Publ. Math. Debrecen
82/2 (2013), 461–471
DOI: 10.5486/PMD.2013.5379

On Cartan torsion of Finsler metrics

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Abstract. In this paper, we find a relation between the norm of Cartan and mean Cartan torsions of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold. Then, we find a subclass of these metrics which have bounded Cartan torsion. It turns out that every C-reducible Finsler metric has bounded Cartan torsion.

1. Introduction

In Finsler geometry, there are several important non-Riemannian quantities. Let (M, F) be a Finsler manifold. The second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ are inner products $\mathbf{g}_y$ and symmetric trilinear forms $\mathbf{C}_y$ on T_xM, respectively. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The Cartan torsion is one of the most important non-Riemannian quantity in Finsler geometry and it was first introduced by Finsler [4] and emphased by Cartan [2]. A Finsler metric reduces to a Riemannian metric if and only if it has vanishing Cartan torsion. Taking a trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}_y$. In [3], Deicke proves that a positive definite Finsler metric F is Riemannian if and only if the mean Cartan torsion vanishes.

One of the fundamental problems in Finsler geometry is whether or not every Finsler manifold can be isometrically immersed into a Minkowski space, which is a finite-dimensional Banach space. The answer is affirmative for Riemannian manifolds. In [10], J. Nash proved that any n-dimensional Riemannian manifold can be isometrically imbedded into a higher dimensional Euclidean space. However

Mathematics Subject Classification: 53B40, 53C60.
Key words and phrases: Cartan torsion, Kropina metric, Randers metric.

for general Finsler manifolds, the problem becomes very difficult. In [5], Ingarden proves that every n-dimensional Finsler manifold can be locally isometrically imbedded into a $2n$-dimensional "Weak" Minkowski space, i.e., a space whose indicatrix is not necessarily strongly convex. Then Burago–Ivanov show that any compact C^r manifold ($r \geq 3$) with a C^2 Finsler metric admits a C^r imbedding into a finite-dimensional Banach spaces [1]. Recently, Shen proved that a Finsler manifold with unbounded Cartan torsion can not be isometrically imbedded into any Minkowski space [14]. Thus the norm of Cartan torsion plays an important role for studying of immersion theory in Finsler geometry.

In this paper, we consider the class of (α,β)-metrics and find the form of Cartan torsion of these metrics. We show that there exists a relation between the norm of Cartan and mean Cartan torsions of an (α,β)-metric. More precisely, we prove the following.

Theorem 1.1. *Let $F=\alpha\phi(s)$ be a non-Riemannian (α,β)-metric on a manifold M of dimension $n\geq 3$. Then the norm of Cartan and mean Cartan torsion of F satisfy in following relation*

$$\|\mathbf{C}\| = \sqrt{\frac{3p^2+6p\,q+(n+1)q^2}{n+1}}\ \|\mathbf{I}\|, \tag{1}$$

where $p=p(x,y)$ and $q=q(x,y)$ are scalar function on TM satisfying $p+q=1$ and given by following

$$p = \frac{n+1}{aA}\big[s(\phi\phi''+\phi'\phi')-\phi\phi'\big] \tag{2}$$

$$a := \phi\{\phi - s\phi'\} \tag{3}$$

$$A = (n-2)\frac{s\phi''}{\phi-s\phi'} - (n+1)\frac{\phi'}{\phi} - \frac{-3s\phi''+(b^2-s^2)\phi'''}{\phi-s\phi'+(b^2-s^2)\phi''}. \tag{4}$$

In [13], Shen proved that the Cartan torsion of Randers metrics $F=\alpha+\beta$ is uniformly bounded by $3/\sqrt{2}$. Then Mo extend his result to a more general Finsler metrics, namely, $F=\frac{(\alpha+\beta)^m}{\alpha^{m-1}}$ ($m\in[1,2]$) [9].

All of above metrics are special Finsler metrics so- called (α,β)-metrics. Let us narrate a brief history of (α,β)-metrics. This marchen originated in 1941 by a physicist G. Randers, who was first introduced the notion of Randers metrics to consider the unified field theory [11]. A Randers metric $F=\alpha+\beta$ on a manifold M is just a Riemannian metric $\alpha=\sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta=b_i(x)y^i$ on M such that $\|\beta\|_\alpha<1$ [15]. In the same time, another event was happened by a geometrician L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by V. K. Kropina [6]. Consequently, other match of Randers metric called Kropina metric $F = \alpha^2/\beta$ was born. Furthermore, by considering Kropina and Randers metrics, Matsumoto introduced the notion of (α, β)-metrics [6]. An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on M. Therefore, a natural question arises:

Is there any class of Finsler metrics which has bounded Cartan torsion?

In this paper, we consider a subclass of (α, β)-metrics which have the following form

$$F = \frac{\alpha^{m+1}}{\beta^m}, \quad (m \neq 0)$$

and called by generalized Kropina metric [6]. Then we prove the following.

Theorem 1.2. *Suppose that $F = \frac{\alpha^{m+1}}{\beta^m}$ be a generalized Kropina metric on a manifold M. Then the Cartan torsion of F is bounded. More precisely, the following holds*

$$\|\mathbf{C}\| = \frac{(2m+1)}{\sqrt{m(m+1)}}.$$

2. Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1. Thus, we must compute the Cartan torsion of an (α, β)-metric. Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$. It is easy to see that the following relations hold

$$\rho' = \alpha\rho_1, \quad -s\rho' = \alpha^2\rho_2, \quad -s\rho_0' = \alpha\rho_1', \quad -s\rho_1' = \alpha\rho_2'. \tag{5}$$

A direct computation shows that the Cartan curvature of F is given by the following

$$\begin{aligned} 2C_{ijk} := {} & \rho_1[a_{ij}b_k + a_{jk}b_i + a_{ki}b_j] + \rho_2[a_{ij}y_k + a_{jk}y_i + a_{ki}y_j] \\ & + \frac{\rho_0'}{\alpha} b_i b_j b_k - \frac{\rho_2'}{\alpha^2}\, s\, y_i y_j y_k + \frac{\rho_1'}{\alpha}[b_i b_j y_k + b_j b_k y_i + b_k b_i y_j] \\ & + \frac{\rho_2'}{\alpha}[b_i y_j y_k + b_j y_k y_i + b_k y_i y_j]. \end{aligned} \tag{6}$$

By (5) and (6), we have

$$\begin{aligned}2C_{ijk} &= [\rho_1 - \rho_2\alpha\epsilon][a_{ij}b_k + a_{jk}b_i + a_{ki}b_j] + \rho_2\alpha[a_{ij}Y_k + a_{jk}Y_i + a_{ki}Y_j]\\ &+ \frac{\rho'_0}{\alpha}b_ib_jb_k - \frac{\rho'_2 s}{\alpha^2}y_iy_jy_k + \frac{\rho'_1}{\alpha}[b_ib_jy_k + b_jb_ky_i + b_kb_iy_j]\\ &+ \frac{\rho'_2}{\alpha}[b_iy_jy_k + b_jy_ky_i + b_ky_iy_j]. \qquad (7)\end{aligned}$$

We can express the angular metric $h_{ij} := g_{ij} - F_{y^i}F_{y^j}$ in the following form

$$h_{ij} = a\ a_{ij} + b\ b_ib_j + c\ [b_i\alpha_j + b_j\alpha_i] + d\ \alpha_i\alpha_j, \qquad (8)$$

where

$$\begin{aligned}a &:= \phi[\phi - s\phi']\\ b &:= \phi\phi''\\ c &:= -s\phi\phi''\\ d &:= -\phi\big[(\phi - s\phi') - s^2\phi''\big].\end{aligned}$$

On the other hand, the mean Cartan torsion is given by

$$I_i = \frac{s}{2\alpha}AY_i, \qquad (9)$$

where

$$A = (n-2)\frac{s\phi''}{\phi - s\phi'} - (n+1)\frac{\phi'}{\phi} - \frac{-3s\phi'' + (b^2 - s^2)\phi'''}{\phi - s\phi' + (b^2 - s^2)\phi''}.$$

Solving (8) for a_{ij}, plugging the result and (9) into (7) and considering $\dim M \geq 3$, implies that the Cartan tensor of an (α,β)-metric is given by following

$$C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{\|\mathbf{I}\|^2}I_iI_jI_k. \qquad (10)$$

where $p = p(x,y)$ and $q = q(x,y)$ are scalar function on TM satisfying $p + q = 1$ and given by following

$$p = \frac{n+1}{aA}\big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\big]. \qquad (11)$$

It is remakable that, a Finsler metric is called semi-C-reducible if its Cartan tensor is given by the equation (10). It is proved that every non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible [6]. By (10) we have

$$C^{ijk} = \frac{p}{1+n}\{h^{ij}I^k + h^{jk}I^i + h^{ki}I^j\} + \frac{q}{\|\mathbf{I}\|^2}I^iI^jI^k. \qquad (12)$$

Then

$$C_{ijk}C^{ijk} = \left[\frac{3p(p+2q)}{n+1} + q^2\right]I_mI^m. \qquad (13)$$

This completes the proof. □

3. Proof of Theorem 1.2

In this section, we are going to prove the Theorem 1.2. Let $F = \alpha\phi(s)$ be an (α,β)-metric on a manifold M of dimension n, where $s = \frac{\beta}{\alpha}$, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Then the fundamental tensor of F is given by

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j,$$

where

$$\rho := \phi(\phi - s\phi'), \qquad \rho_0 := \phi\phi'' + \phi'\phi'$$

$$\rho_1 := -\big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\big], \quad \rho_2 := s\big[s(\phi\phi'' + \phi'\phi') - \phi\phi'\big]$$

$$\alpha_i := \frac{a_{ij}y^j}{\alpha}.$$

Put

$$A_{ij} := a_{ij} + \delta b_i b_j, \quad \delta := \frac{\rho_0 - \varepsilon^2\rho_2}{\rho}, \quad \varepsilon := \frac{\rho_1}{\rho_2}.$$

Then

$$A^{ij} := (A_{ij})^{-1} = a^{ij} - \tau b^i b^j, \quad \tau := \frac{\delta}{1+\delta b^2}.$$

By a simple calculation, we get

$$g^{ij} = \rho^{-1}[a^{ij} - \tau b^i b^j - \eta Y^i Y^j],$$

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]\det(a_{ij}),$$

where

$$\eta = \frac{\mu}{1+Y^2\mu}, \quad \mu := \frac{\rho_2}{\rho}, \quad Y := \sqrt{A_{ij}Y^iY^j},$$

$$Y_i = \alpha_i + \varepsilon b_i, \quad Y^i := A^{ij}Y_j = \frac{y^i}{\alpha} + \lambda b^i, \quad \lambda := \frac{\varepsilon - \delta s}{1+\delta b^2}.$$

By putting $\phi := \frac{1}{s}$, we compute the above relations for the Kropina metric as follows

$$\rho = \frac{2}{s^2}, \quad \rho_0 = \frac{3}{s^4}, \quad \rho_1 = \frac{-4}{s^3}, \quad \rho_2 = \frac{4}{s^2}, \quad \varepsilon = \frac{-1}{s}, \quad \mu = 2$$

$$\delta = \frac{-1}{2s^2}, \quad \lambda = \frac{s}{b^2 - s^2}, \quad Y^2 = \frac{s^2 - b^2}{b^2 - 2s^2}, \quad \tau = \frac{1}{b^2 - 2s^2}, \quad \eta = 4\frac{s^2}{b^2} - 2,$$

which implies that

$$g_{ij} = \frac{2}{s^2}\left[a_{ij} + \frac{3}{2}\frac{b_i b_j}{s^2} - \frac{2}{s}(b_i\alpha_j + b_j\alpha_i) + \frac{2y_i y_j}{\alpha^2}\right], \tag{14}$$

$$g^{ij} = \frac{s^2}{2}\left[a^{ij} - \frac{b^i b^j}{b^2} + \frac{2s}{\alpha b^2}(b^i y^j + b^j y^i) + \frac{2(b^2 - 2s^2)}{b^2\alpha^2} y^i y^j\right], \tag{15}$$

$$\det(g_{ij}) = \frac{2^{n-1}b^2}{s^{2n+2}}\det(a_{ij}), \tag{16}$$

$$I_i = \frac{\partial}{\partial y^i}\ln\sqrt{\det(g_{ij})} = (n+1)\left[\frac{y_i}{\alpha^2} - \frac{b_i}{\beta}\right]. \tag{17}$$

For a Finsler metric F, one can defines the norm of the mean Cartan torsion $\mathbf{I}$ and the Cartan torsion $\mathbf{C}$ as follows

$$\|\mathbf{I}\| = \sup_{F(y)=1, v\neq 0}\frac{|\mathbf{I}_y(v)|}{[\mathbf{g}_y(v,v)]^{\frac{1}{2}}}, \quad \|\mathbf{C}\| = \sup_{F(y)=1, v\neq 0}\frac{|\mathbf{C}_y(v,v,v)|}{[\mathbf{g}_y(v,v)]^{\frac{3}{2}}}. \tag{18}$$

Lemma 3.1. *Let (M, F) be an n-dimensional Finsler manifold. Suppose that $F = \frac{\alpha^2}{\beta}$ be the Kropina metric. Then the norm of mean Cartan tensor of F is given by following*

$$\|\mathbf{I}\| = \frac{(n+1)}{\sqrt{2}}. \tag{19}$$

Proof. Let $F = \frac{\alpha}{s}$, $s = \frac{\beta}{\alpha}$, $|s| < 1$. Then by (17) we have

$$\begin{aligned} g^{ij}I_iI_j &= \frac{(n+1)^2 s^2}{2}\left[\left(\frac{2\beta^2 - b^2\alpha^2}{b^2\alpha^4}\right)y^j - \frac{\beta}{b^2\alpha^2}b^j\right]\left[\frac{y_j}{\alpha^2} - \frac{b_j}{\beta}\right] \\ &= \frac{(n+1)^2 s^2 (b^2 - s^2)}{2b^2\alpha^2}. \end{aligned}$$

Thus

$$\sup_{v\neq 0}\frac{|\mathbf{I}_y(v)|}{[\mathbf{g}_y(v,v)]^{\frac{1}{2}}} = \sqrt{I^i I_i} = \frac{(n+1)s\sqrt{(b^2 - s^2)}}{\sqrt{2}b\alpha} = \frac{(n+1)}{\sqrt{2}\, bF}\sqrt{(b^2 - s^2)}$$

which yields

$$\begin{aligned} \|\mathbf{I}\| &= \sup_{F(y)=1, v\neq 0}\frac{|\mathbf{I}_y(v)|}{[\mathbf{g}_y(v,v)]^{\frac{1}{2}}} = \sup_{F(y)=1}\left[\sup_{v\neq 0}\frac{|\mathbf{I}_y(v)|}{[\mathbf{g}_y(v,v)]^{\frac{1}{2}}}\right] \\ &= \sup_{|s|<b}\frac{(n+1)}{\sqrt{2}\, b}\sqrt{(b^2 - s^2)} = \frac{(n+1)}{\sqrt{2}}. \end{aligned} \tag{20}$$

Thus the mean Cartan torsion of Kropina metric is bounded. □

Now, we are going to find the norm of Cartan torsion. First, we consider the case of $\dim M = 2$. Let us remark the Lemma 1.2.2 of [12].

Lemma 3.2 ([12]). *Let (V, F) be a Minkowski plane and $V_0 := V - \{0\}$. For a vector $y \in V$ with $L(y) \neq 0$, there is a vector $y^{\perp} \in V_0$ such that*

$$\mathbf{g}_y(y, y^{\perp}) = 0, \quad \mathbf{g}_y(y^{\perp}, y^{\perp}) = \epsilon L(y),$$

where $\epsilon = ind(L)$ denotes the index of L.

Lemma 3.3. *Let (V, F) be an n-dimensional Minkowski space. Suppose that $F = \frac{\alpha^2}{\beta}$ be the Kropina metric. Then the norm of Cartan torsion of F is bounded as follows*

$$\|\mathbf{C}\| \leq \frac{3\sqrt{2}}{2}. \tag{21}$$

Proof. First, let us consider the case that $\dim(M) = 2$. Take an oriented basis $\{e_1, e_2\}$ for V which determines a global coordinate system (u, v) in V. Let $L(u, v) := L(ue_1 + ve_2)$. Then for a vector $y = ue_1 + ve_2 \in V_0$, define the vector $y^{\perp} \in V$ as follows

$$y^{\perp} = \frac{-L_v e_1 + L_u e_2}{\sqrt{L_{uu}L_{vv} - L_{uv}L_{uv}}}.$$

Thus we have

$$L_v L_{uu} - L_u L_{uv} = [L_{uu}L_{vv} - L_{uv}L_{uv}]v$$
$$L_u L_{vv} - L_v L_{uv} = [L_{uu}L_{vv} - L_{uv}L_{uv}]u,$$

which yield

$$\begin{aligned} L_v^2 L_{uu} - 2L_u L_v L_{uv} + L_u^2 L_{vv} &= (uL_u + vL_v)[L_{uu}L_{vv} - L_{uv}L_{uv}] \\ &= 2L[L_{uu}L_{vv} - L_{uv}L_{uv}]. \end{aligned}$$

Then we get

$$g_y(y, y^{\perp}) = 0, \quad g_y(y^{\perp}, y^{\perp}) = \frac{L_v^2 L_{uu} - 2L_u L_v L_{uv} + L_u^2 L_{vv}}{2|L_{uu}L_{vv} - L_{uv}L_{uv}|} = \epsilon L(y).$$

The basis $\{y, y^{\perp}\}$ is called the Berwald frame at y. Define

$$I(y) := \frac{\mathbf{C}_y(y^{\perp}, y^{\perp}, y^{\perp})}{L(y)}.$$

It is remarkable that I is 0-homogeneous function and called by the main scalar of L. Let L is positive definite on V. For $y = ue_1 + ve_2$, we have

$$I(y) = \frac{2L^2 L_{vvv}}{(2LL_{vv} - L_v L_v)^{\frac{3}{2}}}.$$

We can express L as $L = [u\phi(\frac{v}{u})]^2$, where $\phi = \phi(s)$ is a positive C^∞ function with $\phi_{\varepsilon\varepsilon}(\varepsilon) > 0$. Then for $y = e_1 + \varepsilon e_2$, we get

$$I(y) = \frac{3\phi_\varepsilon \phi_{\varepsilon\varepsilon\varepsilon} + \phi\phi_{\varepsilon\varepsilon\varepsilon}}{2\phi^{\frac{1}{2}} \phi_{\varepsilon\varepsilon}^{\frac{3}{2}}}.$$

Now, we take an orthonormal basis $\{e_1, e_2\}$ for (V, α) such that $\beta(ue_1+ve_2) = bu$, where $b = \|\beta\|_\alpha := \sup_{\alpha(y)=1} \beta(y)$. For the Kropina metric, we have

$$F = \frac{\alpha^2}{\beta} = \frac{u^2 + v^2}{bu} = u\left(\frac{1 + (\frac{v}{u})^2}{b}\right).$$

If $\phi(\varepsilon) = \frac{1+\varepsilon^2}{b}$, then

$$L = F^2 = \left[u\phi\left(\frac{v}{u}\right)\right]^2.$$

By a simple calculation, we get

$$\phi_\varepsilon = \frac{2\varepsilon}{b}, \quad \phi_{\varepsilon\varepsilon} = \frac{2}{b}, \quad \phi_{\varepsilon\varepsilon\varepsilon} = 0.$$

Thus the main scalar of Kropina metric in the point $y = e_1 + \varepsilon e_2$ is given by

$$I(y) = \frac{3\phi_\varepsilon \phi_{\varepsilon\varepsilon\varepsilon} + \phi\phi_{\varepsilon\varepsilon\varepsilon}}{2\phi^{\frac{1}{2}} \phi_{\varepsilon\varepsilon}^{\frac{3}{2}}} = \frac{3\varepsilon}{\sqrt{2(1+\varepsilon^2)}},$$

which implies that

$$\max |I| = \frac{3}{\sqrt{2}}.$$

Note that in the dimension two, $\|\mathbf{C}\| = \max |I|$ and then $\|\mathbf{C}\| = \frac{3}{\sqrt{2}}$.

Now, let $\dim(M) > 2$. Base on the definition of norm of Cartan torsion, there exist the vectors y_0 and v_0 such that $\|\mathbf{C}\| = \mathbf{C}_{y_0}(v_0, v_0, v_0)$. Put $\bar{V} := \mathrm{span}\{y_0, v_0\}$ and $\bar{F} := F_{|\bar{V}}$. Let $\bar{\mathbf{C}}$ denote the Cartan tensor of $\bar{F}$ on $\bar{V}$. Then

$$\mathbf{C}_{y_0}(v_0, v_0, v_0) = \frac{1}{4}\frac{\partial^3 F^2(y_0 + sv_0)}{\partial s^3} = \bar{\mathbf{C}}_{y_0}(v_0, v_0, v_0).$$

If we put $\bar{\beta} := \beta|_{\bar{V}}$ and $\bar{\alpha} := \alpha|_{\bar{V}}$, then $\|\bar{\beta}\| := \sup_{\bar{\alpha}(y)=1} \bar{\beta}(y) \leq \|\beta\|$. Let $\bar{I} := I|_{\bar{V}}$ denotes the main scalar of $\bar{F}$ on $\bar{V}$. Since $\|\bar{\mathbf{C}}\| = \max |\bar{I}| \leq \frac{3}{\sqrt{2}}$, then

$$\|\mathbf{C}\| = \bar{\mathbf{C}}_{y_0}(v_0, v_0, v_0) \leq \|\bar{\mathbf{C}}\| \leq \frac{3\sqrt{2}}{2}.$$

This completes the proof. □

It is remarkable that, regarding the Cartan tensors of the Randers metric $F=\alpha+\beta$ and the Kropina metric $F=\frac{\alpha^2}{\beta}$, Matsumoto introduced the notion of C-reducibility and proved that any Randers and Kropina metrics are C-reducible [7]. In [8], Matsumoto–Hōjō proved that the converse is true. A Finsler metric F is called C-reducible if its Cartan tensor is given by

$$C_{ijk}=\frac{1}{1+n}\big\{h_{ij}I_k+h_{jk}I_i+h_{ki}I_j\big\}, \tag{22}$$

where $h_{ij}:=FF_{y^iy^j}$ is the angular metric. On the other hand, Shen proved that the Cartan torsion of a Randers metric is bounded [12]. Thus by Lemma 3.3, we conclude the following.

Corollary 3.1. *Every C-reducible Finsler metric on a manifold M od dimension $n\geq 3$ has bounded Cartan torsion.*

Now, let F be a C-reducible Finsler metric. Then we have

$$C^{ijk}=\frac{1}{1+n}\big\{h^{ij}I^k+h^{jk}I^i+h^{ki}I^j\big\}, \tag{23}$$

where $h^{ij}=g^{ij}-F^{-2}y^iy^j$. By (22) and (23), we have

$$C^{ijk}C_{ijk}=\frac{3}{n+1}I^iI_i.$$

Thus we conclude the following.

Corollary 3.2. *Let (M,F) be a n-dimensional C-reducible Finsler manifold. Then*

$$\|\mathbf{C}\|=\sqrt{\frac{3}{n+1}}\ \|\mathbf{I}\|. \tag{24}$$

In the case $n=2$, we obtain $\|\mathbf{C}\|=\|\mathbf{I}\|$, which proved in the previous Lemmas 3.1 and 3.3.

Now, we are going to prove the Theorem 1.2.

Proof of Theorem 1.2. For the generalized Kropina metric $F=\frac{\alpha^{m+1}}{\beta^m}$ on a 2-dimensional plane V, put

$$\phi(\varepsilon)=\frac{[1+\varepsilon^2]^{\frac{m+1}{2}}}{b^m}.$$

For $L=F^2=[u\phi(\frac{v}{u})]^2$, we get

$$\phi_\varepsilon=\frac{(m+1)\varepsilon[1+\varepsilon^2]^{\frac{m-1}{2}}}{b^m},$$

$$\phi_{\varepsilon\varepsilon} = \frac{(m+1)(1+m\varepsilon^2)[1+\varepsilon^2]^{\frac{m-3}{2}}}{b^m},$$

$$\phi_{\varepsilon\varepsilon\varepsilon} = \frac{(m+1)(m-1)(3+m\varepsilon^2)\varepsilon[1+\varepsilon^2]^{\frac{m-5}{2}}}{b^m}.$$

In the point $(1,\varepsilon)$, we have

$$I(y) = \frac{m\varepsilon[(2m+1)\varepsilon^2+3]}{(m+1)^{\frac{1}{2}}(1+m\varepsilon^2)^{\frac{3}{2}}},$$

which implies that

$$\|\mathbf{C}\| = \max|I| = \frac{(2m+1)}{\sqrt{m(m+1)}}.$$

The proof for the higher dimensions, is the same of 2-dimensional case and we omit it. □

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(Received November 9, 2011; revised March, 5 2012)