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Unifications on a type of continuity

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Abstract. In this paper, we investigate different properties of weakly (τ, μ)-continuous functions. We also investigate the relations among (τ, μ)-continuity, weak (τ, μ)-continuity and strong (τ, μ)-continuity.

1. Introduction

The notion of continuity is one of the basic concepts in the study of general topology. This concept has been generalized by many mathematicians from different point of view. Using the concepts of sets like semi-open sets, pre-open sets, α-open sets, β-open sets several authors have introduced and investigated different weak forms of continuity. In this paper we generalize several characterizations and properties of a type of weak continuous functions called weakly (τ, μ)-continuous functions from a topological space (X, τ) to a GTS (Y, μ). The aim of this paper is to investigate some fundamental properties of weakly (τ, μ)-continuous functions.

We first recall some notion defined in [3]. Let X be a non-empty set and $\exp X$ be the power set of X. We call a class $\mu \subsetneqq \exp X$ a generalized topology [3], (briefly, GT) if $\varnothing \in \mu$ and union of elements of μ belongs to μ. A set X, with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ). For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A i.e., the smallest

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μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A i.e., the largest μ-open set contained in A (see [2], [3] for details).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma : \exp X \to \exp X$ is said to be idempotent iff $A \subseteqq B \subseteqq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic iff $\gamma(A) \subseteqq \gamma(B)$. It is also well known from [2], [5] that if μ is a GT on X and $A \subseteqq X$ with $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \varnothing$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$. Throughout the paper, by (X, τ) and (Y, μ) we shall always mean a topological space and a generalized topological space respectively.

2. Properties of weakly (τ, μ)-continuous functions

Definition 1 ([4]). Let (X, μ) be a GTS. The $\mu(\theta)$-closure of a subset A of a GTS (X, μ) is denoted by $c_{\mu(\theta)}(A)$ and is defined to be the set of all points x of X such that for each $U \in \mu(x)$, $c_\mu(U) \cap A \neq \varnothing$, where $\mu(x) = \{U : U \text{ is a } \mu\text{-open set containing } x\}$.

Definition 2. A function $f : (X, \tau) \to (Y, \mu)$ is said to be weakly (τ, μ)-continuous if for each $x \in X$ and each μ-open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subseteqq c_\mu(V)$.

Theorem 1. *For a function $f : (X, \tau) \to (Y, \mu)$ the following are equivalent:*

(a) *f is weakly (τ, μ)-continuous.*

(b) *$f^{-1}(V) \subseteqq \int(f^{-1}(c_\mu(V)))$ for each μ-open set V of Y.*

(c) *$\mathrm{cl}(f^{-1}(i_\mu(F))) \subseteqq f^{-1}(F)$ for each μ-closed subset F of Y.*

(d) *$\mathrm{cl}(f^{-1}(V)) \subseteqq f^{-1}(c_\mu(V))$ for each μ-open subset V of Y.*

(e) *$f(\mathrm{cl}(A)) \subseteqq c_{\mu(\theta)}(f(A))$ for any subset A of X.*

(f) *$\mathrm{cl}(f^{-1}(B)) \subseteqq f^{-1}(c_{\mu(\theta)}(B))$ for any $B \subseteqq Y$.*

Proof. (a) $\Rightarrow$ (b): Let V be a μ-open subset of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$. Thus by (a), there exists an open set U in X containing x such that $f(U) \subseteqq c_\mu(V)$. Then $x \in U \subseteqq f^{-1}(c_\mu(V))$. So $x \in \int(f^{-1}(c_\mu(V)))$.

(b) $\Rightarrow$ (c): Let F be any μ-closed subset of Y. Then $Y \setminus F$ is μ-open in Y and by (b) we have $f^{-1}(Y \setminus F) \subseteqq \int(f^{-1}(c_\mu(Y \setminus F)))$ i.e., $X \setminus f^{-1}(F) \subseteqq \int(f^{-1}(c_\mu(Y \setminus F))) = \int(f^{-1}(Y \setminus i_\mu(F))) = \int(X \setminus f^{-1}(i_\mu(F))) = X \setminus \mathrm{cl}(f^{-1}(i_\mu(F)))$. Thus $\mathrm{cl}(f^{-1}(i_\mu(F))) \subseteqq f^{-1}(F)$.

(c) $\Rightarrow$ (d): Let V be any μ-open subset of Y. Then $c_\mu(V)$ is μ-closed in Y. Thus by (c), $\mathrm{cl}(f^{-1}(V)) \subseteqq \mathrm{cl}(f^{-1}(i_\mu(c_\mu(V)))) \subseteqq f^{-1}(c_\mu(V))$.

(d)$\Rightarrow$ (a): Let $x \in X$ and V be any μ-open set of Y containing $f(x)$. Then $f(x) \notin c_\mu(Y \setminus c_\mu(V))$. Thus by (d), $x \notin \mathrm{cl}(f^{-1}(Y \setminus c_\mu(V)))$. Hence there exists an open set U in X containing x such that $U \cap f^{-1}(Y \setminus c_\mu(V)) = \varnothing$. Thus $f(U) \cap (Y \setminus c_\mu(V)) = \varnothing$ which implies that $f(U) \subseteq c_\mu(V)$.

(a) $\Rightarrow$ (e): Let $A \subseteq X$ and $x \in \mathrm{cl}(A)$. Let V be any μ-open set containing $f(x)$. Then there exists an open set U in X containing x such that $f(U) \subseteq c_\mu(V)$. Then $U \cap A \neq \varnothing$ and hence $\varnothing \neq f(U) \cap f(A) \subseteq c_\mu(V) \cap f(A)$. Thus $f(x) \in c_{\mu(\theta)}(f(A))$ i.e., $x \in f^{-1}(c_{\mu(\theta)}(f(A)))$. Thus $f(\mathrm{cl}(A)) \subseteq c_{\mu(\theta)}(f(A))$.

(e) $\Rightarrow$ (f): Let B be any subset of Y. By (e), $f(\mathrm{cl}(f^{-1}(B))) \subseteq c_{\mu(\theta)}(B) \Rightarrow \mathrm{cl}(f^{-1}(B)) \subseteq f^{-1}(c_{\mu(\theta)}(B))$.

(f) $\Rightarrow$ (a): Let $x \in X$ and V be any μ-open set of Y containing $f(x)$. Then $f(x) \notin c_{\mu(\theta)}(Y \setminus c_\mu(V))$ i.e., $x \notin \mathrm{cl}\, f^{-1}(Y \setminus c_\mu(V))$ (by (f)). Thus there exists an open set U in X containing x such that $U \cap f^{-1}(Y \setminus c_\mu(V)) = \varnothing$ i.e., $f(U) \cap (Y \setminus c_\mu(V)) = \varnothing$. Thus $f(U) \subseteq c_\mu(V)$. $\square$

Definition 3. A GTS (X, μ) is called

(i) μ-Urysohn [1] if for any pair of distinct points $x, y \in X$ there exist μ-open sets U and V such that $x \in U$, $y \in V$ with $c_\mu(U) \cap c_\mu(V) = \varnothing$.

(ii) μ-T_2 [8] if for any pair of distinct points $x, y \in X$ there exist two disjoint open sets U and V such that $x \in U$ and $y \in V$.

Theorem 2. *If $f, g : (X, \tau) \to (Y, \mu)$ are weakly (τ, μ)-continuous and Y is μ-Urysohn, then $A = \{x \in X : f(x) = g(x)\}$ is closed in X.*

Proof. Let $x \in X \setminus A$. Then $f(x) \neq g(x)$. Thus there exist two disjoint μ-open sets V_1 and V_2 containing $f(x)$ and $g(x)$ respectively such that $c_\mu(V_1) \cap c_\mu(V_2) = \varnothing$. Since f, g are weakly (τ, μ)-continuous there exist open sets U_1, U_2 containing x such that $f(U_1) \subseteq c_\mu(V_1)$ and $f(U_2) \subseteq c_\mu(V_2)$. Let $U = U_1 \cap U_2$. Then U is an open set in X containing x such that $U \cap A = \varnothing$. Thus $x \notin \mathrm{cl}(A)$ i.e., A is closed. $\square$

The next corollary is a generalization of the well-known principle of extension of identities.

Corollary 1. *Let $f, g : (X, \tau) \to (Y, \mu)$ be two weakly (τ, μ)-continuous functions and Y be μ-Urysohn. If A is dense in X with $f = g$ on A, then $f = g$ on X.*

Theorem 3. *If $f : (X, \tau) \to (Y, \mu)$ is a weakly (τ, μ)-continuous function and Y is μ-Urysohn, then $A = \{(x, y) \in X \times X : f(x) = f(y)\}$ is closed in $X \times X$.*

Proof. Let $(x, y) \in (X \times X) \setminus A$. Then $f(x) \neq f(y)$. As Y is μ-Urysohn, there exist μ-open sets V_1 and V_2 containing $f(x)$ and $f(y)$ respectively such that $c_\mu(V_1) \cap c_\mu(V_2) = \varnothing$. Since f is a weakly (τ, μ)-continuous function, there exist open sets U_1, U_2 in X containing x and y respectively such that $f(U_1) \subseteqq c_\mu(V_1)$ and $f(U_2) \subseteqq c_\mu(V_2)$. Let $U = U_1 \times U_2$. Then U is an open set in $X \times X$ with $(x, y) \in U$ and $U \cap A = \varnothing$. Thus $(x, y) \notin \mathrm{cl}(A)$ i.e., A is closed in $X \times X$. □

Theorem 4. *Let $f : (X, \tau) \to (Y, \mu)$ be a weakly (τ, μ)-continuous injection. Then the following holds:*

(a) *If Y is μ-Urysohn, then X is T_2.*

(b) *If Y is μ-T_2, then X is T_1.*

Proof. (a) Let x_1 and x_2 be any two distinct points of X. Then $f(x_1) \neq f(x_2)$ and thus there exist μ-open sets V_1, V_2 containing $f(x_1)$ and $f(x_2)$ respectively such that $c_\mu(V_1) \cap c_\mu(V_2) = \varnothing$. Since f is weakly (τ, μ)-continuous, there exist open sets U_1, U_2 containing x_1 and x_2 respectively such that $f(U_1) \subseteqq c_\mu(V_1)$ and $f(U_2) \subseteqq c_\mu(V_2)$. Since $f^{-1}(c_\mu(V_1))$ and $f^{-1}(c_\mu(V_2))$ are disjoint, we have $U_1 \cap U_2 = \varnothing$. Thus X is T_2.

(b) Let x_1 and x_2 be any two distinct points of X. Then $f(x_1) \neq f(x_2)$ and thus there exist disjoint μ-open sets V_1, V_2 containing $f(x_1)$ and $f(x_2)$ respectively. Then $f(x_1) \notin c_\mu(V_2)$ and $f(x_2) \notin c_\mu(V_1)$. Since f is weakly (τ, μ)-continuous there exist open sets U_i in X containing x_i such that $f(U_i) \subseteqq c_\mu(V_i)$ for $i = 1, 2$. Thus we have $x_2 \notin U_1$ and $x_1 \notin U_2$. Hence X is T_1. □

Example 1. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{a, c\}, \{a, b\}, X\}$ and $\mu = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then (X, τ) is a topological space and (X, μ) is a GTS. Consider the constant mapping $f : (X, \tau) \to (X, \mu)$ defined by $f(x) = a$ for all $x \in X$. It can be easily checked that f is weakly (τ, μ)-continuous which is not injective. We note that (X, μ) is μ-Urysohn (and hence μ-T_2) but (X, τ) is not even T_1.

Definition 4. A function $f : (X, \tau) \to (Y, \mu)$ is said to have a strong (τ, μ)-closed graph if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist an open set U in X and a μ-open set V in Y such that $(x, y) \in U \times V$ and $(U \times c_\mu(V)) \cap G(f) = \varnothing$ where $G(f)$ denotes the graph of f.

Theorem 5. *If $f : (X, \tau) \to (Y, \mu)$ is weakly (τ, μ)-continuous and Y is μ-Urysohn, then $G(f)$ is strongly (τ, μ)-closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and by hypothesis there exist μ-open sets V and W in Y containing $f(x)$ and y respectively such that $c_\mu(V) \cap c_\mu(W) = \varnothing$. Since f is weakly (τ, μ)-continuous, there exists an open set U in X containing x such that $f(U) \subsetneqq c_\mu(V)$. Thus $f(U) \cap c_\mu(W) = \varnothing$. Hence $(U \times c_\mu(W)) \cap G(f) = \varnothing$. Thus $G(f)$ is strongly (τ, μ)-closed in $X \times Y$. □

We call $f : (X, \tau) \to (Y, \mu)$ to be (τ, μ)-open if $f(U)$ is μ-open for each open set U in X.

Theorem 6. *Let $f : (X, \tau) \to (Y, \mu)$ be a weakly (τ, μ)-continuous function having a strong closed graph $G(f)$. Then*

(a) *If f is injective, then X is T_2.*

(b) *If f is a (τ, μ)-open surjection, then Y is μ-T_2.*

Proof. (a) Let x_1 and x_2 be any two distinct points of X. Since f is injective, $f(x_1) \neq f(x_2)$ and thus $(x_1, f(x_2)) \notin G(f)$. Since $G(f)$ is strongly (τ, μ)-closed, there exist an open set U in X and a μ-open set V in Y such that $(x_1, f(x_2)) \in (U \times V)$ and $(U \times c_\mu(V)) \cap G(f) = \varnothing$ and hence $f(U) \cap c_\mu(V) = \varnothing$. Since f is weakly (τ, μ)-continuous, there exists an open set W in X containing x_2 such that $f(W) \subsetneqq c_\mu(V)$. Therefore, $f(U) \cap f(W) = \varnothing$ and hence $U \cap W = \varnothing$. Thus X is T_2.

(b) Let y_1 and y_2 be any two distinct points of Y. Since f is surjective there exists $x \in X$ such that $y_1 = f(x)$ and $(x, y_2) \notin G(f)$. By the strong (τ, μ)-closedness of the graph $G(f)$, there exist an open set U in X and a μ-open set V in Y such that $(x, y_2) \in U \times V$ and $f(U) \cap c_\mu(V) = \varnothing$. Thus $f(U) \cap V = \varnothing$. Now, since f is (τ, μ)-open, $f(U)$ is μ-open and hence $f(x) = y_1 \in f(U)$. Thus Y is μ-T_2. □

Example 2. Consider the Example 1. By Theorem 5, the graph $G(f)$ of f is strongly (τ, μ)-closed. As f is not an injective function, X is not T_2.

Definition 5. A GTS (X, μ) is said to be μ-connected [5], [7] if there does not exist any non-empty disjoint μ-open sets U_1 and U_2 such that $X = U_1 \cup U_2$.

Theorem 7. *If $f : (X, \tau) \to (Y, \mu)$ is a weakly (τ, μ)-continuous surjection and X is connected, then Y is μ-connected.*

Proof. Suppose that Y is not μ-connected. Then there exist non-empty disjoint μ-open sets V_1 and V_2 in Y such that $Y = V_1 \cup V_2$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \varnothing$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Clearly, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are non-empty (as f is surjective). By Theorem 1, we have $f^{-1}(V_i) \subsetneqq \int(f^{-1}(c_\mu(V_i)))$ for $i = 1, 2$. Since V_i's are μ-open and also μ-closed, we have $f^{-1}(V_i) \subsetneqq \int(f^{-1}(V_i))$ for $i = 1, 2$. Hence $f^{-1}(V_i)$ are open sets in X for $i = 1, 2$.

Thus X has been decomposed into two non-empty disjoint open sets. This contradicts the fact that X is connected. Thus Y is μ-connected. $\square$

The next example shows that surjection in the above theorem is not a sufficient condition.

Example 3. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{b\}, \{b, c\}, X\}$ and $\mu = \{\varnothing, \{a\}\}$. Then (X, τ) is a connected topological space and (X, μ) is a μ-connected GTS. Consider the mapping $f : (X, \tau) \to (X, \mu)$ defined by $f(a) = f(c) = a$ and $f(b) = b$. Then it is easy to verify that f is weakly (τ, μ)-continuous which is not surjective.

Definition 6. A function $f : (X, \tau) \to (Y, \mu)$ is said to be (τ, μ)-continuous (resp. $\theta(\tau, \mu)$-continuous) if for each $x \in X$ and each μ-open set V containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteqq V$ (resp. $f(\mathrm{cl}\, U) \subseteqq c_\mu(V)$).

Remark 1. It follows from Definitions 2 and 6 that (τ, μ)-continuity $\Rightarrow$ weak (τ, μ)-continuity and also $\theta(\tau, \mu)$-continuity $\Rightarrow$ weak (τ, μ)-continuity. None of the implications are reversible as shown by the next example.

Example 4. (a) Let $X = \{a, b, c, d\}$, $\tau = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ and $\mu = \{\varnothing, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Then (X, τ) is a topological space and (X, μ) is a GTS. Consider the mapping $f : (X, \tau) \to (X, \mu)$ defined by $f(a) = b$, $f(b) = f(d) = d$ and $f(c) = c$. Then it is easy to verify that f is weakly (τ, μ)-continuous but not (τ, μ)-continuous.

(b) Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{c\}, \{b, c\}, X\}$ and $\mu = \{\varnothing, \{c\}, \{a, c\}, \{b, c\}, X\}$. Then (X, τ) is a topological space and (X, μ) is a GTS. The identity map $f : (X, \tau) \to (X, \mu)$ is weakly (τ, μ)-continuous but not $\theta(\tau, \mu)$-continuous.

Definition 7. A GTS (X, μ) is said to be μ-regular [6], [8] if for each μ-closed set F of X not containing x, there exist disjoint μ-open sets U and V such that $x \in U$ and $F \subseteqq V$.

It is known from [6, 8] that a GTS (X, μ) is μ-regular iff for each $x \in X$ and each $U \in \mu$ containing x, there exists $V \in \mu$ such that $x \in V \subseteqq c_\mu(V) \subseteqq U$.

Theorem 8. *Let (Y, μ) be a μ-regular space. Then for a function $f : (X, \tau) \to (Y, \mu)$ the following are equivalent:*

(a) *f is (τ, μ)-continuous.*

(b) *f is $\theta(\tau, \mu)$-continuous.*

(c) *f is weakly (τ, μ)-continuous.*

Proof. (a) $\Rightarrow$ (b) : We shall first show that if f is (τ,μ)-continuous, then for any subset A of X, $f(\mathrm{cl}(A)) \subseteqq c_\mu(f(A))$. Let $x \in \mathrm{cl}(A)$ and V be any μ-open set containing $f(x)$. Then by (τ,μ)-continuity of f, there exists an open set U in X containing x such that $f(U) \subseteqq V$. Since $x \in \mathrm{cl}(A)$, $A \cap U \neq \varnothing$. Hence $f(A) \cap f(U) \subseteqq f(A) \cap V \neq \varnothing$. Thus $f(x) \in c_\mu(f(A))$ and $f(\mathrm{cl}(A)) \subseteqq c_\mu(f(A))$. Now let W be a μ-open set containing $f(x)$. Then by μ-regularity of Y, there exists a μ-open set V such that $f(x) \in V \subseteqq c_\mu(V) \subseteqq W$. Also there exists an open set U in X containing x such that $f(U) \subseteqq V$. Thus $f(\mathrm{cl}(U)) \subseteqq c_\mu(f(U)) \subseteqq c_\mu(V) \subseteqq W$.

(b) $\Rightarrow$ (c): Follows from Remark 1.

(c) $\Rightarrow$ (a): Let $x \in X$ and V be any μ-open set containing $f(x)$. Since (Y,μ) is μ-regular, there exists a μ-open set W such that $f(x) \in W \subseteqq c_\mu(W) \subseteqq V$. Since f is weakly (τ,μ)-continuous there exists an open set U in X containing x such that $f(U) \subseteqq c_\mu(W) \subseteqq V$. □

Theorem 9. *If (X,τ) is regular then the function $f : (X,\tau) \to (Y,\mu)$ is $\theta(\tau,\mu)$-continuous if and only if f is weakly (τ,μ)-continuous.*

Proof. One part of the theorem follows from Remark 1. Suppose that f is weakly (τ,μ)-continuous. Let $x \in X$ and V is a μ-open set containing $f(x)$. Then there exists an open set U_0 in X containing x such that $f(U_0) \subseteqq c_\mu(V)$. Now by regularity of X, there exists an open set U such that $x \in U \subseteqq \mathrm{cl}(U) \subseteqq U_0$. Thus we have $f(\mathrm{cl}(U)) \subseteqq c_\mu(V)$. This shows that f is $\theta(\tau,\mu)$-continuous. □

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