$

$$f_U(x) = \begin{cases} q, & \text{if } x \in U \\ p, & \text{if } x \notin U. \end{cases}$$

Then τ is the topology induced on X via $\{f_U\}_{U \in \tau}$. Conversely, suppose F is a family of topological spaces satisfying $IH(F) = \mathcal{TOP}$. Then the Sierpinski space S can be induced from a family of spaces $\{(X_i, \tau_i)\}_{i \in I} \subseteq F$ via a family of functions $\{f_i : S \to X_i\}_{i \in I}$. Since S is not indiscrete, there exists $j \in I$ such that f_j is injective and $f_j(S)$ is not indiscrete. Furthermore, since S is not discrete, $f_j(S)$ is not discrete. Thus $f_j(S)$ is a Sierpinski space which is a subspace of X_j. $\qquad\qquad\square$

Accordingly, for instance, a family consisting of a single space which has a Sierpinski space as a subspace will generate all of $\mathcal{TOP}$ by its initial hull.

Example 5. Let $X = \{1, 2, 3\}$ with topology τ generated by the basis $\{\{1, 2\}, \{2, 3\}\}$, and let $F = \{(X, \tau)\}$. Then $IH(F) = \mathcal{TOP}$ since $\{1, 2\}$ endowed with the subspace topology is a Sierpinski space.

If we denote by $T_0-\mathcal{TOP}$ the full subcategory of $\mathcal{TOP}$ consisting of all T_0-spaces, then $IH(T_0-\mathcal{TOP}) = \mathcal{TOP}$ since a Sierpinski space is always T_0. Consequently, Proposition 4 suggests that collections of spaces with non-trivial initial hulls will consist of spaces which possess separation properties stronger than T_0, as in the following two examples.

Proposition 6. *In $\mathcal{TOP}$, denote by D_0 the collection of all discrete spaces, and by $0-\mathrm{Dim}$ the collection of all 0 dimensional spaces. Then $IH(D_0) = 0-\mathrm{Dim}$.*

PROOF. Suppose that (X, τ) is 0-dimensional with basis $\{U_i\}_{i \in I}$ consisting of clopen sets. For each $i \in I$, let $X_i = \{0, 1\}$ with the discrete topology, and define $f_i : X \to X_i$ by

$$f_i(x) = \begin{cases} 0, & \text{if } x \in U_i \\ 1, & \text{if } x \notin U_i. \end{cases}$$

Then τ is the topology induced on X by $\{X_i\}_i$ via $\{f_i\}_i$, so that $0 - \mathrm{Dim} \subseteq IH(D_0)$. For the reverse inclusion, suppose that τ is the topology induced on a set X from a family of discrete spaces $\{(X_i, \tau_i)\}_{i \in I}$ via $f_i : X \to X_i$, and that $U \in \tau$. Without loss of generality we can assume that U is a basis element, so there exists $i_1, \dots, i_n \in I$ and there exisis $U_{i_1} \in \tau_{i_1}, \dots, U_{i_n} \in \tau_{i_n}$ such that $U = \bigcap_{j=1}^n f_{i_j}^{-1}(U_{i_j})$, which implies that $U^C = (\bigcap_{j=1}^n f_{i_j}^{-1}(U_{i_j}))^C = \bigcup_{j=1}^n f_{i_j}^{-1}(U_{i_j}^C)$. Since $U_{i_j}^C$ is open in X_{i_j} we have that U^C is open in X and, hence, X is 0-dimensional. $\square$

Recall [18] that a topological space X is called a $T_{i,j}$-space (for $0 \le i < j \le 2$) provided each pair of points $a, b \in X$ which has a T_i-separation in X also has a T_j-separation in X.

Note. $T_{0,2}$-spaces are also called pre-Hausdorff spaces in the literature. Denote by $T_{i,j}-\mathcal{TOP}$ (resp. $T_i-\mathcal{TOP}$) the full subcategory of $\mathcal{TOP}$ consisting of the $T_{i,j}$-spaces (resp. T_i-spaces). Note that a T_i-space is $T_{0,i}$.

Proposition 7. *For $i = 1, 2$, $IH(T_i-\mathcal{TOP}) = T_{0,i}-\mathcal{TOP}$*

PROOF. It is shown in [18] that inducing on a set from a family of $T_{i,j}$-spaces yields a $T_{i,j}$-space; i.e., $T_{i,j}-\mathcal{TOP}$ is initially closed in $\mathcal{TOP}$ and, hence, topological. So, since a T_i-space is always $T_{0,i}$, we have that $IH(T_i-\mathcal{TOP}) \subseteq T_{0,i}-\mathcal{TOP}$. To prove the reverse inclusion, suppose (X, τ) is a $T_{0,i}$-space (i either 1 or 2) and we manufacture a T_i-space which induces the topology τ as follows. Define a relation R on X by:

$(x, y) \in R$ if and only if x and y have no T_0-separation in τ.

It can be readily shown that R is an equivalence relation. We claim that, for $i = 1, 2$, we have (1) if (X, τ) is $T_{0,i}$, then for each $U \in \tau$, $q^{-1}(q(U)) = U$, where q is the cannonical quotient map which sends each $x \in X$ to its equivalence class $[x] \in \frac{X}{R}$. Note that this property implies that q is an open map; (2) $\frac{X}{R}$ is T_i iff (X, τ) is $T_{0,i}$; and (3) τ is the topology induced on X by $\frac{X}{R}$ via q. To prove (1), suppose that $x \in q^{-1}q(U)$. Then $q(x) \in q(U)$, which implies that there exists $y \in U$ with $[x] = [y]$. Then x and y have no T_0-separation in τ, so $x \in U$. Thus $q^{-1}q(U) \subseteq U$, and the reverse inclusion always holds. To prove (2), suppose that (X, τ) is $T_{0,i}$, and that $[x] \neq [y]$ in $\frac{X}{R}$. Then x and y have T_0-separation in τ, which implies that they have a T_i-separation by, say, $U_x, U_y \in \tau$. Since q is an open map, $q(U_x)$ and $q(U_y)$ provide a T_i-separation in $\frac{X}{R}$ and the implication to the left is established. The proof of the converse is similar using inverse images by q. For (3), let $\widetilde{\tau}$ be the topology induced on X by $\frac{X}{R}$ via q. Clearly $\widetilde{\tau} \subseteq \tau$ since $\widetilde{\tau}$ is initial, and the reverse inclusion follows immediately from (1).   □

Notes. (i) With different terminology and without proof, the results of Proposition 7 are stated in [11]. There, the result above for the case $i = 1$ is stated as $IH(T_1-\mathcal{TOP})$ coincides with the class of R_0-spaces (see Section 4 below). Since it is shown in [18] that a topological space is an R_0-space if and only if it is a $T_{0,1}$-space, these results are equivalent. Also, the $T_{0,2}$-spaces mentioned above are refered to as R-spaces in [11].

(ii) The relation R can be defined equivalently as: $(x, y) \in R$ if and only if for all $Y \in T_0-\mathcal{TOP}$, and for all continuous functions $f : X \to Y$, we have $f(x) = f(y)$.

(iii) It is shown in [18] that the assignment $X \mapsto X/R$ defines a functor which is left adjoint to the inclusion functor $T_0-\mathcal{TOP} \hookrightarrow \mathcal{TOP}$, so that $q : X \to X/R$ is the T_0-reflection of the space (X, τ). Accordingly, claim (2) in the proof above states that a topological space is $T_{0,i}$ if and only if its T_0-reflection is T_i.

Because initial hulls are initially closed, and since compositions of initial lifts are initial, clearly the initial hull of any collection of objects E_0 is a topological

category with initial lifts formulated as in the original category $\mathcal{E}$. Furthermore, it is shown in [1] that an initially closed subcategory is always concretely reflective, and hence, reflective. Consequently, we have the following.

Theorem 8. $IH(E_0)$ *is itself topological over* $\mathcal{B}$ *and the inclusion functor* $\mathcal{I}_0 : IH(E_0) \hookrightarrow \mathcal{E}$ *is initially preserving (hence continuous). Moreover,* $IH(E_0)$ *is a reflective subcategory of* $\mathcal{E}$.

In the case where E_0 is reflective in $\mathcal{E}$, the following example gives an explicit description of the left adjoint to $\mathcal{I}_0$. Note that in the construction of the left adjoint $\mathcal{L}_0$, the base category $\mathcal{B}$ is arbitrary. Also in this construction we use the fact that $\mathcal{E}$, being a topological category, has the (bimorphism, initial morphism) factorization structure; i.e., each morphism $f : a \to b$ in $\mathcal{E}$ can be factored as $f = \overline{f} \circ \xi_a$, where the initial morphism $\overline{f} : \overline{a} \to b$ is the initial lift of $\mathcal{U}f : \mathcal{U}a \to \mathcal{U}b$, and the bimorphism $\xi_a : a \to \overline{a}$ is the resulting lift of the identity morphism on $\mathcal{U}a$. Finally we note that, by virtue of Proposition 7, the following construction of $\mathcal{L}_0$ includes the functors $\mathcal{L}_{0,1} : \mathcal{TOP} \to T_{0,1}-\mathcal{TOP}$ and $\mathcal{L}_{0,2} : \mathcal{TOP} \to T_{0,2}-\mathcal{TOP}$ found in [18] as special cases.

Example 9. Suppose $\mathcal{E}_0$ is a reflective subcategory of $\mathcal{E}$ (with object collection E_0) and that $\mathcal{L} : \mathcal{E} \to \mathcal{E}_0$ is left adjoint to the inclusion functor $\mathcal{I} : \mathcal{E}_0 \hookrightarrow \mathcal{E}$. Then to each object $e \in \mathcal{E}$ there is a universal arrow $\eta_e : e \to \mathcal{L}e$. Let $\overline{\eta_e} \circ \xi_e$ be the (bimorphism, initial morphism) factorization of η_e, where $\xi_e : e \to \mathcal{L}_0 e$ and $\overline{\eta_e} : \mathcal{L}_0 e \to \mathcal{L}e$. Then $\mathcal{L}_0 e \in IH(E_0)$. We claim that $\xi_e : e \to \mathcal{L}_0 e$ is a universal arrow and, consequently, $\mathcal{L}_0$ defines a left adjoint to the inclusion functor $\mathcal{I}_0 : IH(E_0) \hookrightarrow \mathcal{E}$. To prove this claim, suppose that $x \in IH(E_0)$ and $f : e \to x$. Then x can be induced from a family of objects in E_0; for notational simplicity we will assume that this family consists of a single object. So there exists $e_0 \in E_0$ such that $x \xrightarrow{f_0} e_0$ is the initial lift of $a \xrightarrow{\mathcal{U}f_0} b_0 = \mathcal{U}(e_0)$. Since η_e is a universal arrow, the composition $f_0 \circ f$ factors through η_e; i.e., there exists $m_0 : \mathcal{L}e \to e_0$ such that $f_0 \circ f = m_0 \circ \eta_e = m_0 \circ \overline{\eta}_e \circ \xi_e$. Thus, we have the commutative square $(m_0 \circ \overline{\eta}_e) \circ \xi_e = f_0 \circ f$. Since ξ_e is a bimorphism and f_0 is an initial morphism, there exists a morphism $m : \mathcal{L}_0 e \to x$ with $f = m \circ \xi_e$ by the unique (bimorphism, initial morphism) diagonalization property (see Definition 14.1 of [1]).

Recall ([1]) that the reflective hull of E_0 in $\mathcal{E}$, hereafter denoted by $\mathcal{R}(E_0)$, is the smallest isomorphism closed full reflective subcategory of $\mathcal{E}$ which contains E_0. It is well-known that $\mathcal{R}(E_0)$ does not always exist. However if E_0 does exist, we note that although $IH(E_0)$ is reflective and is the smallest initially closed subcategory of $\mathcal{E}$ containing E_0, $IH(E_0)$ may or may not coincide with $\mathcal{R}(E_0)$.

Indeed, if E_0 is reflective but not initially closed, then clearly $IH(E_0) \neq \mathcal{R}(E_0)$. This is the case, for instance, in $\mathcal{TOP}$ with $E_0 = T_i - \mathcal{TOP}$, for $i = 1, 2$, as in Proposition 7. Alternatively if E_0 is both reflective and initially closed, then trivially $IH(E_0) = \mathcal{R}(E_0)$. An example of a non-trivial case where the initial and reflective hulls coincide arises in $\mathcal{TOP}$ when $E_0 = D_0$, where it is well-known that $\mathcal{R}(D_0) = 0 - \mathrm{Dim}$ (see [6]), and $IH(D_0) = 0 - \mathrm{Dim}$ by Proposition 6.

3. Dimension 0 in topological categories

In this section, we employ initial hulls to extend the notion of dimension zero to a topological category. To this end, recall ([15]) that each topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ has a left adjoint $\mathcal{D} : \mathcal{B} \to \mathcal{E}$ called the discrete functor, and that $\mathcal{D}(b)$ is the minimum element in $\mathcal{U}^{-1}(b)$ for each $b \in \mathcal{B}$. We denote the collection of all these discrete objects by $\mathcal{D}(\mathcal{B})$. Notice that Proposition 6 gives a characterization of dimension 0 which involves only initial structures and discrete objects. It shows that a topological space is zero dimensional provided it can be induced from a family of discrete spaces. This justifies the following.

Definition 10. Suppose that $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is topological with discrete functor (left adjoint) $\mathcal{D} : \mathcal{B} \to \mathcal{E}$. An object $x \in \mathcal{E}$ has dimension 0 provided $x \in IH(\mathcal{D}(\mathcal{B}))$; i.e., there are families of $\mathcal{B}$-objects $\{b_i\}_{i \in I}$ and $\mathcal{B}$-morphisms $\{f_i : \mathcal{U}(x) \to b_i\}_{i \in I}$ such that $\{x \xrightarrow{\overline{f_i}} \mathcal{D}(b_i)\}$ is the initial lift of $\{\mathcal{U}(x) \xrightarrow{f_i} b_i = \mathcal{U}(\mathcal{D}(b_i))\}$. If so; i.e., if x can be induced from a family of discrete objects, then we write $\dim(x) = 0$.

Accordingly, in any topological category, discrete objects are always zero dimensional. The next example shows that they may be the only zero dimensional objects in a topological category.

Example 11. In CP, the category of pairs (objects are pairs of sets (A, C) with $C \subseteq A$, morphisms from (A_1, C_1) to (A_2, C_2) are functions $f : A_1 \to A_2$ with $f(C_1) \subseteq C_2$, and $\{(A, B) \xrightarrow{f_i} (A_i, B_i)\}$ is initial if and only if $B = \bigcap_i f_i^{-1}(B_i)$, the discrete structure on a set A is $D(A) = (A, \varnothing)$ (see [2]). Consequently, in CP, $\dim(x) = 0$ iff x is discrete.

The next example provides a novel interpretation of an equivalence relation on a set; namely, a set with an equivalence relation is a zero dimensional object in a particular topological category. Recall that a preordered set is a pair (A, R) where A is a set and R is a relation on A which is both reflexive and transitive.

Example 12. In Pr Ord, the category of preordered sets (objects are preordered sets, morphisms from (A_1, R_1) to (A_2, R_2) are functions $f : A_1 \to A_2$ such that if aR_1b then $f(a)R_2f(b)$, and $\left\{(A, R) \xrightarrow{f_i} (A_i, R_i)\right\}_{i \in I}$ is initial provided aRb iff $f_i(a)R_if_i(b)$, $\forall i \in I$), the discrete structure on a set A is $D(A) = (A, \Delta_A)$ where $\Delta_A = \{(a, a) \mid a \in A\}$ is the diagonal on A (see [2]). We claim that $(A, R) \in$ Pr Ord has dimension 0 iff R is an equivalence relation; for if (A, R) has dimension 0, then clearly R is symmetric. Conversely, suppose that R is an equivalence relation on A and let $q : A \to \frac{A}{R}$ be the cannonical quotient map. Then (A, R) is induced from $\left(\frac{A}{R}, \Delta_{\frac{A}{R}}\right)$ via q.

It is shown in [18] that a principal topological space (also sometimes called an Alexandroff space; i.e., a space in which arbitrary intersections of open sets are open) has dimension 0 if and only if it is pre-Hausdorff. For any topological space (X, τ) we have the following result, the proof of which is straightforward.

Proposition 13. *If X has dimension 0, then X is pre-Hausdorff.*

The pre-Hausdorff separation condition has been generalized to arbitrary topological categories. Indeed, there are two categorical conditions, equivalent in some categories but generally distinct, which both reduce to the pre-Hausdorff condition in $\mathcal{TOP}$. For convienience, after developing some notation which follows that of [13], we recall the categorical definitions of pre-Hausdorff objects.

Suppose $\mathcal{B}$ is a category with finite powers (denoted b^n, and whose projection morphisms are denoted $p_i : b^n \to b$) and pushouts (a topos, for instance). Given morphisms $f_i : a \to b$, $i = 1, 2, \ldots, n$, $\langle f_1, \ldots, f_n \rangle : a \to b^n$ denotes the unique morphism f such that $p_if = f_i$, $i = 1, \ldots, n$. The diagonal $d : b \to b^2$ is then $\langle 1_b, 1_b \rangle$, where $1_b : b \to b$ is the identity morphism on b. If we form the pushout of this diagonal d against itself, the resulting object will be denoted $b^2 \bigvee b^2$ (this object is denoted by $b^2 + b^2$ in [13]); i.e., $i_1 \circ d = i_2 \circ d$ is a pushout diagram, where i_1 and i_2 are the cannonical injection morphisms. Since this diagram is a pushout and $\langle p_1, p_1, p_2 \rangle \circ d = \langle p_1, p_2, p_1 \rangle \circ d$, there exists a unique morphism $A : b^2 \bigvee b^2 \to b^3$ such that $A \circ i_1 = \langle p_1, p_2, p_1 \rangle$ and $A \circ i_2 = \langle p_1, p_1, p_2 \rangle$. Similarly, since $\langle p_1, p_1, p_2 \rangle \circ d = \langle p_1, p_2, p_2 \rangle \circ d$, there exists a unique morphism $S : b^2 \bigvee b^2 \to b^3$ such that $S \circ i_1 = \langle p_1, p_2, p_2 \rangle$ and $S \circ i_2 = \langle p_1, p_1, p_2 \rangle$. Note: the notation A and S follows the notation in [2], [3], and [19], whereas [14] labels these morphisms as m_1 and m_2 (resp.), and they are labled $p(2, 2)^*$ and $p(1, 2)^*$ (resp.) in [13].

Definition 14. Suppose $\mathcal{B}$ is a category with finite powers and pushouts, $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is a topological functor, and that e is an $\mathcal{E}$-object with $\mathcal{U}(e) = b$.

(i) e is pre T_2 provided the initial lift of $\{A : b^2 \bigvee b^2 \to b^3 = U(e^3)\}$ = the initial lift of $\{S : b^2 \bigvee b^2 \to b^3 = U(e^3)\}$. The full subcategory of $\mathcal{E}$ consisting of the pre T_2 objects will be denoted pre $T_2(\mathcal{E})$.

(ii) e is pre T_2' provided initial lift of $\{S : b^2 \bigvee b^2 \to b^3 = U(e^3)\}$ = final lift of $\{i_1, i_2 : U(e^2) = b^2 \rightrightarrows b^2 \bigvee b^2\}$. The full subcategory of $\mathcal{E}$ consisting of the pre T_2' objects will be denoted pre $T_2'(\mathcal{E})$.

These pre T_2 and pre T_2' objects were originally introduced in [?] where they arise naturally while studying the image of a topos in a topological category by a geometric functor (i.e., a right adjoint functor whose left adjoint preserves finite limits), including geometric realizations in $\mathcal{TOP}$ (also see [13]). A general relationship between these objects can also be found in [13], where Mielke proves that pre $T_2'(\mathcal{E}) \subseteq$ pre $T_2(\mathcal{E})$ for any topological category $\mathcal{E}$ in which these objects are defined. The fact that these generally distinct concepts coincide and both reduce to the classical pre-Hausdorff separation axiom in $\mathcal{TOP}$ is proved in [2]. Further remarks concerning the relationship between pre T_2 and pre T_2' objects appear in Section 4 below.

In order to generalize Proposition 13, recall ([13]) that a geometric topological functor is a topological functor $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ for which the discrete functor $\mathcal{D} : \mathcal{B} \to \mathcal{E}$ preserves finite limits.

Theorem 15. *Suppose that $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is a geometric topological functor into a topos $\mathcal{B}$.*

(i) *If $m : b \rightarrowtail a$ is a mono in $\mathcal{B}$, then the initial lift of $b \xrightarrow{m} a = \mathcal{U}(\mathcal{D}(a))$ is $\mathcal{D}(b) \xrightarrow{\mathcal{D}(m)} \mathcal{D}(a)$.*

(ii) *For any object $b \in \mathcal{B}$, $\mathcal{D}(b)$ is pre T_2.*

(iii) *Every 0-dimensional object in a geometric topological category over a topos is pre T_2.*

PROOF. (i) Suppose that $m : b \rightarrowtail a$ *is a mono in* $\mathcal{B}$. Since $\mathcal{B}$ is a topos, we have that m is a regular monomorphism; i.e., m is the equalizer of a parallel pair of morphisms ([12], p. 167). Then applying $\mathcal{D}$ yields a regular monomorphism (in $\mathcal{E}$) since $\mathcal{D}$ is left exact. But all regular monomorphisms (being limits) are initial, so $\mathcal{D}(b) \xrightarrow{\mathcal{D}(m)} \mathcal{D}(a)$ is initial.

(ii) For each $b \in \mathcal{B}$ we have that $(\mathcal{D}(b))^3 = \mathcal{D}(b^3)$ since $\mathcal{D}$ is left exact. Then the initial lift of $\{A : b^2 \bigvee b^2 \to b^3 = \mathcal{U}(e^3)\} = \mathcal{D}(b^2 \bigvee b^2)$ = the initial lift of $\{S : b^2 \bigvee b^2 \to b^3 = \mathcal{U}(e^3)\}$ by (i), since both A and S are monos (see [14]).

(iii) The result follows immediately from (ii) above, together with the fact that pre $T_2(\mathcal{E})$ is initially closed (see the proof of Theorem 2.1.3 in [19]). $\square$

We note that 0-dimensional objects are not always pre T_2. In fact, somewhat surprisingly, discrete objects need not be pre T_2. This is surprising in view of the situation in $\mathcal{TOP}$ where pre T_2 is a relatively weak separation property, and discrete is as separated as a space can be. The following example, originally found in [19], illustrates these comments. Recall ([2], [3]) that a stack on a set A is a family of subsets of A which is closed under the formation of supersets, and a stack convergence space is a pair (A, K) where A is a set and K is a function which assigns to each $a \in A$ a set of stacks on A such that 1. $[a] = \{S \mid S \subseteq A$ and $a \in S\} \in K(a)$, $\forall a \in A$, and 2. if $\alpha \in K(a)$ and β is a stack on A with $\alpha \subseteq \beta$, then $\beta \in K(a)$, $\forall a \in A$. A stack convergence space (A, K) where K is a constant function is called a constant stack convergence space. In a constant stack convergent space, the roll of each $K(a)$ will be played generically by a set of stacks K in the following example.

Example 16. The category of constant stack convergent spaces, *conSCO*, has as its objects the constant stack convergence spaces, morphisms $f : (A, K) \to (B, L)$ are functions $f : A \to B$ such that if $\alpha \in K$, then $f\alpha \in L$, where $f\alpha = \{V \mid V \subset B$ and $f(C) \subset V$ for some $C \in \alpha\}$, and $\{f_i : (A, K) \to (B_i, L_i)\}$ is initial provided $\alpha \in K$ iff $f_i\alpha \in L_i, \forall i$. It is shown in [2] that an object $(A, K) \in conSCO$ is discrete iff $K = \{\alpha \mid \alpha \supset [b]$ for some $b \in A\}$. It is also shown there that $(A, K) \in$ pre $T_2(conSCO)$ iff (A, K) is indiscrete; i.e., $K = \{$all stacks on $A\}$. Clearly then, discrete objects in *conSCO* are not necessarily pre T_2. We note that [2] also proves that $(A, K) \in$ pre $T_2'(conSCO)$ iff A contains at most one point. Consequently, neither discrete nor indiscrete objects need be pre T_2' either. Moreover, since we can induce on a set B which has more than one element by a one-element set A via the unique constant map $c : B \to A$, *conSCO* also shows that the full subcategory of pre T_2' objects need not be topological.

4. R_0 and other objects via initial hulls

In this final section we give more examples of objects of possible interest which, similar to dimension zero objects, are definable in topological categories via induced closure. We leave the theoretical development of these objects as possible directions of further study.

In [16], PREUSS defines a topological space to be an R_0-space provided that $x \in \overline{\{y\}}$ (the closure of $\{y\}$) implies $y \in \overline{\{x\}}$ for all pairs of elements x, y in the space, and then proves that $R_0-\mathcal{TOP}$, the full subcategory of $\mathcal{TOP}$ consisting of all R_0-spaces, is isomorphic to the category of topological nearness spaces.

As noted above, the R_0-spaces are exactly the $T_{0,1}$ spaces. Consequently, by Proposition 7, $R_0-\mathcal{TOP} = IH(T_1-\mathcal{TOP})$. Similar to the pre T_2 condition in $\mathcal{TOP}$, the classical T_1 separation axiom has been generalized to arbitrary topological categories. Recall (see [2] or [3]) that if $\mathcal{B}$ is a category with finite powers and pushouts and $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is a topological functor with left adjoint $\mathcal{D} : \mathcal{B} \to \mathcal{E}$, then an $\mathcal{E}$-object e (with $\mathcal{U}(e) = b$) is T_1 provided the initial lift of $\{S : b^2 \bigvee b^2 \to b^3 = \mathcal{U}(e^3)$ and $\nabla : b^2 \bigvee b^2 \to b^2 = \mathcal{UD}(b^2)\}$ is discrete, where ∇ is the unique morphism such that $\nabla \circ i_1 = \nabla \circ i_2 = 1_{b^2}$ (the identity morphism on b^2). We denote the collection of all such T_1 objects in $\mathcal{E}$ by $T_1(\mathcal{E})$.

Now suppose that $\mathcal{B}$ is a category with finite powers and pullbacks and $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ is a topological functor. The discussion in the previous paragraph justifies the following.

Definition 17. e is an R_0-object in $\mathcal{E}$ provided $e \in IH(T_1(\mathcal{E}))$; i.e., there exists a family $\{e_i\}_{i \in I}$ of T_1 objects in $\mathcal{E}$ and a family of $\mathcal{B}$-morphisms $\{f_i : \mathcal{U}(e) = b \to b_i = \mathcal{U}(e_i)\}_{i \in I}$ whose initial lift is $\{e \xrightarrow{\overline{f_i}} e_i\}$.

Hausdorff separation has also been generalized to topological categories; indeed, many different characterizations of "Hausdorff objects" have appeared in the literature ([2], [3], [5]). All of these characterizations are equivalent to the T_2 axiom in $\mathcal{TOP}$, yet they may define distinct collections in the topological categories in which they are defined. Thus, each of these characterizations defines a class of Hausdorff objects whose initial hull could justifyably (by virtue of Proposition 7) be used to define a new type of pre-Hausdorff objects; i.e., if $\Gamma_2(\mathcal{E})$ denotes a collection of all Hausdorff objects in a topological category $\mathcal{E}$, then we define pre $\Gamma_2(\mathcal{E}) = IH(\Gamma_2(\mathcal{E}))$. We note that, as a consequence of Theorem 8, any of these collections pre $\Gamma_2(\mathcal{E})$ are reflective, initially closed (hence topological) subcategories of $\mathcal{E}$. As mentioned in the proof of Theorem 15, pre $T_2(\mathcal{E})$ is always topological, while Example 16 above shows that pre $T_2'(\mathcal{E})$ may not be topological. The fact that pre $T_2'(\mathcal{E}) \subseteq$ pre $T_2(\mathcal{E})$ (see [13]) makes pre $T_2(\mathcal{E})$ a candidate for being the initial hull of pre $T_2'(\mathcal{E})$. An open problem is to determine whether or not $IH(\text{pre } T_2'(\mathcal{E})) = \text{pre } T_2(\mathcal{E})$; if not, then find necessary and sufficient conditions on $\mathcal{E}$ which makes them coincide. Solution of this problem may lead to a solution of another open problem; namely, to characterize those topological categories in which pre $T_2'(\mathcal{E})$ is topological. There are many well-known topological categories, including $\mathcal{TOP}$, in which pre $T_2(\mathcal{E}) = $ pre $T_2'(\mathcal{E})$ (see [2] for several other examples), and in these cases, of course, pre $T_2'(\mathcal{E})$ will be topological. According to the following observation, the proof of which is a triviality, these may be the only categories in which pre $T_2'(\mathcal{E})$ is topological.

Remark 18. Suppose $IH(\operatorname{pre} T_2'(\mathcal{E})) = \operatorname{pre} T_2(\mathcal{E})$. Then $\operatorname{pre} T_2'(\mathcal{E})$ is topological iff $\operatorname{pre} T_2(\mathcal{E}) = \operatorname{pre} T_2'(\mathcal{E})$.

Since $\operatorname{pre} \Gamma_2(\mathcal{E})$, the new types of pre-Hausdorff objects defined above, are initially closed (hence topological), we note that $\operatorname{pre} T_2(\mathcal{E})$ in the preceeding remark could be replaced by any of the categories $\operatorname{pre} \Gamma_2(\mathcal{E})$ for which $\operatorname{pre} T_2'(\mathcal{E}) \subseteq \operatorname{pre} \Gamma_2(\mathcal{E})$.

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