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Chen inequalities for submanifolds of a Riemannian manifold of nearly quasi-constant curvature

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Abstract. The object of the present paper is to study Chen first inequality and k-Ricci curvatures for submanifolds of a Riemannian manifold of nearly quasi-constant curvature.

1. Introduction

Let (M, g) be a Riemannian manifold. If its curvature tensor satisfies the condition

$$\begin{aligned}R(X, Y, Z, W) = a[&g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]\\ &+ b[g(X, W)A(Y)A(Z) - g(X, Z)A(Y)A(W)\\ &+ g(Y, Z)A(X)A(W) - g(Y, W)A(X)A(Z)], \qquad (1.1)\end{aligned}$$

where a, b are scalar functions and A is a 1-form defined by

$$g(X, P) = A(X), \qquad (1.2)$$

P is a unit vector field, then we say that (M, g) is a Riemannian manifold of *quasi-constant curvature* [10]. If $b = 0$ then the manifold reduces to a space of constant curvature.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a *quasi-Einstein manifold* if its Ricci tensor satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y),$$

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where a, b are scalar functions and A is a non-zero 1-form such that $g(X,U) = A(X)$ for every vector field X and U is a unit vector field. If $b = 0$ then the manifold reduces to an Einstein manifold. It can be easily seen that every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

In 2009, A. K. Gazi and U. C. De [12] introduced the notion of a Riemannian manifold of *nearly quasi-constant curvature* as a Riemannian manifold with the curvature tensor satisfying the condition

$$\begin{aligned} R(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + q[g(X,W)B(Y,Z) - g(X,Z)B(Y,W) \\ & + g(Y,Z)B(X,W) - g(Y,W)B(X,Z)] \end{aligned} \tag{1.3}$$

where p, q are scalar functions and B is a non-zero symmetric tensor of type $(0,2)$.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a *nearly quasi-Einstein manifold* if its Ricci tensor satisfies the condition

$$S(X,Y) = ag(X,Y) + bE(X,Y),$$

where a and b are non zero scalar functions and E is a non-zero symmetric tensor of type $(0,2)$ [11]. It can be easily seen that every Riemannian manifold of nearly quasi-constant curvature is a *nearly quasi-Einstein manifold.*

It is known that the outer product of two covariant vectors is a covariant tensor of type $(0,2)$ but the converse is not true, in general. Hence a Riemannian manifold of quasi-constant curvature is a manifold of nearly quasi-constant curvature, but there are existence of manifolds of nearly quasi-constant curvature which are not of quasi-constant curvature. It can be easily seen that a conformally flat manifold of dimension > 3 is a manifold of nearly quasi-constant curvature since the Ricci tensor S is a symmetric $(0,2)$ tensor. But the converse is not necessarily true, in general. On the other hand, a manifold of quasi-constant curvature is conformally flat. Also, we can construct examples of a manifold of nearly quasi-constant curvature which is not a manifold of quasi-constant curvature. Hence, a Riemannian manifold of nearly quasi-constant curvature is a more general idea than a Riemannian manifold of quasi-constant curvature.

Example 1.1. Let us consider a Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

Then the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}}$$

$$R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{\frac{2}{3}}},$$

$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{\frac{2}{3}}$$

and the components obtained by the symmetry properties.

The non-vanishing components of the Ricci tensors are:

$$R_{11} = R_{22} = R_{33} = \frac{2}{3(x^4)^{\frac{2}{3}}}, \quad R_{44} = -\frac{2}{3(x^4)^2}.$$

The scalar curvature of the resulting manifold $(\mathbb{R}^4, g)$ is

$$g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44} = \frac{4}{3(x^4)^2},$$

which is non-vanishing and non-constant.

Let us now consider the associated scalars as follows:

$$p = -\frac{2}{9(x^4)^2}, \quad q\frac{1}{3(x^4)^{\frac{2}{3}}}. \tag{1.4}$$

We choose the associated nonzero symmetric $(0,2)$ tensor B as follows:

$$\begin{aligned} B_{ij}(x) &= (x^4)^2, && \text{for } i = j = 1, 2, 3 \\ &= -(x^4)^{\frac{2}{3}}, && \text{for } i = j = 4, \end{aligned}$$

for any point $x \in \mathbb{R}^4$.

In terms of local coordinates, the defining condition of a nearly quasi-constant curvature can be written as

$$R_{ijkl} = p[g_{jk}g_{il} - g_{ik}g_{jl}] + q[g_{jk}B_{il}g_{il}B_{jk} - g_{ik}B_{jl} - g_{jl}B_{ik}], \tag{1.5}$$

for $i, j, k, l = 1, 2, 3, 4$.

By virtue of (1.4) and choice of the $(0,2)$ tensor B, it can be easily seen that equation (1.5) holds for $i, j, k, l = 1, 2, 3, 4$. Therefore, $(\mathbb{R}^4, g)$ is a manifold of nearly quasi-constant curvature [11].

We shall now show that this manifold is not a manifold of quasi-constant curvature.

If possible, suppose this manifold is of quasi-constant curvature. Then in terms of local coordinates, the curvature tensor R of type $(0,4)$ can be written as

$$Rijkl = p[g_{jk}g_{il} - g_{ik}g_{jl}] + q[g_{jk}A_iA_l + g_{il}A_jA_K - g_{ik}A_jA_l - g_{jl}A_iA_k],$$

for $i, j, k, l = 1, 2, 3, 4$, where p, q are scalars of which $q \neq 0$ and A is a non-zero 1-form.

Now, for $i = l = 1$, $j \neq k$ and $j, k \neq 1$, we have

$$R_{1jk1} = p[g_{jk}g_{11} - g_{1k}g_{1j}] + q[g_{jk}A_1A_1 + g_{11}A_jA_K - g_{1k}A_jA_1 - g_{j1}A_1A_k],$$

which implies,

$$0 = 0 + qA_jA_k,$$

which is a contradiction. Hence, the assumption is wrong. So, the manifold is not a manifold of quasi-constant curvature.

Example 1.2. Let $\tilde{\nabla}$ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T is given by

$$T\,(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y].$$

The connection $\tilde{\nabla}$ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. If there is a Riemannian metric g in M such that $\tilde{\nabla} g = 0$, then the connection $\tilde{\nabla}$ is a metric connection, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi–Civita connection.

A linear connection $\tilde{\nabla}$ is said to be a *semi-symmetric connection* [16] if its torsion tensor T is of the form

$$T(X, Y) = \omega(Y)X - \omega(X)Y, \tag{1.6}$$

where the 1-form ω is defined by

$$\omega(X) = g(X, U),$$

and U is a vector field.

If ∇ is the Levi–Civita connection of a Riemannian manifold M, then we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X - g(X, Y)U, \tag{1.7}$$

where

$$\omega(X) = g(X, U),$$

and X, Y, U are vector fields on M [16]. Let R and $\widetilde{R}$ denote the Riemannian curvature tensor of ∇ and $\widetilde{\nabla}$, respectively. Then from [16] we know that

$$\begin{aligned}\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) &- \theta(Y, Z)g(X, W) + \theta(X, Z)g(Y, W)\\ &- g(Y, Z)\theta(X, W) + g(X, Z)\theta(Y, W),\end{aligned} \tag{1.8}$$

where

$$\theta(X,Y) = g(AX,Y) = (\nabla_X\omega)Y - \omega(X)\omega(Y) + \frac{1}{2}g(X,Y). \tag{1.9}$$

Assume that ω is a closed 1-form. Then $(\nabla_X\omega)Y = (\nabla_Y\omega)X$. Hence θ is a symmetric $(0,2)$-tensor field. Now let $M(c)$ be a Riemannian space of constant curvature c. If $M(c)$ has a semi-symmetric metric connection with closed associated 1-form ω, then the curvature tensor of $M(c)$ with respect to the semi-symmetric metric connection is

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {} & c(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \\ & - \theta(Y,Z)g(X,W) + \theta(X,Z)g(Y,W) \\ & - g(Y,Z)\theta(X,W) + g(X,Z)\theta(Y,W).\end{aligned}$$

Then $M(c)$ is a space of nearly quasi-constant curvature with respect to the semi-symmetric metric connection.

In [5]–[8], B. Y. Chen established some sharp inequalities between intrinsic invariants like Ricci curvatures and the squared mean curvatures, an extrinsic invariant in a submanifold immersed in a Riemannian manifold. Afterwards, various authors studied the inequality in different ambient spaces, for example, see [1], [2], [13], [14] and references therein.

Recently, in [15], the first author studied Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature. In the present paper, we generalize the results of the paper [15] to submanifolds of a Riemannian manifold of nearly quasi-constant curvature.

2. Preliminaries

Let M be an n-dimensional submanifold of an $(n+m)$-dimensional Riemannian manifold N^{n+m}. The Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and h is the second fundamental form related to the shape operator A by $g(h(X,Y),N) = g(A_N X, Y)$. The equation of Gauss is given by

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {} & R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) \\ & + g(h(X,Z),h(Y,W))\end{aligned} \tag{2.1}$$

for all $X, Y, Z, W \in TM$, where R is the curvature tensor of M. The mean curvature vector H is given by $H = \frac{1}{n}\operatorname{trace}(h)$.

Using (1.3), the Gauss equation for the submanifold M^n of a Riemannian manifold of nearly quasi-constant curvature is

$$\begin{aligned} R(X,Y,Z,W) = p\,&[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q[g(X,W)B(Y,Z) \\ &- g(X,Z)B(Y,W) + g(Y,Z)B(X,W) - g(Y,W)B(X,Z)] \\ &+ g\left(h(X,W), h(Y,Z)\right) - g\ \left(h(X,Z), h(Y,W)\right). \end{aligned} \tag{2.2}$$

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_xM^n, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We recall the following algebraic lemma:

Lemma 2.1 ([4]). *Let $a_1, a_2, \ldots, a_n, b$ be $(n+1)$ $(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L.

We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$.

Ones define [6] the *Ricci curvature* (or *k-Ricci curvature*) of L at X by

$$\operatorname{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k},$$

where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on M^n is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} \operatorname{Ric}_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

3. Chen first inequality

In this section, we study submanifolds of a Riemannian manifold of nearly quasi-constant curvature and find Chen first inequality.

Theorem 3.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n+m)$ -dimensional Riemannian manifold of nearly quasi-constant curvature $^{n+m}$. Then we have:*

$$\tau - K(\pi) \leq (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{p}{2}\right] + q\left[(n-2)\lambda + \operatorname{trace} B|_{\pi^\perp}\right], \quad (3.1)$$

where π is a 2-plane section of $T_xM^n, x \in M^n$ and $\lambda = \operatorname{trace} B$. The equality case of inequality (3.1) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, \dots, e_{n+m}\}$ of $T_x^\perp M^n$ such that the shape operators of M^n in N^{n+m} at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_{n+i}} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 2 \leq i \leq m,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r)$, $1 \leq i, j \leq n$ and $n+1 \leq r \leq n+m$.

Proof. Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{n+m}\}$ be an orthonormal basis of T_xM^n and $T_x^\perp M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, from the Gauss equation (2.2) it follows that

$$\begin{aligned} R(e_i, e_j, e_j, e_i) = {} & p\left[g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\right] \\ & + q\left[g(e_i, e_i)B(e_j, e_j) - g(e_i, e_j)B(e_j, e_i) \right. \\ & \left. + g(e_j, e_j)B(e_i, e_i) - g(e_j, e_i)B(e_i, e_j)\right] \\ & + g\left(h(e_i, e_i), h(e_j, e_j)\right) - g\left(h(e_i, e_j), h(e_j, e_i)\right). \end{aligned} \quad (3.2)$$

By summation after $1 \leq i, j \leq n$, from the previous relation we get

$$2\tau + \|h\|^2 - n^2 \|H\|^2 = 2q(n-1)\lambda + (n^2 - n)p, \tag{3.3}$$

where

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

One takes

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1} \|H\|^2 - (n^2 - n)p - 2q(n-1)\lambda. \tag{3.4}$$

Then, from (3.3) and (3.4) we get

$$n^2 \|H\|^2 = (n-1)\big(\|h\|^2 + \varepsilon\big). \tag{3.5}$$

Let $x \in M^n$, $\pi \subset T_x M^n$, $\dim \pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and using (3.5) we obtain:

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left(\sum_{i,j=1}^{n} \sum_{r=n+1}^{n+m} (h_{ij}^r)^2 + \varepsilon\right),$$

or equivalently,

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left\{\sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^r)^2 + \varepsilon\right\}. \tag{3.6}$$

By the use of Lemma 2.1 in view of (3.6):

$$2h_{11}^{n+1} h_{22}^{n+1} \geq \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^r)^2 + \varepsilon. \tag{3.7}$$

Gauss equation for $X = W = e_1$, $Y = Z = e_2$ gives us

$$\begin{aligned} K(\pi) = R(e_1, e_2, e_2, e_1) &= p + q\left[B(e_{1,}e_1) + B(e_2, e_2)\right] \\ &+ \sum_{r=n+1}^{m} \left[h_{11}^r h_{22}^r - (h_{12}^r)^2\right] \geq p + q\left[B(e_{1,}e_1) + B(e_2, e_2)\right] \\ &+ \frac{1}{2}\left[\sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^r)^2 + \varepsilon\right] \end{aligned}$$

$$+\sum_{r=n+2}^{n+m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+m} (h_{12}^r)^2 = p + q\,[B(e_1,e_1) + B(e_2,e_2)]$$

$$+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{n+m}(h_{ij}^r)^2 + \frac{1}{2}\varepsilon + \sum_{r=n+2}^{n+m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+m}(h_{12}^r)^2$$

$$= p + q\,[B(e_1,e_1) + B(e_2,e_2)] + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{n+m}\sum_{i,j>2}(h_{ij}^r)^2$$

$$+\frac{1}{2}\sum_{r=n+2}^{n+m}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon$$

$$\geq p + q\,[B(e_1,e_1) + B(e_2,e_2)] + \frac{\varepsilon}{2},$$

which implies

$$K(\pi) \geq p + q\,[B(e_1,e_1) + B(e_2,e_2)] + \frac{\varepsilon}{2}. \tag{3.8}$$

By the use of (3.4), from (3.8), we find

$$K(\pi) \geq \tau - (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{p}{2}\right] - q[(n-2)\lambda + \operatorname{trace} B|_{\pi^\perp}]$$

hence we obtain (3.1).

The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

$$h_{ij}^{n+1} = 0, \quad \forall i \neq j, i, j > 2,$$

$$h_{ij}^r = 0, \quad \forall i \neq j, i, j > 2,\ r = n+1, \ldots, n+m,$$

$$h_{11}^r + h_{22}^r = 0, \quad \forall r = n+2, \ldots, n+m,$$

$$h_{1j}^{n+1} = h_{2j}^{n+1} = 0, \quad \forall j > 2,$$

$$h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}.$$

We may chose $\{e_1,e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^r, b = h_{22}^r, \mu = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}$.

It follows that the shape operators take the desired forms. □

4. *k*-Ricci curvature

In this section, we consider the k-Ricci curvature which is an intrinsic invariant and find a relation with the squared mean curvature $\|H\|^2$.

Theorem 4.1. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n+m)$ -dimensional space of nearly quasi-constant curvature N^{n+m}. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - p - \frac{2q}{n}\lambda. \tag{4.1}$$

Proof. Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ be orthonormal basis of T_xM^n. From (3.3) we can write

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - 2q(n-1)\lambda - (n^2 - n)p. \tag{4.2}$$

We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+m}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, \dots, e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}, \tag{4.3}$$

$$A_{e_r} = (h_{ij}^r), \; i, j = 1, \dots, n; \; r = n+2, \dots, n+m, \operatorname{trace} A_r = 0. \tag{4.4}$$

From (4.2), we get

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - 2q(n-1)\lambda - (n^2 - n)p. \tag{4.5}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2, \tag{4.6}$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2.$$

From (4.5) we get

$$n^2 \|H\|^2 \geq 2\tau + n \|H\|^2 - 2q(n-1)\lambda - (n^2 - n)p \tag{4.7}$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - p - \frac{2q}{n}\lambda,$$

this proves the theorem. □

Theorem 4.2. *Let $M^n, n \geq 3$, be an n-dimensional submanifold of an $(n+m)$ -dimensional Riemannian manifold of nearly quasi-constant curvature N^{n+m}. Then for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2 \geq \Theta_k(x) - p - \frac{2q}{n}\lambda. \tag{4.8}$$

Proof. Let $\{e_1, \dots e_n\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1 \dots i_k}$ the k-plane section spanned by $e_{i_1}, \dots, e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1 \dots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \dots, i_k\}} \operatorname{Ric}_{L_{i_1 \cdots i_k}}(e_i),$$

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \leq i_1 < \dots < i_k \leq n} \tau(L_{i_1 \dots i_k}).$$

From (4.1) and the above relations, one derives

$$\tau(x) \geq \frac{n(n-1)}{2} \Theta_k(x),$$

which implies (4.8). □

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