

## Integral type operators between logarithmic Bloch-type space and $F(p, q, s)$ space on the unit ball

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**Abstract.** Let  $H(\mathbb{B}_N)$  be the space of all holomorphic functions on the unit ball  $\mathbb{B}_N$  in  $\mathbb{C}^N$ , and  $S(\mathbb{B}_N)$  the collection of all holomorphic self-maps of  $\mathbb{B}_N$ . Let  $\varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N)$ , the generalized composition operator is defined by

$$C_\varphi^g(f)(z) = \int_0^1 \Re f(\varphi(tz))g(tz) \frac{dt}{t},$$

and the product of composition and integral operators is defined by

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz) \frac{dt}{t}.$$

In this paper, we characterize the boundedness and compactness of these two integral operators, acting from the logarithmic Bloch-type space to  $F(p, q, s)$  space on the unit ball  $\mathbb{B}_N$ .

### 1. Introduction

Let  $\mathbb{B}_N$  be the unit ball in the  $N$ -dimensional complex space  $\mathbb{C}^N$ ,  $H(\mathbb{B}_N)$  the space of all holomorphic functions on  $\mathbb{B}_N$ , and  $S(\mathbb{B}_N)$  the collection of all

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holomorphic self-maps of  $\mathbb{B}_N$ . For  $f \in H(\mathbb{B}_N)$ , let

$$\Re f(z) = \langle \nabla f(z), \bar{z} \rangle = \sum_{j=1}^N z_j \frac{\partial f}{\partial z_j}(z)$$

be the radial derivative of  $f$ , where  $\nabla f = \left( \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_N} \right)$ .

For  $\alpha > 0, \beta \geq 0$ , the logarithmic weighted-type space  $H_{\log^\beta}^\alpha$  is the space of all  $f \in H(\mathbb{B}_N)$  such that

$$\|f\|_{H_{\log^\beta}^\alpha} = \sup_{z \in \mathbb{B}_N} |f(z)|(1 - |z|^2)^\alpha \left( \log \frac{e}{1 - |z|^2} \right)^\beta < \infty. \quad (1)$$

The logarithmic Bloch-type space  $\mathcal{B}_{\log^\beta}^\alpha$ , which was introduced in [14] and [15], consists of all  $f \in H(\mathbb{B}_N)$  such that

$$b_{\alpha, \beta}(f) = \sup_{z \in \mathbb{B}_N} |\Re f(z)|(1 - |z|^2)^\alpha \left( \log \frac{e}{1 - |z|^2} \right)^\beta < \infty.$$

The norm on the logarithmic Bloch-type space is defined as

$$\|f\|_{\mathcal{B}_{\log^\beta}^\alpha} = |f(0)| + \sup_{z \in \mathbb{B}_N} |\Re f(z)|(1 - |z|^2)^\alpha \left( \log \frac{e}{1 - |z|^2} \right)^\beta. \quad (2)$$

For  $a \in \mathbb{B}_N$ , let  $h(z, a) = \log |\varphi_a(z)|^{-1}$  be the Green's function on  $\mathbb{B}_N$  with logarithmic singularity at  $a$ , where  $\varphi_a$  is the Möbius transformation of  $\mathbb{B}_N$  with  $\varphi_a(0) = a, \varphi_a(a) = 0$  and  $\varphi_a = \varphi_a^{-1}$ .

Let  $0 < p, s < \infty, -n - 1 < q < \infty$  and  $q + s > -1$ . We say that  $f$  is a function in  $F(p, q, s)$  if  $f \in H(\mathbb{B}_N)$  and

$$\|f\|_{F(p, q, s)}^p = |f(0)|^p + \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} |\Re f(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) < \infty. \quad (3)$$

It is known that if  $q + s \leq -1$ , then  $F(p, q, s)$  is the space of constant functions.

Every  $\varphi \in S(\mathbb{B}_N)$  induces a composition operator  $C_\varphi$  defined by  $C_\varphi f = f \circ \varphi$  for  $f \in H(\mathbb{B}_N), z \in \mathbb{B}_N$ . It is of interest to provide function-theoretic characterizations for when  $\varphi$  induces a bounded or compact composition operator on various spaces. For some results on composition operators, see, e.g. [1] and the references therein.

Let  $g \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$ , where  $\mathbb{D}$  is the unit disk of  $\mathbb{C}$ . The generalized composition operator is defined by

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi$$

for  $z \in \mathbb{D}$  and  $f \in H(\mathbb{D})$ . When  $g = \varphi'$ , we see that this operator is essentially composition operator, since the following difference  $C_\varphi^g - C_\varphi$  is a constant. Therefore,  $C_\varphi^g$  is a generalization of the composition operator  $C_\varphi$ . The generalized composition operator was introduced in [2] and [7]. For related results and operators, see, e.g., [3], [4], [5] and the references therein.

Let  $\varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N)$  with  $g(0) = 0$ . The following operator, so called, generalized composition operator on the unit ball

$$(C_\varphi^g f)(z) = \int_0^1 \Re f(\varphi(tz))g(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}_N), z \in \mathbb{B}_N.$$

was recently introduced by S. STEVIĆ and X. ZHU and studied in [6], [8], [9], [10], [17], [18], [20], [23], [25], [26], [28].

Let  $g \in H(\mathbb{D})$  and  $\varphi \in S(\mathbb{D})$ . The product of integral and composition operators on  $H(\mathbb{D})$ , was introduced and studied by S. LI and S. STEVIĆ in [3] and [4]. The operator is defined as follows:

$$J_g C_\varphi f(z) = \int_0^z f(\varphi(\xi))g(\xi)d\xi. \tag{4}$$

In [11], S. STEVIĆ has extended the operator in (4) to the unit ball setting as follows: let  $\varphi \in S(\mathbb{B}_N)$ ,  $g \in H(\mathbb{B}_N)$  and  $g(0) = 0$ , the product of composition and integral operators in the unit ball  $\mathbb{B}_N$  is defined in this way:

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz) \frac{dt}{t} \tag{5}$$

for  $f \in H(\mathbb{B}_N)$ ,  $z \in \mathbb{B}_N$ .

If  $N = 1$ , then  $g \in H(\mathbb{D})$  and  $g(0) = 0$ , so that  $g(z) = zg_0(z)$ , for some  $g_0 \in H(\mathbb{D})$ . By the change of variable  $\xi = tz$ , it follows that

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))tzg_0(tz) \frac{dt}{t} = \int_0^z f(\varphi(\xi))g_0(\xi)d\xi.$$

Thus operator (5) is a natural extension of the operator  $J_g C_\varphi$  in (4). For some recent results on this operator, see, e.g. [11], [12], [13], [16], [19], [21] and so on.

Most of these papers are devoted to the discussions of the operators  $C_\varphi^g$  and  $P_\varphi^g$ , whose the image spaces are Bloch-type or weighted-type spaces, but there are few results if the element of the image space is given by an integral condition, such as Hardy space, Bergman space, Dirichlet space and so on. More recently, S. STEVIĆ [7], [13] characterized the equivalent conditions about the boundedness and compactness of  $C_\varphi^g$  and  $P_\varphi^g$  acting from logarithmic Bloch-type space to the mixed-norm space in the unit ball, respectively.

The present paper continues this and the line of research in [17], and characterizes the boundedness and compactness of  $C_\varphi^g$  and  $P_\varphi^g$  acting from logarithmic Bloch-type space to  $F(p, q, s)$  space. The paper is organized as follows: Some necessary lemmas will be presented in Section 2. Sections 3 and 4 are devoted to characterizing the conditions about the boundedness and compactness of the operators  $C_\varphi^g$  and  $P_\varphi^g$  acting from logarithmic Bloch-type space to  $F(p, q, s)$  space in the unit ball respectively.

Throughout the remainder of this paper,  $C$  will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation  $a \preceq b$  means that there is a positive constant  $C$  such that  $a \leq Cb$ . We say  $a \asymp b$ , if both  $a \preceq b$  and  $b \preceq a$  hold.

## 2. Auxiliary results

In this section we present several auxiliary results which will be used in the proofs of some results in the next sections.

**Lemma 2.1** (Lemma 4, [16]). *Let  $f \in B_{\log^\beta}^\alpha$ . Then*

$$|f(z)| \leq C \begin{cases} \|f\|_{\mathcal{B}_{\log^\beta}^\alpha} & \alpha \in (0, 1) \text{ or } \alpha = 1, \beta > 1 \\ |f(0)| + b_{\alpha, \beta}(f) \max \left\{ 1, \log \log \frac{e^{\alpha/\beta}}{1 - |z|} \right\} & \alpha = \beta = 1 \\ |f(0)| + b_{\alpha, \beta}(f) \left( \log \frac{e^{\alpha/\beta}}{1 - |z|} \right)^{1-\beta} & \alpha = 1, \beta \in (0, 1) \\ |f(0)| + \frac{b_{\alpha, \beta}(f)}{(1 - |z|)^{\alpha-1} \left( \log \frac{e^{\alpha/\beta}}{1 - |z|} \right)^\beta} & \alpha > 1, \beta \geq 0 \end{cases}$$

for some  $C > 0$  independent of  $f$ .

**Lemma 2.2** (Lemma 6, [13]). Assume  $\alpha > 1$  and  $\beta \geq 0$ . Then there exist  $n = n(N) \in \mathbb{N}$  and functions  $f_1, f_2, \dots, f_n \in \mathcal{B}_{\log \beta}^\alpha$  such that

$$|f_1(z)| + |f_2(z)| + \dots + |f_n(z)| \geq \frac{C}{(1 - |z|^2)^{\alpha-1} \left(\log \frac{e}{1-|z|^2}\right)^\beta}, \quad z \in \mathbb{B}_N,$$

where  $C$  is a positive constant.

**Lemma 2.3.** Assume  $\alpha > 1$  and  $\beta \geq 0$ . Let  $f_m, m \in \{1, 2, \dots, n\}$  be the functions satisfying the conditions in Lemma 2.2. Then the function sequence  $\{F_k^{\{m,l\}}\}_{k=0}^\infty$  such that  $F_k^{\{m,l\}}(z) = z_l^k f_m(z)$  is a bounded sequence in the space  $\mathcal{B}_{\log \beta}^\alpha$ , and it converges to 0 uniformly on compact subsets of  $\mathbb{B}_N$  as  $k \rightarrow \infty$ . Here  $z_l$  is  $l$ -th component of  $z \in \mathbb{B}_N$ .

PROOF. We have

$$\begin{aligned} \|F_k^{\{m,l\}}\|_{\mathcal{B}_{\log \beta}^\alpha} &= \sup_{z \in \mathbb{B}_N} (1 - |z|^2)^\alpha \left(\log \frac{e}{1 - |z|^2}\right)^\beta |\Re(z_l^k f_m(z))| \\ &\leq \sup_{z \in \mathbb{B}_N} (1 - |z|^2)^\alpha \left(\log \frac{e}{1 - |z|^2}\right)^\beta (|\Re f_m(z)| + |k z_l^k f_m(z)|) \\ &\leq b_{\alpha,\beta}(f_m) + C_2 \sup_{z \in \mathbb{B}_N} \frac{(1 - |z|^2)^\alpha \left(\log \frac{e}{1 - |z|^2}\right)^\beta b_{\alpha,\beta}(f_m)}{(1 - |z|)^{\alpha-1} \left(\log \frac{e^{\alpha/\beta}}{1 - |z|}\right)^\beta} \cdot |k z_l^k| \\ &\leq \|f_m\|_{\mathcal{B}_{\log \beta}^\alpha} + C_3 \sup_{z_l \in \mathbb{D}} k(1 - |z_l|) |z_l|^k \|f_m\|_{\mathcal{B}_{\log \beta}^\alpha}. \end{aligned}$$

Let  $h(x) = k(1 - x)x^k, x \in (0, 1), k \geq 1$ . Then  $h'(x) = kx^{k-1}[k - (k + 1)x]$ , it is easy to show that  $h(x) \leq h(\frac{k}{k+1}) = (\frac{k}{k+1})^{k+1}$ , so there is a positive constant  $M$ , such that  $\|F_k^{\{m,l\}}\|_{\mathcal{B}_{\log \beta}^\alpha} \leq M$ .

Since for a compact subset  $K$  of  $\mathbb{B}_N$ , there is a positive constant  $r$ , such that  $|z| < r < 1$  when  $z \in K$ . By Lemma 2.1 and Lemma 2.2, it follows that

$$\begin{aligned} |F_k^{\{m,l\}}(z)| &\leq |z_l^k f_m(z)| \\ &\leq M \frac{b_{\alpha,\beta}(f)}{(1 - |z|)^{\alpha-1} \left(\log \frac{e^{\alpha/\beta}}{1 - |z|}\right)^\beta} |z_l^k| \leq M \frac{r^k}{\inf_{0 \leq t \leq r} (1 - t)^{\alpha-1} \left(\log \frac{e^{\alpha/\beta}}{1 - t}\right)^\beta}, \end{aligned}$$

from which it is obvious that the sequence  $\{F_k^{\{m,l\}}\}_{k=0}^\infty$  converges to 0 uniformly on compact subsets of  $\mathbb{B}_N$  as  $k \rightarrow \infty$ . This ends the proof of this lemma.  $\square$

The following lemma was obtained by STEVO STEVIĆ in [14].

**Lemma 2.4** (Lemma 7, [14]). Assume  $\alpha > 0$  and  $\beta \geq 0$ . Then there exist  $n = n(N) \in \mathbb{N}$  and functions  $f_1, f_2, \dots, f_n \in H_{\log^\beta}^\alpha$  such that

$$|f_1(z)| + \dots + |f_n(z)| \geq \frac{C}{(1 - |z|^2)^\alpha \left(\log \frac{e}{1 - |z|^2}\right)^\beta}, \quad z \in \mathbb{B}_N,$$

where  $C$  is a positive constant.

By the above lemma and Lemma 10 in [17], we have the following result.

**Corollary 2.1.** Assume  $\alpha > 0$ ,  $\beta \geq 0$  and  $r \in (0, 1)$ . Then there exist  $n = n(N) \in \mathbb{N}$  and functions  $F_1, F_2, \dots, F_n \in \mathcal{B}_{\log^\beta}^\alpha$  such that

$$|\Re F_1(z)| + \dots + |\Re F_n(z)| \geq \frac{C}{(1 - |z|^2)^\alpha \left(\log \frac{e}{1 - |z|^2}\right)^\beta},$$

for  $|z| > r$ , where  $C$  is a positive constant depending only on  $r$ .

According to the lemmas in [8], [10], [12], we have the following lemma.

**Lemma 2.5.** Assume that  $\varphi \in S(\mathbb{B}_N)$ , and  $g \in H(\mathbb{B}_N)$  with  $g(0) = 0$ . Then for every  $f \in H(\mathbb{B}_N)$  it holds

$$\Re C_\varphi^g(f)(z) = \Re f(\varphi(z))g(z)$$

and

$$\Re P_\varphi^g(f)(z) = f(\varphi(z))g(z).$$

**Lemma 2.6.** Assume  $\alpha > 0$ ,  $\beta \geq 0$ . Let  $\{f_m\}_{m=1}^n$  be the functions in Lemma 2.4. Then there is a bounded sequence  $\{F_k\}_{k=1}^\infty$  in  $\mathcal{B}_{\log^\beta}^\alpha$  such that  $\Re F_k(z) = z_l^k f_m(z)$ ,  $z_l$  is  $l$ -th component of  $z \in \mathbb{B}_N$ , and it converges to 0 uniformly on compact subsets of  $\mathbb{B}_N$  as  $k \rightarrow \infty$ .

PROOF. For fixed  $m$  and  $l$ , let  $F_k^{\{m,l\}}(z) = \int_0^1 \frac{z_l^k t^k f_m(tz)}{t} dt$  ( $k = 1, 2, \dots$ ). It is easy to see that  $F_k^{\{m,l\}}(0) = 0$  and that by Lemma 2.5,  $\Re F_k^{\{m,l\}}(z) = z_l^k f_m(z)$ . Therefore,

$$\|F_k^{\{m,l\}}(z)\|_{\mathcal{B}_{\log^\beta}^\alpha} = \sup_{z \in \mathbb{B}_N} (1 - |z|^2)^\alpha \left(\log \frac{e}{1 - |z|^2}\right)^\beta |z_l^k f_m(z)| \leq \|f_m\|_{H_{\log^\beta}^\alpha} < \infty.$$

For a compact subset  $K$  of  $\mathbb{B}_N$ , there is a positive constant  $r$ , such that  $|z| < r < 1$  when  $z \in K$ . By Lemma 2.1 and Lemma 2.4, it follows that

$$|F_{k,l}^{\{m\}}(z)| \leq r^k \int_0^1 \frac{t^k |f_m(tz)|}{t} dt \leq M r^k \|f_m\|_{H_{\log^\beta}^\alpha}$$

for  $z \in K$ . From which we can easily get that  $F_k^{\{m,l\}}$  converges to 0 on compact subset of  $\mathbb{B}_N$  as  $k \rightarrow \infty$ . □

**Lemma 2.7** (Lemma 2.1, [24]). *For  $0 < p, s < \infty, -p - 1 < q < \infty, q + s > -1$ , if  $f \in F(p, q, s)$ , then  $f \in \mathcal{B}^{\frac{n+1+q}{p}}$  and*

$$\|f\|_{\mathcal{B}^{\frac{n+1+q}{p}}} < C\|f\|_{F(p,q,s)}.$$

The next lemma is well-known.

**Lemma 2.8.** *For  $0 < p < \infty$ , there is a positive constant  $C_p$  depending on  $p$  and  $n$ , such that  $(\sum_{i=1}^n x_i)^p \leq C_p(\sum_{i=1}^n x_i^p)$ , when  $x_i \in (0, \infty), i \in \{1, 2, \dots, n\}$ .*

The following lemma can be proved in a standard way (see, for example, Proposition 3.11 in [1]), we omit its proof.

**Lemma 2.9.** *Suppose that  $\varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N), g(0) = 0$ . Then the operator  $C_\varphi^g : \mathcal{B}_{\log^\beta}^\alpha \rightarrow F(p, q, s)$  is compact if and only if for any bounded sequence  $\{f_k\}_{k \in \mathbb{N}}$  in  $\mathcal{B}_{\log^\beta}^\alpha$  which converges to zero uniformly on compact subsets of  $\mathbb{B}_N$ , we have  $\|C_\varphi^g f_k\|_{F(p,q,s)}^p \rightarrow 0$  as  $k \rightarrow \infty$ .*

Lemma 2.9 also holds if operator  $C_\varphi^g$  is replace by  $P_\varphi^g$ .

### 3. Boundedness and compactness of $C_\varphi^g : \mathcal{B}_{\log^\beta}^\alpha \rightarrow F(p, q, s)$

In this section we study the boundedness and compactness of the operator  $C_\varphi^g : \mathcal{B}_{\log^\beta}^\alpha \rightarrow F(p, q, s)$ .

**Theorem 3.1.** *Suppose that  $\alpha > 0, \beta \geq 0, \varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N), g(0) = 0$ . Then  $C_\varphi^g$  is bounded from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if and only if*

$$\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |\varphi(z)g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) < \infty. \tag{6}$$

PROOF. Suppose that (6) holds, we prove  $C_\varphi^g$  is bounded. By Lemma 2.5 and the equivalence of norm on  $\mathcal{B}_{\log^\beta}^\alpha$ , it follows that there is a constant  $M$  such that

$$\begin{aligned} \|C_\varphi^g f\|_{F(p,q,s)}^p &= \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |\Re f(\varphi(z))g(z)|^p h^s(z, a) dv(z) \\ &\leq \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |\varphi(z)g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} \\ &\quad \cdot \left| (1 - |\varphi(z)|^2)^\alpha \left( \log \frac{e}{1 - |\varphi(z)|^2} \right)^\beta \nabla f(\varphi(z)) \right|^p dv(z) \end{aligned}$$

$$\leq C \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |\varphi(z)g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) \cdot \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}^p \leq M \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}^p.$$

So we have the boundedness of  $C_\varphi^g$ .

Conversely, we assume that there exists a positive constant  $M$  such that  $\|C_\varphi^g f\|_{F(p,q,s)} \leq M \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}$  for every  $f \in \mathcal{B}_{\log^\beta}^\alpha$ . For some  $r_0 \in (0, 1)$ , by Corollary 2.1, we can find  $f_1, \dots, f_n$  in  $\mathcal{B}_{\log^\beta}^\alpha$  such that

$$\sum_{k=1}^n |\Re f_k(z)| \geq \frac{C}{(1 - |z|^2)^\alpha (\log \frac{e}{1 - |z|^2})^\beta}, \quad |z| > r_0.$$

Then using Lemma 2.8, there are two constants  $C_p$  and  $C$  such that

$$\begin{aligned} \infty &> \sum_{k=1}^n \|C_\varphi^g f_k\|_{F(p,q,s)}^p = \sum_{k=1}^n \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |\Re f_k(\varphi(z))|^p dv(z) \\ &\geq \sum_{k=1}^n \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |\Re f_k(\varphi(z))|^p dv(z) \\ &\geq \frac{1}{C_p} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} (1 - |z|^2)^q |g(z)|^p h^s(z, a) \left( \sum_{k=1}^n |\Re f_k(\varphi(z))| \right)^p dv(z) \\ &\geq C \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) \\ &\geq C \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} \frac{(1 - |z|^2)^q |\varphi(z)g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z). \end{aligned}$$

When  $|\varphi(z)| \leq r_0$ , put  $G_j(z) = z_j$  for  $j = 1, 2, \dots, N$ , it is obvious that  $G_j(z) \in \mathcal{B}_{\log^\beta}^\alpha$ . Since  $|\varphi(z)|^2 = |\varphi_1(z)|^2 + \dots + |\varphi_N(z)|^2 \leq (|\varphi_1(z)| + \dots + |\varphi_N(z)|)^2$  and by Lemma 2.8, we have

$$\begin{aligned} &\sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| \leq r_0} \frac{(1 - |z|^2)^q |\varphi(z)g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) \\ &\leq \sup_{a \in \mathbb{B}_N} \sup_{0 \leq r \leq r_0} \left[ (1 - r^2)^\alpha \left( \log \frac{e}{1 - r^2} \right)^\beta \right]^{-p} \\ &\quad \cdot \int_{|\varphi(z)| \leq r_0} (1 - |z|^2)^q |\varphi(z)g(z)|^p h^s(z, a) dv(z) \\ &\leq C \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| \leq r_0} (1 - |z|^2)^q \left| \sum_{j=1}^N \varphi_j(z) \right|^p |g(z)|^p h^s(z, a) dv(z) \end{aligned}$$

$$\begin{aligned} &\leq C \sum_{j=1}^N \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| \leq r_0} (1 - |z|^2)^q |\varphi_j(z)g(z)|^p h^s(z, a) dv(z) \\ &\leq C \sum_{j=1}^N \|C_\varphi^g G_j\|_{F(p,q,s)}^p < \infty. \end{aligned}$$

Thus we get (6). □

Now we characterize the compactness of the operator.

**Theorem 3.2.** *Suppose that  $\alpha > 0$ ,  $\beta \geq 0$ ,  $\varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N)$ ,  $g(0) = 0$ . Then  $C_\varphi^g$  is compact from  $\mathcal{B}_{\log \beta}^\alpha$  to  $F(p, q, s)$  if and only if (6) holds and*

$$\lim_{r \rightarrow 1} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) = 0. \tag{7}$$

PROOF. First, assume that  $C_\varphi^g$  is compact, then it is bounded, and (6) follows by Theorem 3.1.

Now we prove (7). Consider the test functions  $F_k^{\{m,l\}}$  in Lemma 2.6.

Write  $\varphi = (\varphi_1, \dots, \varphi_N)$ , since  $C_\varphi^g$  is compact, by Lemma 2.9, it follows that as  $k \rightarrow \infty$ ,

$$\begin{aligned} &\|C_\varphi^g F_k^{\{m,l\}}\|_{F(p,q,s)}^p \\ &= \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} |\varphi_l(z)|^{kp} |f_m(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \rightarrow 0. \end{aligned} \tag{8}$$

Note that  $|\varphi(z)| \leq |\varphi_1(z)| + \dots + |\varphi_N(z)|$ , by the relation (8) and Lemma 2.8, we have

$$\begin{aligned} &\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} |\varphi(z)|^{kp} |f_m(\varphi(z))g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \\ &\leq \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \left( \sum_{l=1}^n |\varphi_l(z)| \right)^{kp} |f_m(\varphi(z))g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \\ &\leq C \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \left( \sum_{l=1}^n |\varphi_l(z)|^{kp} \right) |f_m(\varphi(z))g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \rightarrow 0, \quad k \rightarrow \infty \end{aligned}$$

for  $m \in \{1, 2, \dots, n\}$ .

Since  $(1 - |z|^2)^\alpha (\log \frac{e}{1 - |z|^2})^\beta \rightarrow 0$  as  $|z| \rightarrow 1$ , this means that for every  $\varepsilon > 0$ , there is a  $k_0 \in \mathbb{N}$  such that for every  $r \in (0, 1)$ ,

$$r^{k_0 p} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{|g(z)|^p (1 - |z|^2)^q h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z)$$

$$\begin{aligned}
 &\leq \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{r^{k_0 p} |g(z)|^p (1 - |z|^2)^q h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) \\
 &\leq C \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} |\varphi(z)|^{k_0 p} \left( \sum_{m=1}^n |f_m(\varphi(z))| \right)^p |g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \\
 &\leq C_p \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} |\varphi(z)|^{k_0 p} \left( \sum_{m=1}^n |f_m(\varphi(z))|^p \right) |g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \\
 &\leq C_p \sum_{m=1}^n \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} |\varphi(z)|^{k_0 p} |f_m(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) < \varepsilon.
 \end{aligned}$$

Thus when  $r > 2^{-\frac{1}{k_0 p}}$ , by the above inequality we obtain

$$\sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) < 2\varepsilon.$$

From which (7) follows.

Conversely, suppose that (7) holds, then for  $\varepsilon > 0$ , we can find a constant  $r_0 \in (0, 1)$  such that

$$\sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) < \varepsilon. \tag{9}$$

Let  $\{f_k\}_{k \in \mathbb{N}}$  be a bounded sequence in  $\mathcal{B}_{\log^\beta}^\alpha$  with

$$\|f_k\|_{\mathcal{B}_{\log^\beta}^\alpha} \leq M, \quad k \in \mathbb{N},$$

and  $f_k \rightarrow 0$  uniformly on any compact subset of  $\mathbb{B}_N$  as  $k \rightarrow \infty$ . Then

$$\begin{aligned}
 \|C_\varphi^g f_k\|_{F(p,q,s)}^p &= \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q h^s(z, a) |\Re f_k(\varphi(z)) g(z)|^p dv(z) \\
 &\leq \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| \leq r_0} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |\Re f_k(\varphi(z))|^p dv(z) \\
 &\quad + \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |\Re f_k(\varphi(z))|^p dv(z) \\
 &\leq M_k^p(r_0) \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| \leq r_0} \frac{(1 - |z|^2)^q |\varphi(z) g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) \\
 &\quad + \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) \cdot \|f_k\|_{\mathcal{B}_{\log^\beta}^\alpha}^p \\
 &= I_1(k) + I_2(k)
 \end{aligned}$$

here  $M_k(r_0) = \sup_{z \in r_0 \mathbb{B}_N} \left\{ (1 - |z|^2)^\alpha \left( \log \frac{e}{1 - |z|^2} \right)^\beta \left| \frac{\Re f_k(z)}{z} \right| \right\}$ .

Since  $f_k \rightarrow 0$  as  $k \rightarrow \infty$ , uniformly on compact subset of  $\mathbb{B}_N$ ,  $M_k(r_0) < \left( \log \frac{e}{1 - r_0} \right)^\beta \sup_{\{|\varphi(z)| \leq r_0\}} |\nabla f_k(\varphi(z))|$ , it follows from (6) and the Weierstrass theorem that  $I_1(k) \rightarrow 0$  as  $k \rightarrow \infty$ .

Since  $\|f_k\|_{\mathcal{B}_{\log^\beta}^\alpha} \leq M$ , it follows from (9) that  $I_2(k) < \varepsilon M^p$ . Then, when  $k \rightarrow \infty$ ,  $\|C_\varphi^g f_k\|_{F(p,q,s)}^p \rightarrow 0$ , hence by Lemma 2.9, we get that  $C_\varphi^g$  is compact. This completes the proof of this theorem.  $\square$

#### 4. Boundedness and compactness of $P_\varphi^g : \mathcal{B}_{\log^\beta}^\alpha \rightarrow F(p, q, s)$

In this section we characterize the boundedness and compactness of  $P_\varphi^g$  according to the four cases depending on the choice of the parameters of  $\alpha$  and  $\beta$ .

##### 4.1. Case $\alpha > 1$ .

**Theorem 4.1.** *Suppose that  $\alpha > 1$ ,  $\beta \geq 0$ ,  $\varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N)$ ,  $g(0) = 0$ . Then  $P_\varphi^g$  is bounded from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if and only if*

$$\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} \left( \log \frac{e}{1 - |\varphi(z)|^2} \right)^\beta]^p} dv(z) < \infty. \tag{10}$$

PROOF. First, we assume that (10) holds. Then by Lemma 2.1 and Lemma 2.5, we have

$$\begin{aligned} \|P_\varphi^g f\|_{F(p,q,s)}^p &= \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |f(\varphi(z))|^p dv(z) \\ &\leq \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |g(z)|^p h^s(z, a) \cdot \left( \frac{C \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}}{(1 - |\varphi(z)|^2)^{\alpha-1} \left( \log \frac{e}{1 - |\varphi(z)|^2} \right)^\beta} \right)^p dv(z) \\ &\asymp \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} \left( \log \frac{e}{1 - |\varphi(z)|^2} \right)^\beta]^p} dv(z) \cdot \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}^p \leq C \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}^p. \end{aligned}$$

From which the boundedness of  $P_\varphi^g$  follows.

For the converse direction, we suppose that  $P_\varphi^g$  is bounded. By Lemma 2.2, we can find  $f_1, f_2, \dots, f_n$  such that

$$\sum_{i=1}^n \|P_\varphi^g f_i\|_{F(p,q,s)}^p < \infty$$

and

$$\sum_{i=1}^n |f_i(z)| \geq \frac{C}{(1 - |z|^2)^{\alpha-1} \left(\log \frac{e}{1-|z|^2}\right)^\beta}.$$

By some elementary inequalities and Lemma 2.8, we obtain

$$\begin{aligned} \infty &> \sum_{i=1}^n \|P_\varphi^g f_i\|_{F(p,q,s)}^p \\ &= \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |g(z)|^p h^s(z, a) \left( \sum_{i=1}^n |f_i(\varphi(z))|^p \right) dv(z) \\ &\geq \frac{1}{C_p} \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |g(z)|^p h^s(z, a) \left( \sum_{i=1}^n |f_i(\varphi(z))| \right)^p dv(z) \\ &\geq \frac{1}{C_p} \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} \left(\log \frac{e}{1-|\varphi(z)|^2}\right)^\beta]^p} dv(z). \end{aligned}$$

The proof of this theorem is completed. □

**Theorem 4.2.** *Let  $\alpha > 1, \beta \geq 0$ . Suppose that  $\varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N)$ ,  $g(0) = 0$ . Then  $P_\varphi^g$  is compact from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if and only if (10) holds and*

$$\lim_{r \rightarrow 1} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} \left(\log \frac{e}{1-|\varphi(z)|^2}\right)^\beta]^p} dv(z) = 0. \tag{11}$$

PROOF. First, assume that  $P_\varphi^g$  is compact, then it is obviously bounded, and the condition in (10) follows by Theorem 4.1.

Next we prove (11). Setting the test functions  $F_k^{\{m,l\}}$  in Lemma 2.3.

$$\begin{aligned} \|P_\varphi^g F_k^{\{m,l\}}\|_{F(p,q,s)}^p &= \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} |\varphi_l(z)|^{kp} |f_m(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \rightarrow 0 \end{aligned}$$

when  $k \rightarrow \infty$ , here  $\varphi = (\varphi_1, \dots, \varphi_N)$ .

From which and by Lemma 2.8, we have

$$\begin{aligned} &\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} |\varphi_l(z)|^{kp} |f_m(\varphi(z))g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \\ &\leq C \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \left( \sum_{l=1}^n |\varphi_l(z)|^{kp} \right) |f_m(\varphi(z))g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) \rightarrow 0, \end{aligned}$$

as  $k \rightarrow \infty$  for  $m \in \{1, 2, \dots, n\}$ .

For every  $\varepsilon > 0$ , there is a  $k_0 \in \mathbb{N}$  such that for every  $r \in (0, 1)$ ,

$$\begin{aligned} & r^{k_0 p} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{|g(z)|^p (1 - |z|^2)^q h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} (\log \frac{e}{1-|\varphi(z)|^2})^\beta]^p} dv(z) \\ & \leq C_p \sum_{m=1}^n \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} |\varphi(z)|^{k_0 p} |f_m(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) < \varepsilon. \end{aligned}$$

Thus when  $r > 2^{-\frac{1}{k_0 p}}$ , by the above inequality we obtain (11).

Conversely, suppose that (11) holds. Then for every  $\varepsilon > 0$ , we can find an  $r_0 \in (0, 1)$  such that

$$\sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} (\log \frac{e}{1-|\varphi(z)|^2})^\beta]^p} dv(z) < \varepsilon. \tag{12}$$

Let  $\{f_k\}_{k \in \mathbb{N}}$  be a bounded sequence in  $\mathcal{B}_{\log^\beta}^\alpha$  with

$$\|f_k\|_{\mathcal{B}_{\log^\beta}^\alpha} \leq M, \quad k \in \mathbb{N},$$

and  $f_k \rightarrow 0$  uniformly on any compact subset of  $\mathbb{B}_N$  as  $k \rightarrow \infty$ . Then by Lemma 2.1,

$$\begin{aligned} \|P_\varphi^g f_k\|_{F(p,q,s)}^p &= \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |f_k(\varphi(z))|^p dv(z) \\ &\leq \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| \leq r_0} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |f_k(\varphi(z))|^p dv(z) \\ &\quad + \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |f_k(\varphi(z))|^p dv(z) \\ &\leq M_k^p(r_0) \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| \leq r_0} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} (\log \frac{e}{1-|\varphi(z)|^2})^\beta]^p} dv(z) \\ &\quad + C \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r_0} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} (\log \frac{e}{1-|\varphi(z)|^2})^\beta]^p} dv(z) \cdot \|f_k\|_{\mathcal{B}_{\log^\beta}^\alpha}^p \\ &= I_1(k) + I_2(k), \end{aligned}$$

here  $M_k(r_0) = \sup_{z \in r_0 \mathbb{B}_N} \{(1 - |z|^2)^{\alpha-1} (\log \frac{e}{1-|z|^2})^\beta |f_k(z)|\}$ .

Since  $f_k \rightarrow 0$  as  $k \rightarrow \infty$ , uniformly on compact subset of  $\mathbb{B}_N$ , so  $M_k(r_0) < (\log \frac{e}{1-r_0})^\beta \sup_{\{|\varphi(z)| \leq r_0\}} |f_k(\varphi(z))|$ , by (10), we get  $I_1(k) \rightarrow 0$  as  $k \rightarrow \infty$ .

Note that  $\|f_k\|_{\mathcal{B}_{\log^\beta}^\alpha} \leq M$  and by (12), we obtain  $I_2(k) < CM^p \varepsilon$ . Thus,  $\|P_\varphi^g f_k\|_{F(p,q,s)}^p \rightarrow 0$  as  $k \rightarrow \infty$ , and from Lemma 2.9 the compactness of  $P_\varphi^g$  follows. So the proof of this theorem is complete.  $\square$

**4.2. Case  $\alpha \in (0, 1)$ , or  $\alpha = 1$  and  $\beta > 1$ .** Because when  $\alpha \in (0, 1)$ , or  $\alpha = 1$  and  $\beta > 1$ , the condition (6) in [17] holds, the results can be consulted in [17].

**4.3. Case  $\alpha = 1$  and  $\beta \in (0, 1)$ .**

**Theorem 4.3.** Assume  $\alpha = 1$  and  $\beta \in (0, 1)$ ,  $\varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N)$ ,  $g(0) = 0$ . Then

(i)  $P_\varphi^g$  is bounded from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if

$$\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{\left(\log \frac{e}{1 - |\varphi(z)|^2}\right)^{p(1-\beta)}} dv(z) < \infty. \tag{13}$$

(ii)  $P_\varphi^g$  is compact from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if (13) holds and

$$\lim_{r \rightarrow 1} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{\left(\log \frac{e}{1 - |\varphi(z)|^2}\right)^{p(1-\beta)}} dv(z) = 0.$$

PROOF. Assume that (13) holds. If  $f \in \mathcal{B}_{\log^\beta}^\alpha$ , then by Lemma 2.1 and Lemma 2.5, we have

$$\begin{aligned} \|P_\varphi^g f\|_{F(p,q,s)}^p &= \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |g(z)|^p h^s(z, a) |f(\varphi(z))|^p dv(z) \\ &\leq \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} (1 - |z|^2)^q |g(z)|^p h^s(z, a) \left( \frac{C \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}}{\left(\log \frac{e^{\alpha/\beta}}{1 - |\varphi(z)|}\right)^{1-\beta}} \right)^p dv(z) \\ &\asymp \sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{\left(\log \frac{e}{1 - |\varphi(z)|^2}\right)^{p(1-\beta)}} dv(z) \cdot \|f\|_{\mathcal{B}_{\log^\beta}^\alpha}^p. \end{aligned}$$

Thus we get the boundedness. The proof of compactness is also similar to that of Theorem 4.2, so it is omitted.  $\square$

**4.4. Case  $\alpha = \beta = 1$ .** By using the same methods as in the proofs of the previous theorems, we can prove the next theorem.

**Theorem 4.4.** Assume  $\alpha = \beta = 1$ ,  $\varphi \in S(\mathbb{B}_N)$  and  $g \in H(\mathbb{B}_N)$ ,  $g(0) = 0$ . Then

(i)  $P_\varphi^g$  is bounded from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if

$$\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{\left(\max\{1, \log \log \frac{e}{1 - |\varphi(z)|^2}\}\right)^p} dv(z) < \infty. \tag{14}$$

(ii)  $P_\varphi^g$  is compact from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if (14) holds and

$$\lim_{r \rightarrow 1} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q |g(z)|^p h^s(z, a)}{\left(\max\{1, \log \log \frac{e}{1 - |\varphi(z)|^2}\}\right)^p} dv(z) = 0.$$

5. Other two operators

If we use the radial derivative of some function  $k$  to instead of  $g$  in operators  $P_\varphi^g$  and  $C_\varphi^g$ , we can get the following two operators:

$$(L_\varphi^k f)(z) = \int_0^1 \Re f(\varphi(tz)) \Re k(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}_N), \quad z \in \mathbb{B}_N,$$

and

$$(V_\varphi^k f)(z) = \int_0^1 f(\varphi(tz)) \Re k(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}_N), \quad z \in \mathbb{B}_N.$$

The operator  $V_\varphi^k$  is called Volterra composition operator and studied in [22], [25], [27], [29]. According the previous sections, we can obtain the results about the operators at once, here we just list partial results.

**Theorem 5.1.** *Suppose that  $\alpha > 0, \beta \geq 0, \varphi \in S(\mathbb{B}_N)$  and  $k \in H(\mathbb{B}_N)$ . Then  $L_\varphi^k$  is bounded from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if and only if*

$$\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |\varphi(z) \Re k(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) < \infty. \tag{15}$$

**Theorem 5.2.** *Suppose that  $\alpha > 0, \beta \geq 0, \varphi \in S(\mathbb{B}_N)$  and  $k \in H(\mathbb{B}_N)$ . Then  $L_\varphi^k$  is compact from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if and only if (15) holds and*

$$\lim_{r \rightarrow 1} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q |\Re k(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^\alpha (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) = 0. \tag{16}$$

**Theorem 5.3.** *Suppose that  $\alpha > 1, \beta \geq 0, \varphi \in S(\mathbb{B}_N)$  and  $k \in H(\mathbb{B}_N)$ . Then  $V_\varphi^k$  is bounded from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if and only if*

$$\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} \frac{(1 - |z|^2)^q |\Re k(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) < \infty. \tag{17}$$

**Theorem 5.4.** *Let  $\alpha > 1, \beta \geq 0$ . Suppose that  $\varphi \in S(\mathbb{B}_N)$  and  $k \in H(\mathbb{B}_N)$ . Then  $V_\varphi^k$  is compact from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if and only if (17) holds and*

$$\lim_{r \rightarrow 1} \sup_{a \in \mathbb{B}_N} \int_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q |\Re k(z)|^p h^s(z, a)}{[(1 - |\varphi(z)|^2)^{\alpha-1} (\log \frac{e}{1 - |\varphi(z)|^2})^\beta]^p} dv(z) = 0. \tag{18}$$

**Theorem 5.5.** *Assume  $\alpha \in (0, 1)$ , or  $\alpha = 1$  and  $\beta > 1, \varphi \in S(\mathbb{B}_N)$  and  $k \in H(\mathbb{B}_N)$ . Then  $V_\varphi^k$  is bounded from  $\mathcal{B}_{\log^\beta}^\alpha$  to  $F(p, q, s)$  if and only if*

$$\sup_{a \in \mathbb{B}_N} \int_{\mathbb{B}_N} |\Re k(z)|^p (1 - |z|^2)^q h^s(z, a) dv(z) < \infty. \tag{19}$$

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