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Some convergence theorems for the q-integral

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Abstract. First, we use Ramanujan's ${}_1\psi_1$ summation formula to obtain an inequality for the bilateral basic hypergeometric series ${}_{r+1}\psi_{r+1}$. Then, we give some convergence theorems for the q-integral.

1. Introduction

Convergence is an important problem in the study of q-series. There are some papers in the literature [7], [8], [9], [10]. For example, Ito used inequality technique to give a sufficient condition for convergence of a special q-series called Jackson integral [7]. In this paper, we gave some convergence theorems for the q-integral. We first recall some definitions, notation and known results in [1] which will be used in this paper. Throughout the whole paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \dots (a_m;q)_n, \tag{1.2}$$

where n is an integer or ∞. We may extend the definition (1.1) of $(a;q)_n$ to

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \tag{1.3}$$

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for any complex number α. In particular,

$$(a;q)_{-n} = \frac{(a;q)_\infty}{(aq^{-n};q)_\infty} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n}{(q/a;q)_n} q^{\binom{n}{2}}. \tag{1.4}$$

The bilateral basic hypergeometric series ${}_r\psi_s$ is defined by

$$\begin{aligned} &{}_r\psi_s \left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z \right) \\ &\qquad = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} (-1)^{(s-r)n} q^{(s-r)\binom{n}{2}} z^n. \end{aligned} \tag{1.5}$$

The following is the well known Ramanujan's ${}_1\psi_1$ summation formula

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1. \tag{1.6}$$

2. An inequality for the bilateral basic hypergeometric series

Inequality technique is one of the useful tools in the study of special functions [2], [3], [4], [8], [9], [10]. In this section, we use Ramanujan's ${}_1\psi_1$ summation formula to derive an inequality for the bilateral basic hypergeometric series ${}_{r+1}\psi_{r+1}$, which can be used to discuss the convergence of the q-series. The main result of this section is the following inequality.

Theorem 2.1. *Let a, b be any real numbers such that $q < b < a < 1$ or $a < b < 0$, and let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \ldots, r$ with $r \geq 1$ and $|b_1 b_2 \ldots b_r| \leq |a_1 a_2 \ldots a_r|$. Then for any $b/a < |z| < 1$, we have*

$$\left| {}_{r+1}\psi_{r+1} \left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix} ; q, z \right) \right| \leq M \frac{(q, b/a, a|z|, q/a|z|; q)_\infty}{(b, q/a, |z|, b/a|z|; q)_\infty}, \tag{2.1}$$

where

$$M = \max \left\{ \prod_{i=1}^{r} \frac{(-|a_i|; q)_\infty}{(|b_i|; q)_\infty}, \prod_{i=1}^{r} \frac{(-q/|b_i|; q)_\infty}{(q/|a_i|; q)_\infty} \right\}.$$

Proof. Let $n \geq 0$. Since,

$$|(a_j;q)_n| = \prod_{i=0}^{n-1} |1 - a_j q^i| \leq \prod_{i=0}^{n-1} (1 + |a_j| q^i) = (-|a_j|;q)_n \leq (-|a_j|;q)_\infty$$

and

$$|(b_j;q)_n| = \prod_{i=0}^{n-1} |1 - b_j q^i| \geq \prod_{i=0}^{n-1} (1 - |b_j| q^i) = (|b_j|;q)_n \geq (|b_j|;q)_\infty > 0,$$

we have

$$\left| \frac{(a_j;q)_n}{(b_j;q)_n} \right| \leq \frac{(-|a_j|;q)_\infty}{(|b_j|;q)_\infty}. \tag{2.2}$$

Hence,

$$\left| \frac{(a_1, a_2, \ldots, a_r;q)_n}{(b_1, b_2, \ldots, b_r;q)_n} \right| \leq \prod_{i=1}^{r} \frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty}. \tag{2.3}$$

On the other hand, using (1.4) and (2.2) gives

$$\left| \frac{(a_i;q)_{-n}}{(b_i;q)_{-n}} \right| = \left| (\frac{b_i}{a_i})^n \frac{(q/b_i;q)_n}{(q/a_i;q)_n} \right| \leq \left| \frac{b_i}{a_i} \right|^n \frac{(-q/|b_i|;q)_\infty}{(q/|a_i|;q)_\infty}. \tag{2.4}$$

Consequently,

$$\begin{aligned} \left| \frac{(a_1, a_2, \ldots, a_r;q)_{-n}}{(b_1, b_2, \ldots, b_r;q)_{-n}} \right| &\leq \left| \frac{b_1 b_2 \ldots b_r}{a_1 a_2 \ldots a_r} \right|^n \prod_{i=1}^{r} \frac{(-q/|b_i|;q)_\infty}{(q/|a_i|;q)_\infty} \\ &\leq \prod_{i=1}^{r} \frac{(-q/|b_i|;q)_\infty}{(q/|a_i|;q)_\infty}. \end{aligned} \tag{2.5}$$

Combining (2.3) and (2.5) gives

$$\left| \frac{(a_1, a_2, \ldots, a_r;q)_n}{(b_1, b_2, \ldots, b_r;q)_n} \right| \leq M = \max \left\{ \prod_{i=1}^{r} \frac{(-|a_i|;q)_\infty}{(|b_i|;q)_\infty}, \prod_{i=1}^{r} \frac{(-q/|b_i|;q)_\infty}{(q/|a_i|;q)_\infty} \right\}, \tag{2.6}$$

where $n = \cdots -2, -1, 0, 1, 2, \ldots$.

Under the condition $q < b < a < 1$ or $a < b < 0$, it is easy to know

$$\frac{(a;q)_n}{(b;q)_n} > 0, \quad n = \cdots -2, -1, 0, 1, 2, \ldots, \tag{2.7}$$

and

$$0 < b/a < 1. \tag{2.8}$$

Using (2.6), (2.7), (2.8), and Ramanujan's ${}_1\psi_1$ summation formula (1.6), we have

$$\left| {}_{r+1}\psi_{r+1} \left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix} ; q, z \right) \right| = \left| \sum_{n=-\infty}^{\infty} \frac{(a, a_1, \ldots, a_r;q)_n}{(b, b_1, \ldots, b_r;q)_n} z^n \right|$$

$$\leq \sum_{n=-\infty}^{\infty} \left| \frac{(a, a_1, \ldots, a_r; q)_n}{(b, b_1, \ldots, b_r; q)_n} z^n \right| = \sum_{n=-\infty}^{\infty} \left\{ \frac{(a;q)_n |z|^n}{(b;q)_n} \cdot \left| \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_r; q)_n} \right| \right\}$$

$$\leq M \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} |z|^n = M \frac{(q, b/a, a|z|, q/a|z|; q)_\infty}{(b, q/a, |z|, b/a|z|; q)_\infty}, \tag{2.9}$$

where $b/a < |z| < 1$. Therefor inequality (2.1) holds. Thus, we complete the proof. □

3. Some convergence theorems for the q-integral

F. H. Jackson defined the q-integral by [6]

$$\int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n) q^n, \tag{3.1}$$

and

$$\int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t. \tag{3.2}$$

In this section, we use the inequality (2.1) to give some sufficient conditions for convergence of the q-integral. First, we give the following lemma:

Lemma 3.1. *Let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \ldots, r$ with $r \geq 1$ and $|b_1 b_2 \ldots b_r| \leq |a_1 a_2 \ldots a_r|$, let $\{u_n\}$ and $\{v_n\}$ be any convergence real number series such that $q < v_n < u_n < 1$ or $u_n < v_n < 0$ and let $\{c_n\}$ and $\{d_n\}$ be any convergence real number series such that $v_n/u_n < |d_n| < 1$. If*

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = p < 1,$$

then the q-series

$$\sum_{n=1}^{\infty} c_n \, {}_{r+1}\psi_{r+1} \left(\begin{matrix} u_n, a_1, \ldots, a_r \\ v_n, b_1, \ldots, b_r \end{matrix}; q, d_n \right) \tag{3.3}$$

converges absolutely.

Proof. Letting $a = u_n$, $b = v_n$ and $z = d_n$ in (2.1) gives

$$\left| {}_{r+1}\psi_{r+1} \left(\begin{matrix} u_n, a_1, \ldots, a_r \\ v_n, b_1, \ldots, b_r \end{matrix}; q, d_n \right) \right| \leq M \frac{(q, v_n/u_n, u_n|d_n|, q/u_n|d_n|; q)_\infty}{(v_n, q/u_n, |d_n|, v_n/u_n|d_n|; q)_\infty}. \tag{3.4}$$

Multiplying both sides of (3.4) by $|c_n|$ gets

$$\left| c_n\ {}_{r+1}\psi_{r+1}\left(\begin{matrix} u_n, a_1, \dots, a_r \\ v_n, b_1, \dots, b_r \end{matrix}; q, d_n\right) \right| \\ \le |c_n| M \frac{(q, v_n/u_n, u_n|d_n|, q/u_n|d_n|; q)_\infty}{(v_n, q/u_n, |d_n|, v_n/u_n|d_n|; q)_\infty}. \quad (3.5)$$

Let

$$e_n = |c_n| M \frac{(q, v_n/u_n, u_n|d_n|, q/u_n|d_n|; q)_\infty}{(v_n, q/u_n, |d_n|, v_n/u_n|d_n|; q)_\infty}.$$

Since

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = p < 1,$$

the ratio test shows that the series

$$\sum_{n=0}^{\infty} e_n$$

is convergent. From (3.5), it is sufficient to establish that the q-series (3.3) is absolutely convergent. □

Using Lemma 3.1, we can easily get some convergence theorems for the q-integral.

Theorem 3.2. *Let a, b be any real numbers such that $q < b < a < 1$ or $a < b < 0$, let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \dots, r$ with $r \ge 1$ and $|b_1 b_2 \dots b_r| \le |a_1 a_2 \dots a_r|$. Then for any $0 < d < 1$ and $\alpha > -1$, the q-integral*

$$\int_0^d t^\alpha \cdot {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, z\right) d_q t \quad (3.6)$$

converges absolutely. Where $z = b/a + (1 - b/a)t$.

Proof. It is easy to see that

$$b/a < b/a + (1 - b/a)dq^n < 1, \quad n = 0, 1, \dots .$$

By the definition of q-integral (3.1), we get

$$\int_0^d t^\alpha \cdot {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, z\right) d_q t \\ = d^{1+\alpha}(1-q) \sum_{n=0}^{\infty} q^{n(1+\alpha)}\ {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \dots, a_r \\ b, b_1, \dots, b_r \end{matrix}; q, b/a + (1 - b/a)dq^n\right). \quad (3.7)$$

Using Lemma 3.1 with $u_n = a$, $v_n = b$, $c_n = q^{n(1+\alpha)}$, $d_n = b/a + (1 - b/a)dq^n$, and noticing

$$\lim_{n\to\infty} \frac{q^{(n+1)(1+\alpha)}}{q^{n(1+\alpha)}} = q^{(1+\alpha)} < 1,$$

we know the q-integral (3.6) converges absolutely. □

By the definition of q-integral (3.2), we immediately get

Corollary 3.3. *Let a, b be any real numbers such that $q < b < a < 1$ or $a < b < 0$, let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \ldots, r$ with $r \geq 1$ and $|b_1 b_2 \ldots b_r| \leq |a_1 a_2 \ldots a_r|$. Then for any $0 < c, d < 1$ and $\alpha > -1$, the q-integral*

$$\int_c^d t^\alpha \cdot {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ b, b_1, \ldots, b_r \end{matrix}; q, z\right) d_q t \tag{3.8}$$

converges absolutely. Where $z = b/a + (1 - b/a)t$.

Theorem 3.4. *Let a, b and c be any real numbers such that $a < 0$, $b < 0$, $b/a < c < 1$, let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \ldots, r$ with $r \geq 1$ and $|b_1 b_2 \ldots b_r| \leq |a_1 a_2 \ldots a_r|$. Then for any $0 < d < 1$ and $\alpha > -1$, the q-integral*

$$\int_0^d z^\alpha \cdot {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ bz, b_1, \ldots, b_r \end{matrix}; q, cz\right) d_q z \tag{3.9}$$

converges absolutely.

Proof. By the definition of q-integral (3.1), we get

$$\begin{aligned} &\int_0^d z^\alpha \cdot {}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ bz, b_1, \ldots, b_r \end{matrix}; q, cz\right) d_q z \\ &\qquad = d^{1+\alpha}(1-q) \sum_{n=0}^{\infty} q^{n(1+\alpha)}{}_{r+1}\psi_{r+1}\left(\begin{matrix} a, a_1, \ldots, a_r \\ bdq^n, b_1, \ldots, b_r \end{matrix}; q, cdq^n\right). \end{aligned} \tag{3.10}$$

Using Lemma 3.1 with $u_n = a$, $v_n = bdq^n$, $c_n = q^n$ and $d_n = cdq^n$ and noticing

$$a < bdq^n < 0, \quad bdq^n/a < cdq^n < 1,$$

we get the q-integral (3.9) converges absolutely. □

By the definition of q-integral (3.2), we immediately get

Corollary 3.5. *Let a, b and c be any real numbers such that $a < 0$, $b < 0$, $b/a < c < 1$, let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \dots, r$ with $r \geq 1$ and $|b_1 b_2 \dots b_r| \leq |a_1 a_2 \dots a_r|$. Then for any $0 < d, e < 1$ and $\alpha > -1$, the q-integral*

$$\int_e^d z^\alpha \cdot {}_{r+1}\psi_{r+1} \left(\begin{matrix} a, a_1, \dots, a_r \\ bz, b_1, \dots, b_r \end{matrix} ; q, cz \right) d_q z \tag{3.11}$$

converges absolutely.

Theorem 3.6. *Let a, b and c be any real numbers such that $a < 0$, $b < 0$, $b/a < c < 1$, let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \dots, r$ with $r \geq 1$ and $|b_1 b_2 \dots b_r| \leq |a_1 a_2 \dots a_r|$. Then for $\alpha > -1$, $\beta \leq 0$ and $1 \leq d < 1/c$, the q-integral*

$$\int_0^d z^\alpha \cdot {}_{r+1}\psi_{r+1} \left(\begin{matrix} az^\beta, a_1, \dots, a_r \\ bz, b_1, \dots, b_r \end{matrix} ; q, cz \right) d_q z \tag{3.12}$$

converges absolutely.

Proof. By the definition of q-integral (3.1), we get

$$\begin{aligned} &\int_0^d z^\alpha \cdot {}_{r+1}\psi_{r+1} \left(\begin{matrix} az^\beta, a_1, \dots, a_r \\ bz, b_1, \dots, b_r \end{matrix} ; q, cz \right) d_q z \\ &\qquad = d^{1+\alpha}(1-q) \sum_{n=0}^{\infty} q^{n(1+\alpha)} {}_{r+1}\psi_{r+1} \left(\begin{matrix} adq^{n\beta}, a_1, \dots, a_r \\ bdq^n, b_1, \dots, b_r \end{matrix} ; q, cdq^n \right). \end{aligned} \tag{3.13}$$

Using Lemma 3.1 with $u_n = adq^{n\beta}$, $v_n = bdq^n$, $c_n = q^{n(1+\alpha)}$ and $d_n = cdq^n$ and noticing

$$aq^{n\beta} < bq^n < 0, \quad bq^{n(1-\beta)}/a < cdq^n < 1,$$

we have the q-integral (3.12) converges absolutely. □

Corollary 3.7. *Let a, b and c be any real numbers such that $a < 0$, $b < 0$, $b/a < c < 1$, let a_i, b_i be any real numbers such that $|a_i| > q$, $|b_i| < 1$ for $i = 1, 2, \dots, r$ with $r \geq 1$ and $|b_1 b_2 \dots b_r| \leq |a_1 a_2 \dots a_r|$. Then for $\alpha > -1$, $\beta \leq 0$ and $1 \leq d$, $e < 1/c$, the q-integral*

$$\int_e^d z^\alpha \cdot {}_{r+1}\psi_{r+1} \left(\begin{matrix} az^\beta, a_1, \dots, a_r \\ bz, b_1, \dots, b_r \end{matrix} ; q, cz \right) d_q z \tag{3.14}$$

converges absolutely.

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