Publ. Math. Debrecen
82/2 (2013), 425–438
DOI: 10.5486/PMD.2013.5342

Common fixed points of Ćirić-type contractions on partial metric spaces

By MUJAHID ABBAS (Lahore), ISHAK ALTUN (Kirikkale)
and SALVADOR ROMAGUERA (Valencia)

Abstract. We obtain a common fixed point theorem of Boyd–Wong type for four mappings satisfying a Ćirić-type contraction on a complete partial metric space. Our result generalizes and unifies, among others, the very recent results of L. Ćirić, B. Samet, H. Aydi and C. Vetro [Common fixed points of generalized contractions on partial metric spaces and an application, Appl. Math. Comput., 218 (2011), 2398-2406], S. Romaguera [Fixed point theorems for generalized contractions on partial metric spaces, Topology Appl., 159 (2012), 194-199], T. Abdeljawad, E. Karapinar and K. Tas [Existence and uniqueness of a common fixed point on partial metric spaces, Appl. Math. Lett. 24 (2011), 1900–1904], and D. Ilić, V. Pavlović and V. Rakočević [Some new extensions of Banach's contraction principle to partial metric space, Appl. Math. Lett. 24 (2011), 1326–1330].

1. Introduction and preliminaries

In his celebrated paper [6], Ćirić introduced a general kind of contractions on metric spaces which he called λ-generalized contractions. Since then, many authors have introduced and discussed several extensions and variants of such contractions, usually called now Ćirić-type contractions, and have obtained, in this way, a lot of fixed point theorems for complete metric spaces (see e.g. [4], [7], [15], [28] for some recent contributions in this direction). Related to our work we

---

Mathematics Subject Classification: 54H25, 47H10, 54E50.
Key words and phrases: common fixed point, coincidence point, weakly compatible pair, Ćirić-type contraction, complete partial metric space.
The third named author is supported by the Ministry of Science and Innovation of Spain, grant MTM2009-12872-C02-01.

also mention the very recent paper [12], which was pointed out to the authors by one of the referees.

On the other hand, it is now highly recognized that partial metric spaces provide an efficient tool both in constructing quantitative computational models for metric spaces and other related structures ([10], [18], [24], [26], [30], etc) and in analyzing the complexity of programs and algorithms by means of contractive self-maps and fixed point methods of denotational semantics on the so-called complexity quasi-metric space ([9], [21], [23], [25], etc).

Partial metric spaces were introduced by MATTHEWS [16] to the study of denotational semantics of dataflow networks. In particular, he proved in [16, Theorem 5.3] a partial metric version of the Banach contraction principle. Later, VALERO [29], and OLTRA and VALERO [17] gave some generalizations of the result of Matthews. In fact, the study of fixed point theorems on partial metric spaces has received a lot of attention in the last three years (see, for instance, [1], [2], [3], [8], [11], [14], [19], [20], [27] and their references).

Throughout this paper the letters $\mathbb{R}^+$, $\mathbb{N}$ and ω will denote the set of all non-negative real numbers, the set of all positive integer numbers and the set of all non-negative integer numbers, respectively.

In the sequel we recall the notion of a partial metric space and some of its properties which will be useful later on. The main part of them may be found in [16] (see also [3], [19]).

Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$ (T_0-separation axiom),

(p_2) $p(x,x) \le p(x,y)$ (small self-distance axiom),

(p_3) $p(x,y) = p(y,x)$ (symmetry),

(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$ (modified triangular inequality).

A partial metric space is a pair (X,p) such that X is a non-empty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then, from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$.

Other examples of partial metric spaces which are interesting from a computational point of view may be found in [16], [21], [22], [26, 30].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

for all $x, y \in X$, is a metric on X.

The following well-known equivalence will be used later on.

$$\lim_{n\to\infty} p^s(x,x_n) = 0 \Longleftrightarrow p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{1.1}$$

A sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$.

A partial metric space (X,p) is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

The following crucial fact was essentially shown in [16, p. 194].

Lemma 1.1. *Let (X,p) be a partial metric space. Then:*

(i) *$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*

(ii) *(X,p) is complete if and only if (X,p^s) is complete.*

In [3, Theorem 1], Altun, Sola and Simsek proved the following fixed point theorem for Ćirić-type contractions on complete partial metric spaces.

Theorem 1.1 ([3])**.** *Let (X,p) be a complete partial metric space and $f : X \to X$ be a map such that*

$$p(fx,fy) \le \varphi\left(\max\left\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2}\Big[p(x,fy)+p(y,fx)\Big]\right\}\right),$$

for all $x, y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function such that $\varphi(t) < t$ for all $t > 0$ and the series $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$ (φ^n denotes the n-th iterate of φ). Then f has a unique fixed point.

Recently, Ćirić, Samet, Aydi and Vetro [8, Theorem 2.1] obtained the following nice extension of Theorem 1.1 to four self maps.

Theorem 1.2 ([8])**.** *Let (X,p) be a complete partial metric space and $A, B, S, T : X \to X$ be maps such that $AX \subseteq TX$, $BX \subseteq SX$ and*

$$p(Ax,By) \le$$
$$\le \varphi\left(\max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}\Big[p(Sx,By)+p(Ax,Ty)\Big]\right\}\right),$$

for all $x, y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function such that $\varphi(t) < t$ for all $t > 0$ and the series $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$.

If one of the ranges AX, BX, SX and TX is a closed subset of (X,p), then

(i) *A and S have a coincidence point*

(ii) *B and T have a coincidence point.*

Moreover, if the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible, then A, B, S and T have a unique common fixed point.

Generalizing Theorem 1.1, Romaguera obtained in [20, Theorem 3] the following fixed point theorem of Boyd–Wong type [5, Theorem 1].

Theorem 1.3 ([20]). *Let (X,p) be a complete partial metric space and $f : X \to X$ be a map such that*

$$p(fx,fy) \leq \varphi\left(\max\left\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2}\Big[p(x,fy)+p(y,fx)\Big]\right\}\right),$$

for all $x,y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a upper semicontinuous from the right function such that $\varphi(t) < t$ for all $t > 0$. Then f has a unique fixed point.

In this paper we prove a fixed point theorem of Boyd–Wong type for four self maps on complete partial metric spaces that, on one hand, extends Theorem 1.3 and, on other hand, generalizes Theorem 1.2, in a unified approach. It also improves, among others, very recent results of [1] and [11], respectively. We illustrate our theorem with some examples.

2. Results and examples

In the sequel, for a partial metric space (X,p) and four maps $A,B,S,T : X \to X$, we define

$$M(x,y) := \max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}\Big[p(Sx,By)+p(Ax,Ty)\Big]\right\},$$

for all $x,y \in X$.

To avoid repetition of arguments already developed in the proof of [8, Theorem 2.1], we state a result that collects some claims obtained in such a proof and that will be useful later on.

Lemma 2.1 ([8]). *Let (X,p) be a partial metric space and $A,B,S,T : X \to X$ be maps such that $AX \subseteq TX$, $BX \subseteq SX$. Then, for each $x_0 \in X$ there exist two sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X such that*

(i) $y_{2n} = Tx_{2n+1} = Ax_{2n}$ *and* $y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ *for all* $n \in \omega$;

(ii$_1$) $M(x_{2n}, x_{2n+1}) = \max\{p(y_{2n-1}, y_{2n}), p(y_{2n}, y_{2n+1})\}$ *for all* $n \in \mathbb{N}$;

(ii$_2$) $M(x_{2n}, x_{2n-1}) = \max\{p(y_{2n-2}, y_{2n-1}), p(y_{2n-1}, y_{2n})\}$ *for all* $n \in \mathbb{N}$.

If, in addition, there is a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$p(Ax, By) \le \varphi\big(M(x, y)\big),$$

for all $x, y \in X$, *then*

(iii) $p(y_n, y_{n+1}) \le \varphi\Big(\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\}\Big)$ *for all* $n \in \mathbb{N}$.

Moreover, if $\varphi(t) < t$ *for all* $t > 0$, *then*

(iv) *if* $p(y_{2k-1}, y_{2k}) = 0$ *for some* $k \in \mathbb{N}$, *it follows that* $y_n = y_m$ *for all* $n, m \ge 2k - 1$.

Remark 2.1. In [8, Theorem 2.1] it was proved that under the hypothesis of Lemma 2.1, one has $M(x_{2n}, x_{2n+1}) \le \max\{p(y_{2n-1}, y_{2n}), p(y_{2n}, y_{2n+1})\}$ for all $n \in \mathbb{N}$. Then, the equality given in Lemma 2.1 (ii$_1$) holds from the facts that $p(y_{2n-1}, y_{2n}) = p(Sx_{2n}, Tx_{2n+1})$ and $p(y_{2n}, y_{2n+1}) = p(Tx_{2n+1}, Bx_{2n+1})$, and the definition of $M(x_{2n}, x_{2n+1})$. The equality in (ii$_2$) is proved similarly. Consequently, we can deduce claims (iii) and (iv) of Lemma 2.1 without using the condition that φ is non-decreasing (see the proof of [8, Theorem 2.1]).

Let us recall that a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is upper semicontinuous from the right provided that for each $t \ge 0$ and each sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \ge t$ and $\lim_{n\to\infty} t_n = t$, it follows that $\limsup_{n\to\infty} \varphi(t_n) \le \varphi(t)$.

On the other hand if X is a non-empty set, $f, g : X \to X$ are self maps of X and $fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g. The pair $\{f, g\}$ is said to be weakly compatible if $fx = gx$ implies $fgx = gfx$.

Theorem 2.1. *Let* (X, p) *be a complete partial metric space and* $A, B, S, T : X \to X$ *be maps such that* $AX \subseteq TX$, $BX \subseteq SX$ *and*

$$p(Ax, By) \le \varphi\big(M(x, y)\big), \tag{2.1}$$

for all $x, y \in X$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a upper semicontinuous from the right function such that* $\varphi(t) < t$ *for all* $t > 0$.

If one of the ranges AX, BX, SX *and* TX *is a closed subset of* (X, p^s), *then*

(i) *A and S have a coincidence point*

(ii) *B and T have a coincidence point.*

Moreover, if the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point of X. By Lemma 2.1 (i), we can construct two sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \quad \text{for all} \quad n \in \omega.$$

We shall show that $(y_n)_{n\in\omega}$ is a Cauchy sequence in (X, p).

Indeed, by Lemma 2.1 (iii), we have

$$p(y_n, y_{n+1}) \le \varphi\Big(\max\big\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\big\}\Big), \tag{2.2}$$

for all $n \in \mathbb{N}$.

Moreover, if $p(y_{2k-1}, y_{2k}) = 0$ for some $k \in \mathbb{N}$, then $y_n = y_m$ for all $n, m \ge 2k-1$, by Lemma 2.1 (iv). So, in this case, $(y_n)_{n\in\omega}$ is obviously a Cauchy sequence in (X, p).

Hence, we shall assume that $p(y_n, y_{n+1}) > 0$ for all $n \in \omega$.

If $p(y_{n_0}, y_{n_0+1}) = \max\big\{p(y_{n_0-1}, y_{n_0}), p(y_{n_0}, y_{n_0+1})\big\}$ for some $n_0 \in \omega$, we deduce, from (2.2), that

$$p(y_{n_0}, y_{n_0+1}) \le \varphi\big(p(y_{n_0}, y_{n_0+1})\big) < p(y_{n_0}, y_{n_0+1})$$

a contradiction.

Therefore $p(y_{n-1}, y_n) = \max\big\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\big\}$ for all $n \in \omega$, and thus

$$p(y_n, y_{n+1}) \le \varphi\big(p(y_{n-1}, y_n)\big) < p(y_{n-1}, y_n), \tag{2.3}$$

for all $n \in \mathbb{N}$. Hence $\big(p(y_n, y_{n+1})\big)_{n\in\omega}$ is a decreasing sequence in $\mathbb{R}^+$, so there is $c \in \mathbb{R}^+$ such that

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = c.$$

Consequently

$$\lim_{n\to\infty} \varphi\big(p(y_n, y_{n+1})\big) = c,$$

by (2.3).

If $c > 0$, it follows $\lim_{n\to\infty} \varphi\big(p(y_n, y_{n+1})\big) \le \varphi(c)$, a contradiction, because $\varphi(c) < c$. We conclude that

$$\lim_{n\to\infty} p(y_n, y_{n+1}) = 0.$$

Thus, by (p4) in Definition 1.1, we have $\lim_{k\to\infty} p(y_{n_k}, y_{n_k+j}) = 0$ for every subsequence $(y_{n_k})_{k\in\omega}$ of $(y_n)_{n\in\omega}$ and $j \in \mathbb{N}$ fixed. This fact will be used in the sequel without explicit mention.

Next we prove that for each $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $p(y_{2n}, y_{2m+1}) < \varepsilon$ whenever $m > n \geq n_\varepsilon$.

Assume the contrary. Then there is $\varepsilon > 0$ and sequences $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $m_k > n_k \geq k$ and $p(y_{2n_k}, y_{2m_k+1}) \geq \varepsilon$ for all $k \in \mathbb{N}$.

Since $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$, we can suppose, without loss of generality, that $p(y_{2n_k}, y_{2m_k-1}) < \varepsilon$ for all $k \in \mathbb{N}$.

We show that $\lim_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = \varepsilon$.

Indeed, from the contraction condition (2.1) and the fact that $\varphi(t) < t$ for $t > 0$, it follows that

$$\begin{aligned}
\varepsilon &\leq p(y_{2n_k}, y_{2m_k+1}) = p(Ax_{2n_k}, Bx_{2m_k+1}) \\
&\leq \varphi\big(M(x_{2n_k}, x_{2m_k+1})\big) \\
&< M(x_{2n_k}, x_{2m_k+1}) \\
&= \max\Big\{p(y_{2n_k-1}, y_{2m_k}), p(y_{2n_k}, y_{2n_k-1}), p(y_{2m_k+1}, y_{2m_k}), \\
&\quad \frac{1}{2}\big[p(y_{2n_k-1}, y_{2m_k+1}) + p(y_{2n_k}, y_{2m_k})\big]\Big\},
\end{aligned}$$

for all $k \in \mathbb{N}$.

Since for each $\delta > 0$ there is $k_\delta \in \mathbb{N}$ such that

$$p(y_{2n_k-1}, y_{2n_k}) < \delta, \quad p(y_{2m_k-1}, y_{2m_k}) < \delta \quad \text{and} \quad p(y_{2m_k}, y_{2m_k+1}) < \delta,$$

whenever $k \geq k_\delta$, we deduce, by using (p4) in Definition 1.1, that

$$\varepsilon < M(x_{2n_k}, x_{2m_k+1}) < \max\Big\{2\delta + \varepsilon, \delta, \delta, \frac{1}{2}\big[(3\delta + \varepsilon) + (\delta + \varepsilon)\big]\Big\} = 2\delta + \varepsilon,$$

for all $k \geq k_\delta$. Thus $\lim_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = \varepsilon$. Hence

$$\lim_{k\to\infty} \sup \varphi\big(M(x_{2n_k}, x_{2m_k+1})\big) \leq \varphi(\varepsilon) < \varepsilon,$$

which contradicts that $\varepsilon \leq \varphi\big(M(x_{2n_k}, x_{2m_k+1})\big)$ for all $k \in \mathbb{N}$.

We conclude that for each $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $p(y_{2n}, y_{2m+1}) < \varepsilon$ whenever $m > n \geq n_\varepsilon$.

By using Lemma 2.1 (ii$_2$), we obtain, similarly, that for each $\varepsilon > 0$ there is $n'_\varepsilon \in \mathbb{N}$ such that $p(y_{2n-1}, y_{2m}) < \varepsilon$ whenever $m > n \geq n'_\varepsilon$.

From these two facts and $\lim_{n\to\infty} p(y_n, y_{n+1}) = 0$, we deduce that

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0.$$

Since (X, p) is complete, by Lemma 1.1 there exists $y \in X$ such that

$$\lim_{n\to\infty} p^s(y, y_n) = 0.$$

In particular

$$\lim_{n\to\infty} p^s(y, Ax_{2n}) = \lim_{n\to\infty} p^s(y, Tx_{2n+1}) = 0, \tag{2.4}$$

and

$$\lim_{n\to\infty} p^s(y, Bx_{2n-1}) = \lim_{n\to\infty} p^s(y, Sx_{2n}) = 0. \tag{2.5}$$

Moreover, from (1.1) we deduce that

$$p(y, y) = \lim_{n\to\infty} p(y, y_n) = \lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{2.6}$$

Now, assume, without loss of generality, that SX is closed in (X, p^s). Then, by (2.5), $y \in SX$, and thus there is $u \in X$ such that $y = Su$.

We claim that $p(y, Au) = 0$. Suppose $p(y, Au) > 0$, and then choose $\delta \in (0, p(y, Au)/2)$. By (2.6) there exists $n_0 \in \mathbb{N}$ such that

$$p(y, y_n) < \delta \quad \textit{and} \quad p(y_{2n}, y_{2n+1}) < \delta,$$

for all $n \geq n_0$. Since

$$\begin{aligned} M(u, x_{2n+1}) \quad &= \quad \max\Big\{p(y, y_{2n}), p(y, Au), p(y_{2n}, y_{2n+1}), \\ &\qquad \frac{1}{2}\big[p(y, y_{2n+1}) + p(Au, y_{2n})\big]\Big\}, \end{aligned}$$

we deduce that, for each $n \geq n_0$,

$$M(u, x_{2n+1}) \leq \max\Big\{\delta, p(y, Au), \delta, \frac{1}{2}\big[\delta + (p(Au, y) + \delta)\big]\Big\} = p(y, Au),$$

so $M(u, x_{2n+1}) = p(y, Au)$ for all $n \geq n_0$. Hence

$$\begin{aligned} p(y, Au) \quad &\leq \quad p(y, y_{2n+1}) + p(Au, y_{2n+1}) \\ &\leq \quad p(y, y_{2n+1}) + \varphi\big(M(u, x_{2n+1})\big) = p(y, y_{2n+1}) + \varphi\big(p(y, Au)\big), \end{aligned}$$

for all $n \geq n_0$.

Since $\lim_{n\to\infty} p(y, y_{2n+1}) = 0$, we deduce that $p(y, Au) \le \varphi\big(p(y, Au)\big)$, which contradicts our assumption that $p(y, Au) > 0$.

Therefore $p(y, Au) = 0$, and thus, $y = Au$.

Since $y = Su$, we conclude that $Au = Su$, i.e., u is a coincidence point of A and S.

From $AX \subseteq TX$ it follows that $y \in TX$, so $y = Tv$ for some $v \in X$.

Exactly as in the proof of [8, Theorem 2.1], we deduce that $y = Bv = Tv$, so v is a coincidence point of B and T.

Finally, suppose that the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. Then, in particular, we have $Ay = ASu = SAu = Sy$.

We show that $p(y, Ay) = 0$. Indeed, we first observe that

$$\begin{aligned} M(y,v) &= \max\Big\{p(Sy,Tv), p(Ay,Sy), p(Bv,Tv), \frac{1}{2}\big[p(Sy,Bv)+p(Ay,Tv)\big]\Big\} \\ &= \max\Big\{p(Ay,y), p(Ay,Ay), p(y,y), \frac{1}{2}\big[p(Ay,y)+p(Ay,y)\big]\Big\} \\ &= p(Ay,y). \end{aligned}$$

Then

$$p(Ay,y) = p(Ay,Bv) \le \varphi\big(M(y,v)\big) = \varphi\big(p(Ay,y)\big),$$

and consequently $p(Ay, y) = 0$. We conclude that $y = Ay = Sy$.

Exactly as in the proof of [8, Theorem 2.1], we deduce that $y = By = Ty$, and that, in fact, y is the unique common fixed point of A, B, S and T. □

In his excellent paper [13], Jachymski showed, among others, the following result.

Lemma 2.2. *([13, Lemma 1]). Let D be a(non-empty) subset of $\mathbb{R}^+ \times \mathbb{R}^+$. Then, the following are equivalent:*

(i) *there exist two continuous and non-decreasing functions $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi^{-1}(0) = \phi^{-1}(0) = \{0\}$, such that $D \subseteq \big\{(t,u) \in \mathbb{R}^+ \times \mathbb{R}^+ : \psi(u) \le \psi(t) - \phi(t)\big\}$.*

(ii) *there exists a continuous and non-decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(t) < t$ for all lt > 0, such that $D \subseteq \big\{(t,u) \in \mathbb{R}^+ \times \mathbb{R}^+ : u \le \varphi(t)\big\}$.*

Combining Theorem 2.1 and Lemma 2.2, we obtain the following.

Corollary 2.1. *Let (X, p) be a complete partial metric space and $A, B, S, T : X \to X$ be maps such that $AX \subseteq TX$, $BX \subseteq SX$ and*

$$\psi\big(p(Ax, By)\big) \le \psi\big(M(x,y)\big) - \phi\big(M(x,y)\big),$$

for all $x, y \in X$, where $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and non-decreasing functions such that $\psi^{-1}(0) = \phi^{-1}(0) = \{0\}$.

If one of the ranges AX, BX, SX and TX is a closed subset of (X, p), then

(i) *A and S have a coincidence point*

(ii) *B and T have a coincidence point.*

Moreover, if the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. Put $D = \Big\{\big(M(x,y), p(Ax, By)\big) : x, y \in X\Big\}$. By Lemma 2.2, (i)⇒(ii), there exists a continuous and non-decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) < t$ for all $t > 0$, and $p(Ax, By) \leq \varphi\big(M(x,y)\big)$ for all $x, y \in X$. Theorem 2.1 concludes the proof. □

Remark 2.2. Theorem 2.1 generalizes Theorem 2.1 of [8], whereas Theorem 3 of [20] is a special case of Theorem 2.1 when $A = B$ and $S = T = \mathrm{id}$. On the other hand, Theorem 5 of [1] is a special case of Corollary 2.1 when $S = T = \mathrm{id}$, and $\psi(t) = t$ for all $t \in \mathbb{R}^+$, and Theorem 3.2 of [11] is a consequence of Theorem 2.1 when $A = B$, $S = T = \mathrm{id}$ and $\varphi(t) = t$ for all $t \in \mathbb{R}^+$.

We conclude the paper with some examples that illustrate Theorem 2.1.

Example 2.1. *Let $X = \mathbb{R}^+$ and $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. It is well known that the partial metric space (X, p) is complete (in fact, p^s is the usual metric on $\mathbb{R}^+$). Let $A, B, S, T : X \to X$ defined by $Ax = Bx = x/(1+x)$, $Sx = Tx = x$ for all $x \in X$. Clearly, the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. Moreover $AX \subseteq TX = X$ and $BX \subseteq SX = X$.*

Now define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t/(1+t)$. Then φ is continuous on $\mathbb{R}^+$ and $\varphi(t) < t$ for all $t > 0$.

For each $x, y \in X$ with $x \geq y$ we have

$$\begin{aligned} p(Ax, By) &= \max\left\{\frac{x}{1+x}, \frac{y}{1+y}\right\} = \frac{x}{1+x} \\ &= \varphi(x) = \varphi\big(p(Sx, Ty)\big) \leq \varphi\big(M(x,y)\big). \end{aligned}$$

This shows that the contraction condition of Theorem 2.1 is satisfied, so all the required conditions of that theorem are verified. Hence A, B, S and T have a unique common fixed point in X. Note that $\varphi^n(t) = t/(1+nt)$ for all $t \in \mathbb{R}^+$ and so $\sum_{n=0}^{\infty} \varphi^n(t)$ is not convergent for $t > 0$. Therefore, we can not apply [8, Theorem 2.1] *to this example.*

Example 2.2. *Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a complete partial metric space (in fact, p^s is the Euclidean metric on X). Let $A, B, S, T : X \to X$, defined by $Ax = x/8$, $Bx = x/4$, $Sx = 5x/8$ and $Tx = 3x/4$ for all $x \in X$. Then, clearly $AX \subseteq TX$ and $BX \subseteq SX$, and AX, BX, SX and TX are closed subsets in (X, p^s). Moreover, the pairs $\{A, S\}$ and $\{B, T\}$ are clearly weakly compatible.*

Now define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t/(1 + t)$ for all $t > 0$. Then φ is continuous on $\mathbb{R}^+$ and satisfies that $\varphi(t) < t$ for all $t > 0$. Note also that it is non-decreasing.

For all $x, y \in X$, with $x \leq y$, we have

$$\begin{aligned} p(Ax, By) &= \max\left\{\frac{x}{8}, \frac{y}{4}\right\} = \frac{y}{4} \leq \frac{5y}{8 + 5y} \\ &= \varphi\left(\frac{5y}{8}\right) \leq \varphi\left(\frac{3y}{4}\right) \\ &= \varphi\left(\max\left\{\frac{5x}{8}, \frac{3y}{4}\right\}\right) \\ &= \varphi\big(p(Sx, Ty)\big) \\ &\leq \varphi\big(M(x, y)\big). \end{aligned}$$

If $x > y$, we distinguish two cases: (i) $x/8 \geq y/4$ and (ii) $x/8 < y/4$.

In case (i) we obtain, as in the case that $x \leq y$, that

$$p(Ax, By) = \frac{x}{8} \leq \frac{5x}{8 + 5x} = \varphi\left(\frac{5x}{8}\right) \leq \varphi\big(p(Sx, Ty)\big) \leq \varphi\big(M(x, y)\big).$$

In case (ii) we obtain

$$\begin{aligned} p(Ax, By) &= \frac{y}{4} \leq \frac{3y}{4 + 3y} = \varphi\left(\frac{3y}{4}\right) \leq \varphi\left(\max\left\{\frac{5x}{8}, \frac{3y}{4}\right\}\right) \\ &= \varphi\big(p(Sx, Ty)\big) \leq \varphi\big(M(x, y)\big). \end{aligned}$$

Hence, the contraction condition is satisfied. Thus all the conditions of Theorem 2.1 are satisfied and 0 is unique common fixed point of A, B, S and T in X.

Note that $\sum_{n=0}^{\infty} \varphi^n(t)$ is not convergent for $t > 0$; in fact, $\varphi^n(t) = t/\,(1 + nt)$ for all $t \in \mathbb{R}^+$. Therefore, we can not apply [8, Theorem 2.1] *to this example.*

Example 2.3. *Let (X, p) be the complete partial metric space of Example 2.2. Let $A, B, S, T : X \to X$, defined by $Ax = x/6$, $Bx = x/9$, $Sx = 2x/3$ and $Tx = 5x/6$ for all $x \in X$. Then, clearly $AX \subseteq TX$ and $BX \subseteq SX$, and AX, BX, SX and TX are closed subsets in (X, p^s). Moreover, the pairs $\{A, S\}$ and $\{B, T\}$ are clearly weakly compatible.*

Now define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = t/2$ for all $t \in [0, 1)$ and $\varphi(t) = n(n+1)/(n+2)$ for $t \in [n, n+1)$, $n \in \mathbb{N}$. Then φ is non-decreasing and (right) upper semicontinuous on $\mathbb{R}^+$, with $\varphi(t) < t$ for all $t > 0$. However it is not continuous at $t = n$ for all $n \in \mathbb{N}$, so we can not apply [8, Theorem 2.1] *to this example.*

For all $x, y \in X$ we have

$$\begin{aligned} p(Ax, By) &= \max\left\{\frac{x}{6}, \frac{y}{9}\right\} \leq \frac{1}{2}\left(\max\left\{\frac{2x}{3}, \frac{5y}{6}\right\}\right) \\ &= \varphi\left(\max\left\{\frac{2x}{3}, \frac{5y}{6}\right\}\right) = \varphi\Big(p(Sx, Ty)\Big) \\ &\leq \varphi\big(M(x, y)\big). \end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied. Note also that, in this case, the series $\sum_{n=0}^{\infty} \varphi^n(t)$ is convergent for all $t \in \mathbb{R}^+$.

Acknowledgment. The authors are grateful to one of the referees for calling our attention about the reference [12].

References

[1] T. Abdeljawad, E. Karapinar and K. Tas, Existence and uniqueness of a common fixed point on partial metric spaces, *Appl. Math. Lett.* **24** (2011), 1900–1904.

[2] I. Altun and A. Erduran, Fixed point theorems for monotone mappings on partial metric spaces, Article ID 508730, *Fixed Point Theory Appl.* **2011** (2011), 10 pages.

[3] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, *Topology Appl.* **157** (2010), 2778-2785.

[4] V. Berinde, Some remarks on a fixed point theorem for Ciric-type almost contractions, *Carpathian J. Math.* **25** (2009), 157-162.

[5] D.W. Boyd and J.S.W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.* **20** (1969), 458-464.

[6] L. Ćirić, Generalized contractions and fixed point theorems, *Publ. Inst. Math.* **12** (1971), 19-26.

[7] L.B. Ćirić, N. Hussain and N. Cakić, Common fixed points for Ciric type f-weak contraction with applications, *Publ. Math. Debrecen* **76** (2010), 31-49.

[8] L. Ćirić, B. Samet, H. Aydi and C. Vetro, Common fixed points of generalized contractions on partial metric spaces and an application, *Appl. Math. Comput* **218** (2011), 2398-2406.

[9] L.M. García-Raffi, S. Romaguera and M.P. Schellekens, Applications of the complexity space to the general probabilistic divide and conquer algorithms, *J. Math. Anal. Appl.* **348** (2008), 346-355.

[10] R. Heckmann, Approximation of metric spaces by partial metric spaces, *Appl. Cat. Struct.* **7** (1999), 71-83.

[11] D. Ilić, V. Pavlović and V. Rakočević, Some new extensions of Banach's contraction principle to partial metric space, *Appl. Math. Lett.* **24** (2011), 1326-1330.

[12] D. Ilić, V. Pavlović and V. Rakočević, Extensions of the Zamfirescu theorem to partial metric spaces, *Math. Comput. Model.* **55** (2012), 801-809.

[13] J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces, *Nonlinear Anal.* **74** (2011), 768-774.

[14] E. Karapinar and I.M. Erhan, Fixed point theorems for operators on partial metric spaces, *Appl. Math. Lett.* **24** (2011), 1894-1899.

[15] T. Lazăr, D. O'Regan and A. Petruşel, Fixed points and homotopy results for Ćirić-type multivalued operators on a set with two metrics, *Bull. Korean Math. Soc.* **45** (2008), 67-73.

[16] S.G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Appl., *Ann. New York Acad. Sci.* **728** (1994), 183-197.

[17] S. Oltra and O. Valero, Banach's fixed point theorem for partial metric spaces, *Rend. Istit. Math. Univ. Trieste* **36** (2004), 17-26.

[18] S. Romaguera, On Computational Models for the Hyperspace, Advances in Mathematics Research, Vol. 8, *Nova Science Publishers*, *New York*, 277-294.

[19] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Article ID 493298, *Fixed Point Theory Appl.* **2010** (2011), 6 pages.

[20] S. Romaguera, Fixed point theorems for generalized contractions on partial metric spaces, *Topology Appl.* **159** (2012), 194-199.

[21] S. Romaguera and M. Schellekens, Quasi-metric properties of complexity spaces, *Topology Appl.* **98** (1999), 311-322.

[22] S. Romaguera and M. Schellekens, Partial metric monoids and semivaluation spaces, *Topology Appl.* **153** (2005), 948-962.

[23] S. Romaguera, M.P. Schellekens and O. Valero, The complexity space of partial functions: a connection between complexity analysis and denotational semantics, *Internat. J. Comput. Math.* **88** (2011), 1819-1829.

[24] S. Romaguera, O. Valero, A quantitative computational model for complete partial metric spaces via formal balls, *Math. Struct. Comput. Sci.* **19** (2009), 541-563.

[25] M. Schellekens, The Smyth completion: A common foundation for denotational semantics and complexity analysis, *Electron. Notes Theor. Comput. Sci.* **1** (1995), 535-556.

[26] M.P. Schellekens, *Theoret. Comput. Sci.* **305** (2003), 409-432.

[27] W. Shatanawi, B. Samet and M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, *Math. Comput. Model.* **55** (2012), 680-687.

[28] R. Sumitra, V. Rhymend Uthariaraj, R. Hemavathy and P. Vijayaraju, Common fixed point theorem for non-self mappings satisfying generalized Ćirić type contraction condition in cone metric space, Article ID 408086, *Fixed Point Theory Appl.* **2010** (2010), 17 pages.

[29] O. Valero, On Banach fixed point theorems for partial metric spaces, *Appl. Gen. Topology* **6** (2005), 229-240.

[30] P. Waszkiewicz, Partial metrisability of continuous posets, *Math. Struct. Comput. Sci.* **16** (2006), 359-372.

MUJAHID ABBAS
DEPARTMENT OF MATHEMATICS
LAHORE UNIVERSITY OF MANAGEMENT SCIENCES
54792 LAHORE
PAKISTAN

E-mail: mujahid@lums.edu.pk

ISHAK ALTUN
FACULTY OF SCIENCES AND ARTS
KIRIKKALE UNIVESITY
71540 YAHSIHAN, KIRIKKALE
TURKEY

E-mail: ishakaltun@yahoo.com

SALVADOR ROMAGUERA
INSTITUTO UNIVERSITARIO DE MATEMÁTICA PURA Y APLICADA
UNIVERSITÀT POLITÈCNICA DE VALÈNCIA
CAMÍ DE VERA S/N
46022 VALENCIA
SPAIN

E-mail: sromague@mat.upv.es

(Received September 26, 2011; revised July 1, 2012)