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Conditional oscillation and principal solution of generalized half-linear differential equation

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Abstract. We establish an explicit formula for conditionally oscillatory potential in the generalized half-linear second order differential equation. We also present an alternative construction of the principal solution of this equation.

1. Introduction

We deal with the second order differential equation of the form

$$x'' + c(t)f(x, x') = 0 \tag{1}$$

where c is a continuous function and the function f satisfies the following assumptions which were introduced in [5]:

(i) The function f is continuous on $\Omega = \mathbb{R} \times [\mathbb{R} \setminus \{0\}]$;
(ii) It holds $xf(x, y) > 0$ if $xy \neq 0$;
(iii) The function f is homogeneous, i.e., $f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda \in \mathbb{R}$ and $(x, y) \in \Omega$;
(iv) The function f is sufficiently smooth in order to ensure the continuous dependence and the uniqueness of solutions of the initial value problem $x(t_1) = x_0$, $x'(t_1) = x_1$ at some $(x_0, x_1) \in \Omega$;

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(v) Let $F(t) := tf(t,1)$, then

$$\int_{-\infty}^{\infty} \frac{dt}{1+F(t)} < \infty \quad \text{and} \quad \lim_{|t|\to\infty} F(t) = \infty.$$

Under these assumptions, the solution space of (1) is homogeneous. This is the reason why this equation was called half-linear, its solution space has just one half of the properties which characterize linearity. The word "generalized" reflects the fact that (1) covers as a special case the "classical" half-linear differential equation

$$\big(\Phi_p(x')\big)' + c(t)\Phi_p(x) = 0, \quad \Phi_p(x) := |x|^{p-2}x,\ p > 1, \tag{2}$$

which was a subject of investigation of many recent papers, see, e.g., [3], [8], [10] and the reference therein. Note also that the generalized half-linear differential equation was introduced in Bihari's papers [1], [2] in the form

$$\big(r(t)x'\big)' + c(t)f(x, r(t)x') = 0 \tag{3}$$

with a positive continuous function r. However, the change of independent variable $s = \int^t r^{-1}(\tau)\, d\tau$ transforms (3) into (1), so we suppose without loss of generality that $r(t) \equiv 1$ in (3).

Let g be the differentiable function given by the formula

$$g(u) = \begin{cases} \int_{1/u}^{\infty} \dfrac{ds}{F(s)} & \text{if } u > 0, \\ -\int_{-\infty}^{1/u} \dfrac{ds}{F(s)} & \text{if } u < 0, \end{cases}$$

and $g(0) = 0$. Then g is increasing and $\lim_{v\to\pm\infty} g(v) = \pm\infty$. If x is a solution of (1) such that $x(t) \neq 0$, then the function $v = g(x'/x)$ solves the Riccati type differential equation

$$v' + c(t) + H(v) = 0, \tag{4}$$

where the function H is given by the formula

$$H(v) = [g^{-1}(v)]^2 g'(g^{-1}(v)) \tag{5}$$

with $H(0) = 0$ (g^{-1} being the inverse function of g). Moreover, the function H satisfies

$$\int_{-\infty}^{-\varepsilon} \frac{dv}{H(v)} < \infty, \qquad \int_{\varepsilon}^{\infty} \frac{dv}{H(v)} < \infty.$$

Following [5], to study oscillatory properties of (1) in more details, we also need the additional assumption:

(vi) The function H given by (5) is strictly *convex*.

This assumption is satisfied e.g. when the function $\log F(t)$ is strictly concave, see [5]. Under this assumption, the function H is decreasing for $u \leq 0$ and increasing for $u \geq 0$. Moreover, it is locally Lipschitzian, hence solutions of (4) are uniquely determined by the initial condition and hence graphs of solutions of this equation cannot intersect.

The aim of our paper is to prove two conjectures posed in [4]. These conjectures concern the conditional oscillation and the construction of the principal solution of (1). We recall these concepts in the next sections.

2. Preliminaries

It is known that many of the properties of the linear Sturm-Liouville differential equation

$$\big(r(t)x'\big)' + c(t)x = 0$$

can be extended to (1). In particular, we have the full analogy of the Sturm separation and comparison theory. This means that equation (1) can be classified as oscillatory or nonoscillatory similarly as in the linear case. This is due to the relationship between (1) and the Riccati type equation (4) via the substitution $v = g(x'/x)$. The existence of a solution x of (1) with consecutive zeros $t_1 < t_2$ means that $v(t_1+) = \infty$, $v(t_2-) = -\infty$ for the associated solution v of (4). This implies that any other solution $\tilde{x}$ of (1) has to have a zero in (t_1, t_2), otherwise the graph of $\tilde{v} = g(\tilde{x}'/\tilde{x})$ intersects the graph of v, which contradicts the unique solvability of (4).

The following statement, which is presented in [5], summarizes the essentials of the application of Riccati technique in oscillation theory of (1). Recall that a *proper solution* of (4) is a solution which exists on some interval $[T, \infty)$, i.e., it is extensible up to ∞.

Proposition 1. *The following statements are equivalent:*

(1) *Equation* (1) *is nonoscillatory.*

(2) *There exists a proper solution of the generalized Riccati equation* (4).

(3) *There exists a continuously differentiable function v defined on some interval $[T, \infty)$ for which*

$$v'(t) + c(t) + H(v(t)) \leq 0, \quad t \in [T, \infty).$$

We will also need the next statement which concerns nonnegativity of proper solutions of (4).

Lemma 1. *If $c(t) \geq 0$, then all possible proper solutions of* (4) *are nonnegative and tend to zero as $t \to \infty$.*

Proof. Under assumptions on the function H, the function $v(t) \equiv 0$ is the minimal solution of the equation

$$v' + H(v) = 0, \tag{6}$$

see [4], and by another result of the same paper all possible proper solutions of (4) are greater than this minimal solution of (6). As for the limit of a proper solution of (4), this limit $v(\infty)$ is a positive number or zero. Since v is decreasing, we have $v(t) > v(\infty)$ and integrating (4) from T to ∞

$$v(\infty) + \int_T^\infty c(t)\,dt + \int_T^\infty H(v(t))\,dt = v(T)$$

Note that the existence of a proper solution of (4) implies that $\int_T^\infty c(t)\,dt < \infty$, see [5]. Hence

$$\int_T^\infty H(v(\infty))\,dt < \int_T^\infty H(v(t))\,dt < \infty.$$

This shows that $v(\infty) = 0$. □

3. Conditional oscillation

We say that equation (1) with a nonnegative function c is *conditionally oscillatory* if there exists a constant $\lambda_0 > 0$, called the *constant of conditional oscillation*, such that (1) with $\lambda c(t)$ instead of $c(t)$ is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. A typical example of a conditionally oscillatory half-linear equation is the Euler differential equation

$$\big(\Phi_p(x')\big)' + \frac{\lambda}{t^p}\Phi_p(x) = 0$$

with the oscillation constant

$$\lambda_0 = \gamma_p := \left(\frac{p-1}{p}\right)^p.$$

This means that the potential $c(t) = \gamma_p/t^p$ is a kind of borderline between oscillation and nonoscillation of equation (2). More precisely, this equation is oscillatory provided

$$\liminf_{t\to\infty} t^p c(t) > \gamma_p$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^p c(t) < \gamma_p.$$

The next theorem answers Conjecture 2 of [4] and shows that conditional oscillation of (1) is determined by the behavior of the function H in a right neighbourhood of $v = 0$.

Theorem 1. *Suppose that*

$$\lim_{v\to 0+} \frac{H(v)}{v^\beta} = L \in (0, \infty)$$

for some $\beta > 1$. Then equation (1) with $c(t) = \lambda t^{-\alpha}$, where $\alpha = \frac{\beta}{\beta-1}$ is the conjugate exponent of β, is conditionally oscillatory with the constant of conditional oscillation

$$\lambda_0 = \left(\frac{L}{\alpha - 1}\right)^{1-\alpha} \gamma_\alpha, \quad \gamma_\alpha := \left(\frac{\alpha - 1}{\alpha}\right)^\alpha.$$

Proof. First consider the case $\lambda > \lambda_0$. There exists $\varepsilon > 0$ such that

$$\lambda > \gamma_\alpha \left(\frac{L - \varepsilon}{\alpha - 1}\right)^{1-\alpha}. \tag{7}$$

Suppose, by contradiction, that (1) with $c(t) = \lambda t^{-\alpha}$ is nonoscillatory, i.e., by Proposition 1, there exists a proper solution of the equation

$$v' + \frac{\lambda}{t^\alpha} + H(v) = 0.$$

Then $v(t) \to 0+$ as $t \to \infty$ by Lemma 1, i.e., there exists T such that

$$H(v(t)) > (L - \varepsilon)(v(t))^\beta$$

for $t > T$, hence

$$v'(t) + \frac{\lambda}{t^\alpha} + (L - \varepsilon)(v(t))^\beta < 0. \tag{8}$$

The left-hand side of the last inequality is the Riccati operator corresponding to the half-linear equation (related to v by the formula $v = \left(\frac{L-\varepsilon}{\alpha-1}\right)^{1-\alpha} \Phi_\alpha\big(x'/x\big)\big)$

$$\left[\left(\frac{L-\varepsilon}{\alpha-1}\right)^{1-\alpha} \Phi_\alpha(x')\right]' + \frac{\lambda}{t^\alpha}\Phi_\alpha(x) = 0,$$

which is the same as the equation

$$\big(\Phi_\alpha(x')\big)' + \lambda\left(\frac{L-\varepsilon}{\alpha-1}\right)^{\alpha-1} t^{-\alpha}\Phi_\alpha(x) = 0.$$

We suppose (7), which means that the coefficient by $t^{-\alpha}\Phi_\alpha(x)$ in the last equation is greater than γ_α and this implies that this equation is oscillatory. But this is a contradiction in view of Proposition 1 since we have found a function satisfying inequality (8) on $[T, \infty)$.

Now we deal with the case $\lambda < \lambda_0$. Let $\varepsilon > 0$ be so small that

$$\lambda < \gamma_\alpha\left(\frac{L+\varepsilon}{\alpha-1}\right)^{1-\alpha}$$

and consider the function

$$v(t) = \left(\frac{L+\varepsilon}{\alpha-1}\right)^{1-\alpha} \Gamma_\alpha t^{1-\alpha}, \quad \Gamma_\alpha := \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}. \tag{9}$$

Then $v(t) \to 0+$ as $t \to \infty$, i.e., there exists T such that $H(v(t)) < (L+\varepsilon)(v(t))^\beta$ for $t > T$. We have

$$v'(t) + \frac{\lambda}{t^\alpha} + H(v(t)) < v'(t) + \gamma_\alpha\left(\frac{L+\varepsilon}{\alpha-1}\right)^{1-\alpha} t^{-\alpha} + (L+\varepsilon)(v(t))^\beta. \tag{10}$$

Substituting for v from (9) into the right-hand side of the last inequality we have

$$\begin{aligned}
v' + {} & \gamma_\alpha\left(\frac{L+\varepsilon}{\alpha-1}\right)^{1-\alpha} + (L+\varepsilon)v^\beta \\
&= t^{-\alpha}\left[\left(\frac{L+\varepsilon}{\alpha-1}\right)^{1-\alpha}(1-\alpha)\Gamma_\alpha + \gamma_\alpha\left(\frac{L+\varepsilon}{\alpha-1}\right)^{1-\alpha} + (L+\varepsilon)\left(\frac{L+\varepsilon}{\alpha-1}\right)^{(1-\alpha)\beta}\right] \\
&= \left(\frac{L+\varepsilon}{\alpha-1}\right)^{-\alpha} t^{-\alpha}\left[\frac{L+\varepsilon}{\alpha-1}(1-\alpha)\Gamma_\alpha + \frac{L+\varepsilon}{\alpha-1}\gamma_\alpha + (L+\varepsilon)\gamma_\alpha\right] \\
&= \left(\frac{L+\varepsilon}{\alpha-1}\right)^{1-\alpha} t^{-\alpha}(L+\varepsilon)\left[-\Gamma_\alpha + \gamma_\alpha\left(\frac{1}{\alpha-1}+1\right)\right] = 0.
\end{aligned}$$

Hence v is a proper solution of the Riccati inequality associated with the equation

$$x'' + \frac{\lambda}{t^\alpha} f(x, x') = 0$$

which means that this equation is nonoscillatory by Proposition 1. □

4. Principal solution and Prüfer transformation

In this concluding section we briefly describe the construction of the so-called principal solution of (1) based on the generalized Prüfer transformation. We follow essentially the idea of the paper [7], where the principal solution of the "classical" nonoscillatory half-linear equation

$$(r(t)\Phi_p(x'))' + c(t)\Phi_p(x) = 0$$

was introduced via the Prüfer type transformation.

Following [6], consider the equation (1) with $c(t) \equiv 1$, i.e., the equation

$$x'' + f(x, x') = 0$$

and denote by $S = S(t)$ its solution given by the initial condition $S(0) = 0$, $S'(0) = 1$. This solution defines *the generalized sine functions* and its derivative the *generalized cosine function*. Then any nontrivial solution of (1) and its derivative can be expressed in the form

$$x(t) = \rho(t)S(\varphi(t)), \quad x'(t) = \rho(t)S'(\varphi(t)), \tag{11}$$

where ρ is the positive radius function. Of course, in the linear case $f(x, x') = x$, this is nothing else than the classical Prüfer transformation. The angle variable φ is a solution the differential equation

$$\varphi' = 1 + (c(t) - 1)G(\varphi). \tag{12}$$

The function G is given by the formula

$$G(\varphi) = \begin{cases} \dfrac{F(T(\varphi))}{1 + F(T(\varphi))}, & \text{if } S'(\varphi) \neq 0, \\ 1, & \text{if } S'(\varphi) = 0, \end{cases} \tag{13}$$

where $T(\varphi) = S(\varphi)/S'(\varphi)$ is the generalized tangent function and the function F is given in the assumption (v) of Section 1. The function G is periodic with the period $2p$, where

$$p = \int_{-\infty}^{\infty} \frac{dt}{1 + F(t)}$$

is the first positive zero of S, and equation (12) is uniquely solvable. Moreover, we have $\varphi'(t) = 1$ at points where $\varphi(t) = 0$, and $S(\varphi) = 0$ if and only if $\varphi = kp$, $k \in \mathbb{Z}$.

Suppose that (1) is nonoscillatory and let T be so large that there is a solution $\hat{x}$ for which $\hat{x}(t) \neq 0$ for $t \geq T$. This solution can be expressed in the form (11) and let $\hat{\varphi}$ be its Prüfer angle. Then $\hat{\varphi}(t) \in (kp, (k+1)p)$ for some integer k and without loss of generality we can suppose that $k = -1$. Now, let $\tau \in (T, \infty)$ and denote by φ_τ the solution of (12) given by the initial condition $\varphi_\tau(\tau) = 0$. The unique solvability of (12) implies that $\hat{\varphi}(t) < \varphi_{\tau_2}(t) < \varphi_{\tau_1}(t)$ for $t \geq T$ whenever $T < \tau_1 < \tau_2$. This means that there exists a finite limit $\varphi^* := \lim_{\tau \to \infty} \varphi_\tau(T)$. The solution $\tilde{x}$ of (1) given by the initial condition

$$\tilde{x}(T) = S(\varphi^*), \quad \tilde{x}'(T) = S'(\varphi^*)$$

we call the *principal solution* of (1).

The concept of the principal solution of a nonoscillatory generalized half-linear equation (1) was introduced in [4] via Mirzov's method (see [9]) of the minimal solution of the associated Riccati equation (4). Next we show that the principal solution defined above using the generalized Prüfer transformation "produces" via the substitution $v = g(x'/x)$ the minimal solution of (4), i.e., both definitions are *equivalent*. Indeed, let x_τ be a nontrivial solution satisfying $x_\tau(\tau) = 0$. According to homogeneity of the solution space of (1), this solution is determined uniquely up to a nonzero multiplicative factor. The solution x_τ can be expressed in the form

$$x_\tau(t) = \rho(t) S(\varphi_\tau(t)), \quad x'_\tau(t) = \rho(t) S'(\varphi_\tau(t)),$$

where φ_τ is the solution of (12) satisfying $\varphi_\tau(\tau) = 0$. The solution $v_\tau(t) = g(x'_\tau(t)/x_\tau(t))$ of (4) satisfies, in view of the properties of the function g, the relation $v_\tau(-\tau) = -\infty$. The minimal solution of (4) is defined by

$$v_{\min}(t) = \lim_{\tau \to \infty} v_\tau(t),$$

i.e., it is just the solution satisfying $v_{\min}(T) = g\big(S'(\varphi^*)/S(\varphi^*)\big)$. Hence $\tilde{v}(t) = v_{\min}(t)$.

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