Two dimensional (α, β)-metrics with reversible geodesics

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Abstract. We study the necessary and sufficient conditions for a Finsler surface with (α, β)-metrics to be with reversible geodesics. We show that such a Finsler structure is with reversible geodesics if and only if it is a Randers change of an absolute homogeneous Finsler metric by a closed one-form.

1. Introduction

It is well-known that in general the geodesics of a Finsler space (M, F) are not reversible, i.e. for a geodesic $\gamma : [0, 1] \to M$, the reverse curve $\bar{\gamma} : [0, 1] \to M$, $\bar{\gamma}(t) := \gamma(1 - t)$ is not necessarily a geodesic. This peculiarity distinguishes Finslerian structures from the Riemannian ones. Of course, if (M, F) is a Riemannian or an absolute homogeneous Finsler space, then geodesics are reversible by definition.

We have studied in the past the problem of finding necessary and sufficient conditions for a Finsler space with (α, β)-metric to be with reversible geodesics (see [MSS], compare with [Cr]). The method used there was inspired by the study of projectively related Finsler spaces with (α, β)-metrics ([BM], [S1], compare also [Ai]). However, the method does not work well in the 2-dimensional case. Indeed, we were not able to give a necessary and sufficient condition for a Finsler surface with (α, β)-metric to be with reversible geodesics.

In the present paper, we approach this open problem in a completely different way by using the method of moving frames. This time, we succeeded to characterize Finsler surface with (α, β)-metrics, other than Riemannian and absolute homogeneous, to be with reversible geodesics in the following main theorem (see Section 2 for notations).

Theorem 1.1. *Let (M, F) be a Finsler surface with (α, β)-metric which is not projectively equivalent to the Riemannian metric a. Then F is with reversible geodesics if and only if F is a Randers change $F = F_0 + \varepsilon\beta$ of an absolute homogeneous Finsler metric F_0 with (α, β)-metric, $\varepsilon \neq 0$, by a closed one form β.*

Here is the content of our paper. In Section 2 we recall the basic notions and notations of Finsler surfaces and describe the (α, β)-metrics. In Section 3 we consider moving frames on a Finsler surface and obtain the characterization of metrics with reversible geodesics in terms of a differential relation with respect to a Riemannian moving frame (Proposition 3.1). In Sections 4, 5 we compute this differential relation in terms of α and β using isothermal coordinates (Theorem 4.2), and give some properties of the functions $\mathcal{E}$ and $\mathcal{F}$. Finally, in the last section, we study the integrability of PDE (6.6) and prove our main Theorem 1.1.

Our results gives an answer to one of the questions in [Ca] referring to the use of moving frames technique in other cases than the Randers metric. It also suggest a method to study the projective relatedness of (α, β)-metrics in the 2-dimensional case left unsolved by BÁCSÓ and MATSUMOTO in [BM]. It is true that formally our findings are similar to AIKOU's in [Ai], but the present paper is not a consequence of his result but links it with other types of formalism (see the Remark in the end of the paper).

2. Finsler surfaces

Let us recall that a Finsler surface is a pair (M, F), where M is a real smooth 2-dimensional manifold and $F : TM \longrightarrow [0, \infty)$ a Finsler norm, i.e. a positive, smooth function on $\widetilde{TM} = TM \setminus \{0\}$, with the homogeneity propryety $F(x, \lambda y) = \lambda \cdot F(x, y)$, for all $\lambda > 0$ and all $(x, y) \in \widetilde{TM}$ and whose Hessian matrix $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive-definite at each point $u = (x, y) \in \widetilde{TM}$.

Equivalently, a Finsler structure on the surface M can be regarded as a smooth hypersurface $\Sigma^3 \subset TM$ for which the canonical projection $\pi : \Sigma \longrightarrow TM$ is a surjective submersion having the property that for each $x \in M$, the π-fiber

$\Sigma_x = \pi^{-1}(x)$ is a smooth, closed, strongly convex curve in $T_x M$ enclosing the origin.

Here, strongly convex means that Σ_x is strictly convex and it has contact of precisely order two with its tangent line in each point. Traditionally, the curve $\Sigma_x \subset TM$ is called *the indicatrix* of the Finsler structure F and it has the property that it is not centrally symmetric about the origin of $T_x M$. If it is, then the Finsler structure (M, F) is called *absolutely homogeneous*, in other words, $F(x, y) = F(x, -y)$, for all $(x, y) \in TM$.

The simplest case of Finsler surface is a Riemannian surface and in this case its indicatrix is a centrally symmetric circle on an ellipse in $T_x M$, as well known.

We are going to construct a canonical moving frame on Σ (see for example [Br1997], [Br2002]).

Let Σ_1 be the unit tangent bundle of a Riemannian metric, say a, on M (it is customary to denote $\alpha := \sqrt{a(y, y)}$, where $(x, y) \in TM$). For any Finsler structure Σ on M, there exists a smooth, positive function $r : \Sigma_1 \longrightarrow \mathbf{R}^+$ such that

$$\Sigma \equiv \Sigma_r = \left\{ \frac{1}{r(u)} \cdot u : u \in \Sigma_1 \right\}.$$

This notations will be used throughly.

In order to assure the strong convexity on Σ, an additional differential condition on r must be given. Conversely, any positive function $p : \Sigma_1 \longrightarrow \mathbf{R}^+$ satisfying a certain differential condition defines a Finsler structure on the surface M in this way. In other words, one can say that a Finsler structures on a surface M dependes on a function of 3 variables, namely the function r on Σ_1. Obviously Σ_p is in fact the indicatrix bundle of (M, F) and the curve $\Sigma_p\big|_x = \left\{ \frac{1}{p(u)} \cdot u : u \in \Sigma_1\big|_x \right\} \equiv \{y \in T_x M : F(x, y) = 1\}$ corresponds to the indicatrix curve described above.

The function $\rho : \Sigma_p \longrightarrow \Sigma_1$, $\rho\left(\frac{1}{p(u)} \cdot u\right) = u$, $\forall u \in \Sigma_1$ is the "inverse" function which takes the Finsler structure (M, F) back to the original Riemannian structure (M, a).

The functions F and p are essentially the same, namely, if one parametrizes the Riemannian indicatrix $\Sigma_1\big|_x$ by the usual Euclidean angle t, then

$$p(x^1, x^2, t) = F(x^1, x^2, \cos t, \sin t),$$

where $\big(y^1(t), y^2(t)\big) = (\cos t, \sin t)$.

Recall that a *Finsler space with (α, β)-metric* (M, F) is given by a Finslerian norm $F = F(x, y) : TM \longrightarrow [0, \infty)$, where F is a positive one-homogeneous function of the two arguments α and β. Hereafter we consider only (α, β)-metrics

obtained by means of a positive definite Riemannian metric (M, a) on M and a linear 1-form $\beta(x, y) = b_i(x)y^i$, such that $b^2 := a(b, b) < 1$.

Following SHEN ([S2]), we can always write F as

$$F = \alpha \cdot \phi\left(\frac{\beta}{\alpha}\right), \tag{2.1}$$

where $\phi : I = [-r, r] \longrightarrow [0, \infty)$ is a C^∞ function and the interval I can be chosen large enough such that $r \geq \left|\frac{\beta}{\alpha}\right|$, for all $x \in M$ and $y \in T_x M$.

We also recall

Lemma 2.1 ([S2])**.** *The function $F = \alpha \cdot \phi(s)$, $s = \frac{\beta}{\alpha}$ is a Finsler metric for any $\alpha = \sqrt{a_{ij}y^i y^j}$ and any $\beta = b_i y^i$ with $\|\beta_x\|_\alpha < b_0$ if and only if $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying the following condition:*

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0. \tag{2.2}$$

Remark 2.1. (1) Lemma 2.1 implies that

$$\phi(s) - s\phi'(s) > 0, \quad |s| < b_0 \tag{2.3}$$

(2) In general, due to the presence of the 1-form β, the function F is not absolute homogeneous.

Classical examples of (α, β)- metrics are: the *Randers metrics*, namely $F = \alpha + \beta$, or *Matsumoto metrics*, i.e. $F = \frac{\alpha^2}{\alpha - \beta}$.

For simplicity, we will use in the following the notations:

$$\phi'(s) = \left.\frac{\partial \phi(t)}{\partial t}\right|_{t=s}, \quad \phi'(-s) = \left.\frac{\partial \phi(t)}{\partial t}\right|_{t=-s}. \tag{2.4}$$

In other words, we have

$$\left[\phi(-s)\right]' = \frac{d\phi(-s)}{ds} = -\phi'(-s), \quad \left[\phi(-s)\right]'' = \frac{d^2\phi(-s)}{d^2 s} = \phi''(-s). \tag{2.5}$$

It can be easily seen that if (M, F) is a Finsler metric, then its *reverse metric* $\bar{F}(x, y) := F(x, -y)$ must be a Finsler metric as well.

Let us remark that Lemma 2.1 implies

Lemma 2.2. *If $F = \alpha \cdot \phi\left(\frac{\beta}{\alpha}\right)$ is an (α, β) Finsler metric, then ϕ cannot be an odd function.*

PROOF. Let us assume that F is Finsler and the corresponding ϕ is odd, i.e. $\phi(-s) = -\phi(s)$, for all $s \in (-b_0, b_0)$, then it follows

$$\big[\phi(-s)\big]' = \big[-\phi(s)\big]' = -\phi'(s), \quad \big[\phi(-s)\big]'' = \big[-\phi(s)\big]'' = -\phi''(s). \qquad (2.6)$$

On the other hand, using the derivation rule of composed functions, we get

$$\big[\phi(-s)\big]' = -\phi'(-s), \quad \big[\phi(-s)\big]'' = \big[-\phi(s)\big]'' = \phi''(-s).$$

and therefore, for odd functions, we obtain $\phi'(-s) = \phi'(s)$, $\phi''(-s) = -\phi''(s)$.

Substituting these formulas in (2.2), we have

$$-\phi(-s) + s\phi'(-s) - (b^2 - s^2)\phi''(-s) > 0, \quad |s| \leq b < b_0,$$

but these formula contradicts (2.2) written by putting $-s$ instead of s, i.e. it is impossible for ϕ to be an odd function. $\qquad \square$

3. Moving frames on Finsler surfaces

The 3-manifold Σ_1 can be regarded as the orthonormal frame bundle over M with respect to a and therefore it has a canonical coframing $\{\alpha^1, \alpha^2, \alpha^3\}$, where α^1, α^2 are the tautological 1-forms and α^3 is the Levi–Civita connection form. The canonical coframing $\{\alpha^1, \alpha^2, \alpha^3\}$ satisfies the structure equations

$$d\alpha^1 = \alpha^2 \wedge \alpha^3, \quad d\alpha^2 = \alpha^3 \wedge \alpha^1, \quad d\alpha^3 = k\alpha^1 \wedge \alpha^2, \qquad (3.1)$$

where the function $k : M \longrightarrow \mathbf{R}$ is the Gauss curvature of the Riemannian structure (M, a).

It is well known that for a Finsler structure (M, F) with indicatrix bundle $\Sigma \subset TM$ a canonical coframing $\{\omega^1, \omega^2, \omega^3\}$ can be as well constructed. The corresponding structure equations are

$$d\omega^1 = -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3, \quad d\omega^2 = -\omega^1 \wedge \omega^3,$$

$$d\omega^3 = K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3, \qquad (3.2)$$

where the functions $I, J, K : \Sigma \longrightarrow \mathbf{R}$ are called *the Cartan, Landsberg* and *flag curvatures*, respectively (see [Br1997], [Br2002], [SSS]). We point out that, unlikely the Riemannian case, all these curvatures live on Σ and not on the base manifold M.

Regarding now the Finslerian indicatrix bundle $\Sigma \equiv \Sigma_p$ as a deformation of the Riemannian unit tangent bundle Σ_1 by $\rho : \Sigma_p \longrightarrow \Sigma_1$, where $p : \Sigma_1 \longrightarrow \mathbf{R}^+$ gives the Finslerian norm, it is quite obvious that the cotangent map $\rho^* : T^*\Sigma_1 \longrightarrow T^*\Sigma_p$, will allow to obtain the Finsler coframing $\{\omega^1, \omega^2, \omega^3\}$ from the Riemannian one $\{\alpha^1, \alpha^2, \alpha^3\}$. Indeed, some computations show

$$\omega^1 = \rho^*\big(\sqrt{p(p + p_{33})}\,\alpha^1\big), \quad \omega^2 = \rho^*(p\alpha^2 + p_3\alpha^1),$$

$$\omega^3 = \rho^*\left(\frac{(p + p_{33})\alpha^3 + (p_{32} - p_1)\alpha^2}{\sqrt{p(p + p_{33})}} + \frac{P_p\alpha^1}{\sqrt{p^3(p + p_{33})^3}}\right), \qquad (3.3)$$

where

$$P_p = \frac{1}{2}(p_3 p_{32} p_{33} - p_3 p_{33} p_1 + p p_{333} p_{32} - p p_1 p_{333} + 2p p_{32} p_3 - 2p p_1 p_3$$

$$- 3p p_2 p_{33} - p^2 p_{332} - 2p^2 p_2 - p_2 p_{33}^2 - p p_{332} p_{33}). \qquad (3.4)$$

It can be seen that the strongly convexity of Σ_r is equivalent to the differentiable condition (see [Br1997], [Ca])

$$p_{33} + p > 0,$$

where the subscript indicate the directional derivatives with respect to the Riemannian coframing $\{\alpha^1, \alpha^2, \alpha^3\}$, i.e. for any differentiable function $f : \Sigma_1 \longrightarrow \mathbf{R}$ we denote $df = f_1 \cdot \alpha^1 + f_2 \cdot \alpha^2 + f_3 \cdot \alpha^3$.

It is known that the geodesics of the Riemannian structure (M, a) are the projections to M of the integral lines of the differential system $\{\alpha^1 = 0, \alpha^3 = 0\}$ defined on Σ_1.

Similarly, for a Finsler structure (M, F) with indicatrix bundle Σ and canonical coframing $\{\omega^1, \omega^2, \omega^3\}$, the Finslerian geodesics are the projections to M of the integral lines of the exterior differential system $\{\omega^1 = 0, \omega^3 = 0\}$ on Σ.

Let us consider now another Finsler structure $\overline{F}$ on the same surface M. This implies that there exists another smooth positive function, say $r : \Sigma_1 \longrightarrow \mathbf{R}^+$, such that $\Sigma_r = \big\{\frac{1}{r(u)} \cdot u : u \in \Sigma_1\big\}$ is the indicatrix bundle of $(M, \overline{F})$. The inverse function $\bar{\rho} : \Sigma_r \longrightarrow \Sigma_1$,
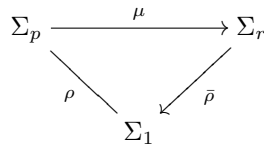
$$\bar{\rho}\left(\frac{1}{r(u)} \cdot u\right) = u, \quad \forall u \in \Sigma_1 \qquad (3.5)$$

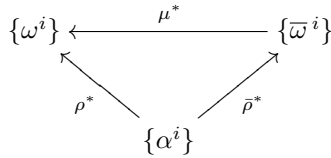allows to recover the original Riemannian structure (M, a).

Obviously, $\bar{\rho}$ is invertible in the sense that we can always define

$$\bar{\rho}^{-1} : \Sigma_1 \longrightarrow \Sigma_r, \quad \bar{\rho}^{-1} = \frac{1}{r(u)} \cdot u, \quad \forall u \in \Sigma_1. \tag{3.6}$$

This means that the following diagram is commutative

$$
\begin{array}{ccc}
\Sigma_p & \xrightarrow{\ \ \mu\ \ } & \Sigma_r \\
& \rho \searrow \quad \swarrow \bar{\rho} & \\
& \Sigma_1 &
\end{array}
$$

where $\mu := \bar{\rho}^{-1} \circ \rho$, and therefore, the

$$
\begin{array}{ccc}
\{\omega^i\} & \xleftarrow{\ \ \mu^*\ \ } & \{\bar{\omega}^{\,i}\} \\
& \rho^* \nwarrow \quad \nearrow \bar{\rho}^* & \\
& \{\alpha^i\} &
\end{array}
$$

where $\{\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3\}$ is the associated canonical coframe of $(M, \overline{F})$ defined in the same way as above. We also have

$$\bar{\omega}^1 = \bar{\rho}^*\big(\sqrt{r(r + r_{33})}\alpha^1\big), \quad \bar{\omega}^2 = \bar{\rho}^*(r\alpha^2 + r_3\alpha^1),$$

$$\bar{\omega}^3 = \bar{\rho}^*\left(\frac{(r + r_{33})\alpha^3 + (r_{32} - r_1)\alpha^2}{\sqrt{r(r + r_{33})}} + \frac{P_r\alpha^1}{\sqrt{r^3(r + r_{33})^3}}\right), \tag{3.7}$$

where

$$P_r = \frac{1}{2}(r_3 r_{32} r_{33} - r_3 r_{33} r_1 + r r_{333} r_{32} - r r_1 r_{333} + 2 r r_{32} r_3 - 2 r r_1 r_3$$

$$- 3 r r_2 r_{33} - r^2 r_{332} - 2 r^2 r_2 - r_2 r_{33}^2 - r r_{332} r_{33}).$$

Similar formulas can be written by means of μ in order to construct the relation between the coframings $\{\omega^1, \omega^2, \omega^3\}$ and $\{\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3\}$, but we do not need to do this.

With this setting, one can see that the Finsler structures F and $\bar{F}$ are projectively equivalent if and only $\mathrm{span}\{\omega^1, \omega^3\} = \mu^*(\mathrm{span}\{\bar{\omega}^1, \bar{\omega}^3\})$. Since both Σ_p, Σ_r are topologically diffeomorphic to projective sphere $SM := \widetilde{TM}/_\sim$ and $\mu : \Sigma_p \longrightarrow \Sigma_r$ is diffeomorphism, we identify here the 3-manifolds Σ_p and Σ_r,

where the equivalence relation $\sim$ is defined by $(x, y) \sim (x, z)$ if and only if y, z are positive multiples of each other.

In terms of the Riemannian canonical coframing $\{\alpha^1, \alpha^2, \alpha^3\}$ the above condition become $\mathrm{span}\{\alpha^1, \mathcal{M}_2 \cdot \alpha^2 + \mathcal{M}_3 \cdot \alpha^3\} = \mathrm{span}\{\alpha^1, \overline{\mathcal{M}_2} \cdot \alpha^2 + \overline{\mathcal{M}_3} \cdot \alpha^3\} = 0$, where we denote for simplicity

$$\begin{cases} \mathcal{M}_2 = \dfrac{p_{32} - p_1}{\sqrt{p(p + p_{33})}} \\ \mathcal{M}_3 = \sqrt{\dfrac{p + p_{33}}{p}} \end{cases} , \qquad \begin{cases} \overline{\mathcal{M}_2} = \dfrac{r_{32} - r_1}{\sqrt{r(r + r_{33})}} \\ \overline{\mathcal{M}_3} = \sqrt{\dfrac{r + r_{33}}{r}}. \end{cases} \tag{3.8}$$

It can be seen easily now that the projective equivalence condition reduce to

$$\frac{\mathcal{M}_3}{\mathcal{M}_2} = \frac{\overline{\mathcal{M}_3}}{\overline{\mathcal{M}_2}}, \Longleftrightarrow \frac{p + p_{33}}{p_{32} - p_1} = \frac{r + r_{33}}{r_{32} - r_1}, \tag{3.9}$$

provided $\mathcal{M}_2 \neq 0$ and $\overline{\mathcal{M}_2} \neq 0$. We observe that the geometrical meaning of $\mathcal{M}_2 = 0$ is that (M, F) and (M, α) are projectively related, i.e. the Finslerian geodesic of (M, F) and Riemannian geodesic of (M, α) coincide. In this case, obviously (M, F), $(M, \overline{F})$ and (M, α) are all projectively equivalent. We consider this case to be trivial and exclude it from our analysis. Therefore, we always assume in the following that the Finslerian structures (M, F) and $(M, \overline{F})$ are not projectively equivalent to (M, α), i.e. $\mathcal{M}_2 \neq 0$ and $\overline{\mathcal{M}_2} \neq 0$.

In order to obtain the condition for (M, F) to be with reversible geodesics, we impose the condition that $\overline{F}(x, y) = F(x, -y)$, for all $(x, y) \in TM$, where $\overline{F}(x, y)$ is the reverse Finsler structure associated to F on M. In this case, with the notations above, we obtain:

Proposition 3.1. *Let (M, F) be a Finsler surface and $(M, \overline{F})$ be the associated reverse Finsler structure on M. We assume that both Finslerian structures F and $\overline{F}$ are not Riemannian projectively equivalent. Then, (M, F) is with reversible geodesics if and only if*

$$\frac{p + p_{33}}{p_{32} - p_1} = \frac{r + r_{33}}{r_{32} - r_1} \tag{3.10}$$

with the notations above.

4. The reversible geodesics condition

We start with the Riemannian surface (M, a) and let us consider the isothermal coordinates $x = (x^1, x^2)$ on M, namely, in these local coordinates $a_{ij} = e^{2\nu}\delta_{ij}$,

where ν is a smooth function on M and δ_{ij} the Kronecker operator. This allows to write the canonical Riemannian coframing $\{\alpha^1, \alpha^2, \alpha^3\}$ as

$$\alpha^1 = -e^{\nu(x_1, x_2)} \sin t \, dx^1 + e^{\nu(x_1, x_2)} \cos t \, dx^2,$$

$$\alpha^2 = e^{\nu(x_1, x_2)} \cos t \, dx^1 + e^{\nu(x_1, x_2)} \sin t \, dx^2,$$

$$\alpha^3 = -\frac{\partial \nu(x_1, x_2)}{\partial x^2} dx^1 + \frac{\partial \nu(x_1, x_2)}{\partial x^1} dx^2 + dt, \qquad (4.1)$$

where $t \in [0, 2\pi)$ is the fiber coordinate. The unit circle $\Sigma_1\big|_x \in T_x M$ of (M, a) is therefore parametrized as

$$y^1 = e^{-\nu(x^1, x^2)} \cdot \cos t, \quad y^2 = e^{-\nu(x^1, x^2)} \cdot \sin t, \qquad t \in [0, 2\pi). \qquad (4.2)$$

One can easily remark that for a vector (y^1, y^2), the opposite vector is given by $-y = (-y^1, -y^2) = \left(e^{-\nu(x^1, x^2)} \cdot \cos(t + \pi), e^{-\nu(x^1, x^2)} \cdot \sin(t + \pi)\right)$. Therefore, if we denote by p and r the Finslerian norms corresponding to $F(x, y)$ and $\overline{F}(x, y) = F(x, -y)$ considered as positive real valued function on Σ_1 as explained before, then we get

Lemma 4.1. *The relation between p and r is given by*

$$r(x^1, x^2, t) = p(x^1, x^2, t + \pi). \qquad (4.3)$$

Straightforward computations give immediately the relations between the directional derivatives of p and r with respect to the Riemannian coframing $\{\alpha^1, \alpha^2, \alpha^3\}$ and the partial derivatives with respect to the natural coordinates (x^1, x^2, t). Since these computations are quite long and annoying we decided to put them in a preliminary version of this paper [MSS2] available on arxiv.org.

Let us remark that in the natural coordinates (x^1, x^2, t) on Σ_1 we have

$$\alpha := \sqrt{a(y, y)} = 1,$$
$$\beta := b_1(x^1, x^2)y^1 + b_2(x^1, x^2)y^2$$
$$= e^{-\nu(x^1, x^2)}\left[b_1(x^1, x^2) \cdot \cos t + b_2(x^1, x^2) \cdot \sin t\right] \qquad (4.4)$$

where $\nu, b_1, b_2 : M \longrightarrow \mathbf{R}$ are smooth functions.

Hence, on the hypersurface $\Sigma_1 \hookrightarrow TM$, we can put $s = \beta$ and therefore

$$p(x^1, x^2, t) = \phi(s)_{|s = \beta} = \phi(b_1(x^1, x^2)e^{-\nu(x_1, x_2)} \cos t$$

$$+ b_2(x^1, x^2)e^{-\nu(x_1, x_2)}\sin t),$$

$$p(x^1, x^2, \pi + t) = r(x^1, x^2, t) = \phi(-s)_{|s=\beta}$$

$$= \phi\big(b_1(x^1, x^2)e^{-\nu(x_1, x_2)}\cos(\pi + t)$$

$$+ b_2(x^1, x^2)e^{-\nu(x_1, x_2)}\sin(\pi + t)\big). \tag{4.5}$$

Remark 4.1. It is useful to see that for $\beta_t' = e^{-\nu(x^1, x^2)}\big(-b_1(x^1, x^2)\sin t + b_2(x^1, x^2)\cos t\big)$ we have $(\beta_t')^2 = b^2 - \beta^2$, where $b^2 = e^{-2\nu(x^1, x^2)}(b_1^2 + b_2^2)$ is the Riemannian length of the vector (b_1, b_2).

Straightforward computations (that can be found in [MSS2]) lead us to

Theorem 4.2. *The necessary and sufficient condition for the Finsler structures $F(x, y)$ and $\overline{F}(x, y) = F(x, -y)$ to be projectively equivalent is*

$$\sqrt{b^2 - s^2} \cdot \mathcal{E}(s) \cdot \mathcal{M} + \mathcal{F}(s) \cdot e^{-\nu(x^1, x^2)}\,\mathrm{curl}_{21} = 0, \tag{4.6}$$

where

$$\mathcal{E}(s) := s\big(\phi'(s)\phi''(-s) + \phi'(-s)\phi''(s)\big) + \big(\phi(-s)\phi''(s) - \phi(s)\phi''(-s)\big), \tag{4.7}$$

$$\mathcal{F}(s) := (b^2 - s^2)\big(\phi'(s)\phi''(-s) + \phi'(-s)\phi''(s)\big) + \big(\phi(-s)\phi'(s) + \phi(s)\phi'(-s)\big) \tag{4.8}$$

and

$$M := e^{-\nu(x^1, x^2)} \cdot \left(\frac{\partial b_1(x^1, x^2)}{\partial x^1}\cos^2 t + \sin t \cos t \left(\frac{\partial b_1(x^1, x^2)}{\partial x^2} + \frac{\partial b_2(x^1, x^2)}{\partial x^1}\right)\right.$$

$$\left. + \frac{\partial b_2(x^1, x^2)}{\partial x^2}\sin^2 t\right) + \beta_t' \left(\frac{\partial \nu(x^1, x^2)}{\partial x^2}\cos t - \frac{\partial \nu(x^1, x^2)}{\partial x^1}\sin t\right)$$

$$- \beta\left(\frac{\partial \nu(x^1, x^2)}{\partial x^1}\cos t + \frac{\partial \nu(x^1, x^2)}{\partial x^2}\sin t\right),$$

$$\mathrm{curl}_{21} := \frac{\partial b_2(x^1, x^2)}{\partial x^1} - \frac{\partial b_1(x^1, x^2)}{\partial x^2}. \tag{4.9}$$

Remark 4.2. Using the above formulas for β and β_t', one can see that $\mathcal{M}$ can be expressed as

$$\mathcal{M} = \mathcal{K}_1 + \mathcal{K}_2 \cdot \cos 2t + \mathcal{K}_3 \cdot \sin 2t, \tag{4.10}$$

where

$$\mathcal{K}_1 := \frac{1}{2}\left(\frac{\partial b_1}{\partial x^1} + \frac{\partial b_2}{\partial x^2}\right), \qquad \mathcal{K}_2 := \frac{1}{2}\left(\frac{\partial b_1}{\partial x^1} - \frac{\partial b_2}{\partial x^2}\right) - \left(\frac{\partial \nu}{\partial x^1}b_1 - \frac{\partial \nu}{\partial x^2}b_2\right),$$

$$\mathcal{K}_3 := \frac{1}{2}\left(\frac{\partial b_2}{\partial x^1} + \frac{\partial b_1}{\partial x^2}\right) - \left(\frac{\partial \nu}{\partial x^2}b_1 + \frac{\partial \nu}{\partial x^1}b_2\right).$$

5. Basic lemmas

In the present section, we are going to give some results to be used later.

Lemma 5.1. *The following relations are equivalent*

(1) $\mathcal{E} = 0$,

(2) $\phi(s) = k_1 \cdot \phi(-s) + k_2 \cdot s$, k_1, k_2 *non vanishing constants,*

(3) $F(\alpha, \beta) = F_0(\alpha, \beta) + \varepsilon\beta$, *where F_0 is an absolute homogeneous (α, β)-metric and ε is a non vanishing constant.*

PROOF. The equivalence of 1 and 2 follows directly from Lemma 3.4 in [MSS]. Indeed, one can easily see that $\mathcal{E} = 0$ is equivalent to the equation $\mathcal{T} - \overline{\mathcal{T}} = 0$ in [MSS] and therefore 2 follows.

We prove now the equivalence of 2 and 3. First of all, we remark that k_1 can take only the value 1. Indeed, by putting $-s$ instead of s in relation 2, it follows

$$\phi(-s) = k_1 \cdot \phi(s) - k_2 \cdot s \tag{5.1}$$

and by adding these formulas, it results

$$\phi(s) + \phi(-s) = k_1 \cdot \big[\phi(s) + \phi(-s)\big], \tag{5.2}$$

i.e.

$$\big[\phi(s) + \phi(-s)\big](k_1 - 1) = 0 \tag{5.3}$$

and we have two cases here. The first case is $\phi(s) = -\phi(-s)$, i.e. ϕ is an odd function, but this is not good due to Lemma 2.2. Therefore, the only possible case is $k_1 = 1$ and the formula in 2 actually reads

$$\phi(s) = \phi(-s) + k_2 \cdot s, \tag{5.4}$$

where $k_2 \neq 0$, because otherwise we would obtain only absolute homogeneous metrics. We will show now that (5.4) is, in fact, equivalent to the relation 3.

Let us recall that the vector space of all real-valued functions is the direct sum of the subspaces of even and odd functions. In other words, any function $\phi(s)$ can be uniquely written as the sum of an even function ϕ_{even} and an odd function ϕ_{odd}, namely

$$\phi(s) = \phi_{even}(s) + \phi_{odd}(s), \tag{5.5}$$

where

$$\phi_{even}(s) = \frac{1}{2}\big[\phi(s) + \phi(-s)\big], \quad \phi_{odd}(s) = \frac{1}{2}\big[\phi(s) - \phi(-s)\big]. \tag{5.6}$$

Using now (5.4) it follows

$$\phi_{odd}(s) = \frac{1}{2}\left[\phi(s) - \phi(-s)\right] = \frac{k_2}{2} \cdot s, \quad \phi(s) = \phi_{even}(s) + \frac{k_2}{2} \cdot s, \qquad (5.7)$$

i.e. the corresponding $F(\alpha, \beta)$ is of the form in 3. $\qquad\square$

We are going to discuss next the equation $\mathcal{F}(s) = 0$, where $\mathcal{F}(s)$ is given in (4.8).

A straightforward computation shows that, for $\phi'(s) \neq 0$, this is equivalent to

$$\frac{(b^2 - s^2) \cdot \bar{\phi}''(s) - s\bar{\phi}'(s) + \bar{\phi}(s)}{\bar{\phi}'(s)} = \frac{(b^2 - s^2) \cdot \phi''(s) - s\phi'(s) + \phi(s)}{\phi'(s)}, \qquad (5.8)$$

where we put $\bar{\phi}(s) := \phi(-s)$. Since both ϕ and $\bar{\phi}$ must be Finsler metrics, from Lemma 2.1 it results that the numerators in both hand sides of (5.8) must be positive and from here it results $\phi'(s) \cdot \bar{\phi}'(s) > 0$, in other words, ϕ and $\bar{\phi}$ must have the same monotonicity.

Let us remark that every even function ϕ is solution of $\mathcal{F} = 0$. Of course, any odd function is also solution, but we can exclude these functions due to Lemma 2.2.

Let us suppose that an arbitrary ϕ, i.e. it is not even, nor odd, is solution of $\mathcal{F} = 0$. Then, $\phi(s)$ and $\bar{\phi}(s)$ must have the same monotonicity. We will show that this is not possible.

Indeed, recall that the composition of two functions with same monotonicity gives an increasing function and the composition of two functions with different monotony gives an decreasing function (this can be easily be seen from the derivation rule of composed functions).

If we write $\bar{\phi}(s) = (\phi \circ \psi)(s)$, where $\psi(s) := -s$, then we have two cases

(1) If ϕ is an increasing function, then, since ψ is decreasing, their composition $\bar{\phi}(s)$ is decreasing, i.e. $\phi(s)$ and $\bar{\phi}(s)$ have different monotonicities, but this is a contradiction.

(2) If ϕ is decreasing, it follows that $\bar{\phi}(s)$ is increasing, but this also implies that $\phi(s)$ and $\bar{\phi}(s)$ have different monotonicities and this is not good again.

We can conclude that the equation $\mathcal{F} = 0$ has no Finslerian solution, except the absolute homogeneous Finsler metrics, provided $\phi'(s) \neq 0$ for all $s \in (-b_0, b_0)$.

Let us consider now the case $\phi'(s) = 0$.

If $\phi'(s) = 0$ for all $s \in [-b_0, b_0]$, then ϕ is linear in s and this is not good because we do not get a genuine Finsler metric.

Therefore, the only possible case is that there exists some $s_0 \in [-b_0, b_0]$ such that $\phi'(s_0) = 0$, i.e. s_0 is a singular point of ϕ. In order to study the metric at the singular point s_0, we need to consider the 2nd order derivative $\phi''(s_0)$.

Let us assume that s_0 is degenerate, i.e. $\phi''(s_0) = 0$. Then, by Taylor's expansion, ϕ must be of the form $\phi(s) = a + c \cdot s^3 +$ *higher order terms*, for $s_0 - \varepsilon < s < s_0 + \varepsilon$. Consequently, by neglecting the higher order terms, we obtain $\phi'(s) = 3cs^2$ and $\bar{\phi}'(s) = -3cs^2$, for $\varepsilon \to 0$. Thus we get $\phi'(s) \cdot \bar{\phi}'(s) < 0$, but this is a contradiction.

Therefore, all singular points s_0 must be non-degenerate, i.e. $\phi : [-b_0, b_0] \longrightarrow \mathbf{R}^+$ is a Morse function. From Morse theory, we know that the set of singular points of ϕ must be finite and contains only isolated points. Then, by means of Morse lemma, it follows that ϕ must be of the form $\phi(s) = a + bs^2 +$ *higher orders terms*, for $s_0 - \varepsilon < s < s_0 + \varepsilon$.

In general, one can see that for arbitrary s, the function

$$\phi(s) = a_0 + a_2 \cdot s^2 + a_4 \cdot s^4 + \cdots + a_{2k} \cdot s^{2k}$$

is solution for $\mathcal{F} = 0$, but this is an even function, i.e. F must be absolute homogeneous, and from the previous analysis it follows that there are no other solutions of the equation $\mathcal{F} = 0$.

Remark 5.1. We point out that for a singular point s_0 of ϕ, there exists a small enough positive constant ε such that there is no other singular point in the ε-neighborhood $(s_0 - \varepsilon, s_0 + \varepsilon)$. Indeed, if the singular points would accumulate, then ϕ must be constant on the ε-neighborhood and it is not good for us because violates the conditions in Lemma 2.1.

Therefore, we can conclude

Proposition 5.2. *The equation $\mathcal{F} = 0$ has no other Finsler solutions except the absolute homogeneous case.*

We also have

Lemma 5.3. *The function $\mathcal{E}(s)$ is an odd function and $\mathcal{F}(s)$ is an even one.*

PROOF. Indeed, if one puts $-s$ instead of s in the definitions of $\mathcal{E}(s)$ and $\mathcal{F}(s)$, then the conclusion follows immediately. Here, we take into account the formulas (2.4) and (2.5). $\qquad\qquad\square$

6. (α, β)-metrics with reversible geodesics

Let us consider the necessary and sufficient condition (4.6) given in Theorem 4.2 for an (α, β)-metric to be with reversible geodesics.

If we put $-s$ instead of s and taking into account Lemma 5.3 it follows

$$\sqrt{b^2 - s^2} \cdot \mathcal{E}(-s) \cdot \mathcal{M} + \mathcal{F}(-s) \cdot e^{-\nu(x^1, x^2)} \operatorname{curl}_{21} = 0 \qquad (6.1)$$

$$-\sqrt{b^2 - s^2} \cdot \mathcal{E}(s) \cdot \mathcal{M} + \mathcal{F}(s) \cdot e^{-\nu(x^1, x^2)} \operatorname{curl}_{21} = 0, \qquad (6.2)$$

and therefore, from relations (4.6) and (6.2) it follows the following system

$$\begin{cases} \mathcal{E}(s) \cdot \mathcal{M} = 0, \\ \mathcal{F}(s) \cdot \operatorname{curl}_{21} = 0. \end{cases} \qquad (6.3)$$

Since, due to Proposition 5.2, the condition $\mathcal{F}(s) = 0$ is not convenient, it follows that geodesic reversibility condition (4.6) is equivalent to one of the following two cases

$$\mathcal{E}(s) = 0, \quad \operatorname{curl}_{21} = 0, \qquad (6.4)$$

or

$$\mathcal{M} = 0, \quad \operatorname{curl}_{21} = 0. \qquad (6.5)$$

The first case was already discussed in Lemma 5.1.

We will discuss next the case $\mathcal{M} = 0$.

Let us assume $\mathcal{M} = 0$, for all $t \in [0, 2\pi)$, i.e. $\mathcal{M} = \mathcal{K}_1 + \mathcal{K}_2 \cdot \cos 2t + \mathcal{K}_3 \cdot \sin 2t = 0$. Evaluating this formula for $t = 0$, $t = \frac{\pi}{2}$ and $\frac{\pi}{4}$, we get $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}_3 = 0$, and taking into account the condition $\operatorname{curl}_{21} = 0$, we obtain

$$\begin{cases} \dfrac{\partial b_2}{\partial x^1} - \dfrac{\partial b_1}{\partial x^2} = 0 \\[2mm] \dfrac{\partial b_1}{\partial x^1} + \dfrac{\partial b_2}{\partial x^2} = 0 \\[2mm] \dfrac{1}{2}\left(\dfrac{\partial b_1}{\partial x^1} - \dfrac{\partial b_2}{\partial x^2}\right) - \left(\dfrac{\partial \nu}{\partial x^1} b_1 - \dfrac{\partial \nu}{\partial x^2} b_2\right) = 0 \\[2mm] \dfrac{1}{2}\left(\dfrac{\partial b_2}{\partial x^1} + \dfrac{\partial b_1}{\partial x^2}\right) - \left(\dfrac{\partial \nu}{\partial x^2} b_1 + \dfrac{\partial \nu}{\partial x^1} b_2\right) = 0. \end{cases} \qquad (6.6)$$

This is a 1st order PDE with 2 unknown functions b_1, b_2, defined on M, where ν is a given function.

One can easily remark that the first two equations of the system are in fact the divergence and the curl of the vector (b_1, b_2) and these are equivalent to

Cauchy–Riemann conditions of differentiability for the function $\mathfrak{b} : \mathbb{C} \to \mathbb{C}$, given by $\mathfrak{b}(x^1, x^2) = (b_1(x^1, x^2), b_2(x^1, x^2))$. In other words, any differentiable complex function of one complex variable on M satisfies the first two equations of the system (6.6).

A straightforward computation shows that this PDE system is integrable if and only if

$$\frac{\partial^2 \nu}{\partial x^1 \partial x^1} + \frac{\partial^2 \nu}{\partial x^2 \partial x^2} = 0, \tag{6.7}$$

provided b_1 and b_2 do not vanish in the same time.

We remark that the same conclusion follows from the Cartan–Kähler theory applied to the system (6.6).

On the other hand, we recall that in the isothermal coordinates x^1, x^2, the Gauss curvature k of the Riemannian metric $e^{2\nu} \delta_{ij}$ is given by

$$k = -e^{-2\nu} \left(\frac{\partial^2 \nu}{\partial x^1 \partial x^1} + \frac{\partial^2 \nu}{\partial x^2 \partial x^2} \right). \tag{6.8}$$

Therefore we can conclude that the PDE system (6.6) is integrable if and only if the Riemannian metric a is flat. But this means that the function $\nu(x^1, x^2)$ must be constant and thus the system (6.6) has only the constant solution, i.e. the functions b_1, b_2 are constant.

In conclusion, condition (4.6) in Theorem 4.2 implies that only two cases are possible: the case (6.4) that is detailed in Lemma 5.1, and the case (6.5) that leads to b_1, b_2, ν constants, but this last case contradicts our assumption in Proposition 3.1, so it should be eliminated.

Conversely, it is trivial to check that F given in Lemma 5.1 is with reversible geodesics and this proves Theorem 1.1.

Remark 6.1. If (M, a) is a flat Riemannian space and $\beta = b_i \cdot y^i$ a linear 1-form on TM, such that b_1, b_2 are constants, then it can be seen directly (see for example (2.18) in [MSS]) that any (α, β) metric $F = F(\alpha, \beta)$ constructed with these α and β is with reversible geodesics and projectively equivalent (M, a). In fact, F is a Minkowski metric on M.

We point out that this property is true in arbitrary dimension.

Remark 6.2. T. AIKOU defines in [Ai] the notion of *"strictly projectively equivalence"* of two Finsler structures F and $\bar{F}$ on M by the relation

$$d\omega_F = d\omega_{\bar{F}}, \tag{6.9}$$

where $\omega_F := \frac{\partial F}{\partial y^i} dx^i$, and shows that this relation is equivalent to $F = \bar{F} + \beta$, with β closed one form on M.

On the other hand, we recall that, regardless dimension, our definition of projectively equivalence is

$$\bar{\Gamma} = \Gamma + \lambda C, \tag{6.10}$$

where Γ is the geodesic spray of F, C is the Liouville vector field $C := y^i \frac{\partial}{\partial y^i}$, and λ a scalar function on TM (see [Cr] and [MSS] for details).

One can easily see that (6.9) implies (6.10), but without other arguments, there is no obvious reason for the inverse implication to be true.

In fact, our results in the present paper and [MSS] are not a consequence of [Ai], but we prove exactly the fact that the inverse statement is indeed true and therefore our notion of "projectively equivalence" and Aikou's notion of "strictly projectively equivalence" are in fact equivalent.

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