Publ. Math. Debrecen
82/3-4 (2013), 511–529
DOI: 10.5486/PMD.2013.4770

Distribution functions of ratio sequences, III

By VLADIMÍR BALÁŽ (Bratislava), LADISLAV MIŠÍK (Ostrava),
OTO STRAUCH (Bratislava) and JÁNOS T. TÓTH (Komárno)

Dedicated to Professor Kálmán Győry on the occasion of his 70th birthday

Abstract. In this paper we study the distribution functions $g(x)$ of the sequence of blocks $X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$, $n = 1, 2, \dots$, where x_n is an increasing sequence of positive integers. Assuming that the lower asymptotic density $\underline{d}$ of x_n is positive, we find the optimal lower and upper bounds of $g(x)$. As an application, we also get the optimal bounds of limit points of $\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}$, $n = 1, 2, \dots$

1. Introduction

Let x_n, $n = 1, 2, \dots$, be an increasing sequence of positive integers (by "increasing" we mean strictly increasing). The double sequence x_m/x_n, $m, n = 1, 2, \dots$ is called *the ratio sequence* of x_n, which has been introduced by T. Šalát [Sa]. He studied its everywhere density. For further study of ratio sequences, O. Strauch and J. T. Tóth [ST] introduced the sequence X_n of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \quad n = 1, 2, \dots$$

and they studied the uniform distribution of X_n in the sense of the monographs [KN] and [DT]. The authors in [ST] further studied the distribution functions

Mathematics Subject Classification: 11K31.
Key words and phrases: blocks sequence, distribution function, asymptotic density.
Supported by the APVV Projects SK-CZ-0098-07, SK-HU-0009-08, by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070) and VEGA 1/1022/12.

$g(x)$ of X_n. The motivation for this is that the existence of strictly increasing $g(x)$ implies everywhere density of x_m/x_n, which is the primary problem of Šalát in [Sa].

In what follows, we will use the following definitions, and basic properties, see [SP, p. 1–28, 1.8.23].

• Denote by $F(X_n, x)$ the step distribution function

$$F(X_n, x) = \frac{\#\{i \le n;\ \frac{x_i}{x_n} < x\}}{n},$$

for $x \in [0, 1)$, and $F(X_n, 1) = 1$. Directly from definition we have

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x\frac{x_m}{x_n}\right) \tag{1}$$

for all pairs of integers $m < n$, and every $x \in [0, 1)$.

• For any increasing sequence of positive integers x_n, $n = 1, 2, \dots$, we define a counting function $A(t)$ as

$$A(t) = \#\{n \in \mathbb{N};\ x_n < t\}.$$

Then for every $x \in (0, 1]$ we have the equality

$$\frac{nF(X_n, x)}{xx_n} = \frac{A(xx_n)}{xx_n}, \tag{2}$$

which we shall use to compute the asymptotic density of x_n. Here, the lower asymptotic density $\underline{d}$, and the upper asymptotic density $\overline{d}$ of x_n, $n = 1, 2, \dots$ are defined as

$$\underline{d} = \liminf_{t\to\infty} \frac{A(t)}{t} = \liminf_{n\to\infty} \frac{n}{x_n}, \quad \overline{d} = \limsup_{t\to\infty} \frac{A(t)}{t} = \limsup_{n\to\infty} \frac{n}{x_n}.$$

• A non-decreasing function $g : [0, 1] \to [0, 1]$, $g(0) = 0$, $g(1) = 1$ is called a distribution function. We shall identify any two distribution functions coinciding at common points of continuity.

• A distribution function $g(x)$ is a distribution function of the sequence of blocks X_n, $n = 1, 2, \dots$, if there exists an increasing sequence $n_1, n_2, \dots$ of positive integers such that

$$\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$$

almost everywhere in $[0, 1]$. This is equivalent to weak convergence; it means that the preceding limit holds for every point $x \in [0, 1]$ of continuity of $g(x)$.

In this paper we frequently use the following two theorems of Helly (see First and Second Helly theorem [SP, Theorem 4.1.0.10 and Theorem 4.1.0.11 , p. 4–5]).

• *Helly's selection principle*: For any sequence $g_n(x)$, $n = 1, 2, \dots$, of distribution functions in $[0, 1]$ there exists a subsequence $g_{n_k}(x)$, $k = 1, 2, \dots$, and a distribution function $g(x)$ such that $\lim_{k\to\infty} g_{n_k}(x) = g(x)$ almost everywhere.

• *Second Helly theorem*: If we have $\lim_{n\to\infty} g_n(x) = g(x)$ almost everywhere in [0,1], then for every continuous function $f : [0,1] \to \mathbb{R}$ we have $\lim\limits_{n\to\infty} \int_0^1 f(x)\mathrm{d}g_n(x) = \int_0^1 f(x)\mathrm{d}g(x)$.

• Note that applying Helly's selection principle, from the sequence $F(X_n, x)$, $n = 1, 2, \dots$, one can select a subsequence $F(X_{n_k}, x)$, $k = 1, 2, \dots$, such that $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$ holds not only for the continuity points x of $g(x)$, but also for all $x \in [0, 1]$.

• Denote by $G(X_n)$ the set of all distribution functions of X_n, $n = 1, 2, \dots$. For a singleton $G(X_n) = \{g(x)\}$, the distribution function $g(x)$ is also called asymptotic distribution function of X_n.

• We will use the one-step distribution function $c_\alpha(x)$ with the step 1 at α defined on $[0, 1]$ via

$$c_\alpha(x) = \begin{cases} 0, & \text{if } x \le \alpha; \\ 1, & \text{if } x > \alpha, \end{cases} \tag{3}$$

while always $c_\alpha(0) = 0$ and $c_\alpha(1) = 1$.

• The lower distribution function $\underline{g}(x)$, and the upper distribution function $\overline{g}(x)$ of a sequence x_n, $n = 1, 2, \dots$ are defined as

$$\underline{g}(x) = \inf_{g\in G(X_n)} g(x), \quad \overline{g}(x) = \sup_{g\in G(X_n)} g(x).$$

In Section 2 of this paper we recall some known theorems, which we shall use and extend. In Section 3 (Theorem 5) we solve Open problem no. 7 from [SN, 1.9. Block sequence] stating that every sequence of blocks X_n has a distribution function $g(x)$ such that $g(x) \ge x$ for all $x \in [0, 1]$. Then, assuming $\underline{d} > 0$, we find (Theorem 6) boundaries $h_1(x) \le g(x) \le h_2(x)$, which hold for every distribution function $g(x) \in G(X_n)$, and which are, in a sense, optimal. As a consequence, we produce boundaries (Theorem 7) for $\frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}$. In the last Section 4 (Example 3), we find the exact values of the lim inf and lim sup of $\frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}$ for integers x_n from the intervals $(\gamma a^k, \delta a^k)$, $k = 1, 2, \dots$.

2. Basic known results

For an increasing sequence x_n, $n = 1, 2, \dots$ of positive integers the following theorems are known.

Theorem 1 ([ST, Theorem 7.1]). *For every sequence of positive integers x_n there exits $g(x) \in G(X_n)$ such that*

$$\int_0^1 g(x)\mathrm{d}x \ge \frac{1}{2}. \tag{4}$$

Theorem 2 ([ST, Theorem 6.2 (ii),(iii)]). *If $\underline{d} > 0$, then there exits $g \in G(X_n)$ such that $g(x) \ge x$ for every $x \in [0,1]$. Furthermore, for every $g(x) \in G(X_n)$, and $x \in [0,1]$ we have*

$$x\frac{\underline{d}}{\overline{d}} \le g(x) \le x\frac{\overline{d}}{\underline{d}}.$$

Theorem 3 ([ST, Propozicion 6.1]). *Assume for a sequence n_k, $k = 1, 2, \dots$ that*

(i) $\lim_{k\to\infty} F(X_{n_k}, x) = g(x)$,

(ii) $\lim_{k\to\infty} \frac{n_k}{x_{n_k}} = d_g$.

Then there exists

(iii) $\lim_{k\to\infty} \frac{A(xx_{n_k})}{xx_{n_k}} = d_g(x)$ *and*

$$\frac{g(x)}{x} d_g = d_g(x). \tag{5}$$

Here the limits (i), *and* (iii) *can be considered for all $x \in (0,1]$, or all continuity points $x \in (0,1]$ of $g(x)$.*

Theorem 4 ([ST, Theorem 4.1, Theorem 6.2]). *Assume that every distribution function in $G(X_n)$ is continuous at 1. Then each distribution function in $G(X_n)$ is continuous in $(0,1]$, i.e. the only point of discontinuity is possibly 0. Furthermore, if $\underline{d} > 0$, then all distribution functions in $G(X_n)$ are continuous in $[0,1]$.*

3. Main results

We start with an extension of Theorem 1, and the first part of Theorem 2.

Theorem 5. *For every increasing sequence of positive integers x_n, $n = 1, 2, \dots$, there exists $g(x) \in G(X_n)$ such that $g(x) \ge x$ for all $x \in [0,1]$.*

Proof. If $\underline{d} > 0$, select n_k so that $\frac{n_k}{x_{n_k}} \to \underline{d} > 0$, and $F(X_{n_k}, x) \to g(x)$. For such $g(x)$, (5) implies

$$\frac{g(x)}{x}\underline{d} \geq \underline{d}.$$

Now, let $\underline{d} = 0$. Select n_k such that

$$\frac{n_k}{x_{n_k}} = \min_{i \leq n_k} \frac{i}{x_i},$$

and $F(X_{n_k}, x) \to g(x)$. Then for every $x \in (0, 1]$,

$$\frac{A(xx_{n_k})}{xx_{n_k}} \geq \frac{n_k - 1}{x_{n_k}}.$$

Applying (2) yields

$$\frac{F(X_{n_k}, x)}{x}\frac{n_k}{x_{n_k}} \geq \frac{n_k - 1}{x_{n_k}},$$

and taking the limit, as $k \to \infty$, we obtain $g(x) \geq x$ for all $x \in [0, 1]$. □

Now we are going to study in more detail the second part of Theorem 2.

Theorem 6. *Let $x_1 < x_2 < \dots$ be a sequence of positive integers with positive lower asymptotic density $\underline{d} > 0$ and upper asymptotic density $\overline{d}$. Then all distribution functions $g(x) \in G(X_n)$ are continuous, non-singular and bounded by $h_1(x) \leq g(x) \leq h_2(x)$, where*

$$h_1(x) = \begin{cases} x\dfrac{\underline{d}}{\overline{d}} & \text{if } x \in \left[0, \dfrac{1-\overline{d}}{1-\underline{d}}\right]; \\ \dfrac{\underline{d}}{\frac{1}{x} - (1 - \underline{d})} & \text{otherwise,} \end{cases} \tag{6}$$

$$h_2(x) = \min\left(x\frac{\overline{d}}{\underline{d}}, 1\right). \tag{7}$$

Moreover, $h_1(x)$ and $h_2(x)$ are the best possible in the following sense: for given $0 < \underline{d} \leq \overline{d}$, there exists $x_1 < x_2 < \dots$ with lower and upper asymptotic densitie $\underline{d}$, $\overline{d}$, such that $\underline{g}(x) = h_1(x)$ for $x \in \left[\frac{1-\overline{d}}{1-\underline{d}}, 1\right]$; also, there exists $x_1 < x_2 < \dots$ with given $0 < \underline{d} \leq \overline{d}$ such that $\overline{g}(x) = h_2(x) \in G(X_n)$.

Proof. For $g(x) \in G(X_n)$, let n_k, $k = 1, 2, \dots$, be an increasing sequence of indices such that $F(X_{n_k}, x) \to g(x)$. From n_k we can select a subsequence (for simplicity written as the original n_k) [1] such that

$$\frac{n_k}{x_{n_k}} \to d_g > 0. \tag{8}$$

[1]We call d_g a *local asymptotic density* related to $g(x)$.

Then, by (5), we have

$$g(x) = x\frac{d_g(x)}{d_g}, \quad \text{where} \quad \frac{A(xx_{n_k})}{xx_{n_k}} \to d_g(x) \tag{9}$$

for arbitrary $x \in (0,1]$.

We will continue in six steps 1^0–6^0.

1^0. We prove the continuity of $g(x)$ at $x = 1$ (improving (iv) in [ST, Theorem 6.2]) for each $g(x) \in G(X_n)$.

In view of the definition of the counting function $A(t)$

$$0 \le A(x_{n_k}) - A(xx_{n_k}) \le x_{n_k} - xx_{n_k};$$

thus,

$$0 \le \frac{A(x_{n_k})}{x_{n_k}} - \frac{A(xx_{n_k})}{x_{n_k}} = \frac{n_k - 1}{x_{n_k}} - \frac{A(xx_{n_k})}{xx_{n_k}}x \le 1 - x,$$

and, as $k \to \infty$, we have $0 \le d_g - d_g(x)x \le 1 - x$, which implies

$$0 \le d_g - d_g(x) + d_g(x)(1 - x) \le 1 - x.$$

Consequently, $\lim_{x\to 1} d_g(x) = d_g$, and so $\lim_{x\to 1} g(x) = \lim_{x\to 1} x\frac{d_g(x)}{d_g} = 1$. Since $g(x) \in G(X_n)$ is arbitrary, Theorem 4 gives continuity of $g(x)$ in the whole unit interval $[0,1]$.

2^0. We prove that $g(x)$ has a bounded right derivative for every $x \in (0,1)$, and for each $g(x) \in G(X_n)$.

For $0 < x < y < 1$ again

$$0 \le A(yx_{n_k}) - A(xx_{n_k}) \le (y - x)x_{n_k},$$

which implies

$$0 \le \frac{A(yx_{n_k})}{yx_{n_k}}y - \frac{A(xx_{n_k})}{xx_{n_k}}x \le y - x.$$

Letting $k \to \infty$, we get $0 \le d_g(y)y - d_g(x)x \le y - x$, hence

$$0 \le g(y) - g(x) = \frac{d_g(y)y - d_g(x)x}{d_g} \le \frac{y - x}{d_g}.$$

Consequently,

$$0 \le \frac{g(y) - g(x)}{y - x} \le \frac{1}{d_g} \tag{10}$$

for all $x, y \in (0,1)$, $x < y$, which gives the upper bound of the right derivatives of $g(x)$ for every $x \in (0,1)$. Note that a singular distribution function (continuous, strictly increasing, having zero derivative almost everywhere) has infinite right Dini derivatives in a dense subset of $(0,1)$.

3^0. We prove a local form of Theorem 5.

As $\underline{d} \le d_g \le \overline{d}$, (9) implies

$$x\frac{\underline{d}}{d_g} \le g(x) \le x\frac{\overline{d}}{d_g} \tag{11}$$

for every $x \in [0,1]$. It follows from (10), that there exists an extreme point $A_g = (x_g, y_g)$ on the line $y = x\frac{\underline{d}}{d_g}$ such that $g(x)$ has no common point with this line for $x > x_g$. This point A_g is the intersection of the lines

$$y = x\frac{\underline{d}}{d_g} \text{ and, } y = x\frac{1}{d_g} + 1 - \frac{1}{d_g} \tag{12}$$

therefore,

$$A_g = (x_g, y_g) = \left(\frac{1-d_g}{1-\underline{d}}, \frac{\underline{d}}{d_g}\frac{1-d_g}{1-\underline{d}}\right). \tag{13}$$

It means that for a given $g(x) \in G(X_n)$, $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, where

$$h_{1,g}(x) = \begin{cases} x\frac{\underline{d}}{d_g} & \text{if } x < y_0 = \frac{1-d_g}{1-\underline{d}}; \\ x\frac{1}{d_g} + 1 - \frac{1}{d_g} & \text{if } y_0 \le x \le 1, \end{cases} \tag{14}$$

$$h_{2,g}(x) = \min\left(x\frac{\overline{d}}{d_g}, 1\right). \tag{15}$$

4^0. Now we find $h_1(x)$, and $h_2(x)$ such that

$$h_1(x) \le h_{1,g}(x) \le h_{2,g}(x) \le h_2(x)$$

for every $g \in G(X_n)$.

In the parametric expression (13) of A_g, the local asymptotic density d_g defined by (8) belongs to the interval $[\underline{d}, \overline{d}\,]$. The well-known Darboux property of the asymptotic density implies that for an arbitrary $d \in [\underline{d}, \overline{d}\,]$ there exists an increasing n_k, $k = 1, 2, \dots$, such that $\frac{n_k}{x_{n_k}} \to d$ [2], and then the Helly selection

[2]A simple proof follows from the fact that for every $d \in (\underline{d}, \overline{d})$ there exist infinitely many $n \in \mathbb{N}$ such that $A(n)/n \le d \le A(n+1)/(n+1)$. These n we denote as n_k.

principle implies the existence of a subsequence of n_k such that $F(X_{n_k}, x) \to g(x)$ for some $g(x) \in G(X_n)$. Thus, if $g(x)$ runs over $G(X_n)$, then d_g runs over the entire interval $[\underline{d}, \overline{d}\,]$. Substituting $d_g = 1 - x_g(1 - \underline{d})$ in $A_g = (x_g, y_g)$ we get

$$y_g = y_g(x_g) = \frac{\underline{d}}{\frac{1}{x_g} - (1 - \underline{d})},$$

where $x_g = \frac{1-d_g}{1-\underline{d}}$ runs through the interval $I = \left[\frac{1-\overline{d}}{1-\underline{d}}, 1\right]$ for $d_g \in [\underline{d}, \overline{d}\,]$. By putting $x_g = x$, and $y_g = h_1$ we find a part of $h_1(x)$ for $x \in I$ in (6). The remaining part of $h_1(x)$, and also the whole $h_2(x)$, follow from the basic inequality (11), see Figure 1.

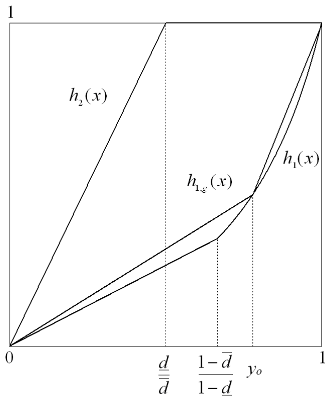

Figure 1. Boundaries of $g(x) \in G(X_n)$

5^0. The optimality of $h_1(x)$ follows from the following example.

The increasing sequence x_n of the integers lying in the intervals

$$(\gamma, \delta), (\gamma a, \delta a), \dots, (\gamma a^k, \delta a^k), \dots,$$

where $1 \le \gamma < \delta \le a$, has been used in [ST, pp. 774–777, Example 11.2]. For its lower, and upper asymptotic densities $\underline{d}$, and $\overline{d}$, it has been shown that

$$\underline{d} = \frac{(\delta - \gamma)}{\gamma(a-1)}, \quad \overline{d} = \frac{(\delta - \gamma)a}{\delta(a-1)}, \tag{16}$$

and that the graph of every $g \in G(X_n)$ lies in the intervals

$$[1/a, 1] \times [1/a, 1] \cup \left[1/a^2, 1/a\right] \times \left[1/a^2, 1/a\right] \cup \dots$$

Moreover, the part of the graph in $\left[1/a^k, 1/a^{k-1}\right] \times \left[1/a^k, 1/a^{k-1}\right]$ is similar to the part of the graph in $\left[1/a^{k+1}, 1/a^k\right] \times \left[1/a^{k+1}, 1/a^k\right]$ with the scale a. It is also proved in [ST], that the graph of $\underline{g}(x)$ in $\left[1/a, 1\right] \times \left[1/a, 1\right]$ has the form $\underline{g}(x) = \left(1 + \frac{1}{\underline{d}}\left(\frac{1}{x} - 1\right)\right)^{-1}$ for $x \in \left[\frac{\gamma}{\delta}, 1\right]$, and it coincides with the graph of $h_1(x)$ in the interval $I = \left[\frac{1-\overline{d}}{1-\underline{d}}, 1\right]$, since $\frac{1-\overline{d}}{1-\underline{d}} = \frac{\gamma}{\delta}$.

6^0. Finally, we prove the optimality of $h_2(x)$. Before proving it in several substeps, note that in 5^0 the graph of the upper distribution function $\overline{g}(x)$ in $\left[1/a, 1\right] \times \left[1/a, 1\right]$ is a straight line which intersects the line $y = 1$ at $x = \frac{\delta}{\gamma a} = \frac{\underline{d}}{\overline{d}}$. Thus, $\overline{g}(\frac{\underline{d}}{\overline{d}}) = h_2(\frac{\underline{d}}{\overline{d}}) = 1$ proving that the point $\left(\frac{\underline{d}}{\overline{d}}, 1\right)$ is optimal.

To complete the proof of 6^0, in the following steps a)- f) we shall construct a sequence of positive integers $x_1 < x_2 < \dots$ with $0 < \underline{d} \le \overline{d}$ such that $h_2(x) \in G(X_n)$. This implies $h_2(x) = \overline{g}(x)$.

a) The condition $h_2(x) \in G(X_n)$ for a sequence of positive integers $x_1 < x_2 < \dots$ is equivalent to the existence of an increasing sequence of indices n_k such that $F(X_{n_k}, x) \to h_2(x)$ for $x \in [0, 1]$, and $\frac{n_k}{n_{k+1}} \to 0$. An application of (1) yields that this is equivalent (see Fig. 2) to the existence of $m'_k < m_k < n_k$ such that the values $x_{m'_k} < x_{m_k} < x_{n_k}$ satisfy

(i) $\frac{x_{m_k}}{x_{n_k}} \to \frac{\underline{d}}{\overline{d}}$,

(ii) $\frac{m_k}{n_k} \to 1$,

(iii) $\frac{x_{m'_k}}{x_{n_k}} \to 0$,

(iv) $\frac{m'_k}{n_k} \to 0$.

Moreover, because the sequence of positive integers x_n increases, we have (see Figure 3)

(v) $x_{m_k} - x_{m'_k} \ge m_k - m'_k$,

vi) $x_{n_k} - x_{m_k} \ge n_k - m_k$,

(vii) $x_{m'_{k+1}} - x_{n_k} \ge m'_{k+1} - n_k$,

(viii) $n_k < m'_{k+1}$,

(ix) $m'_1 \le x_{m'_1}$.

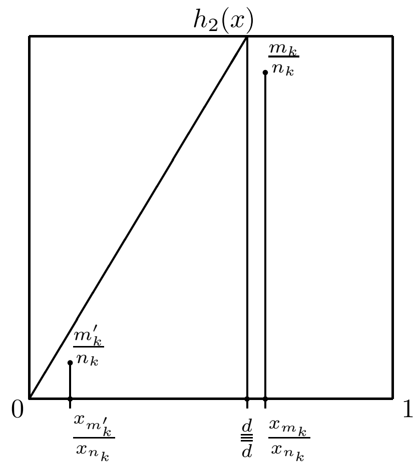

Figure 2: $F(X_{n_k}, x) \to h_2(x)$, the properties (i)–(iv).

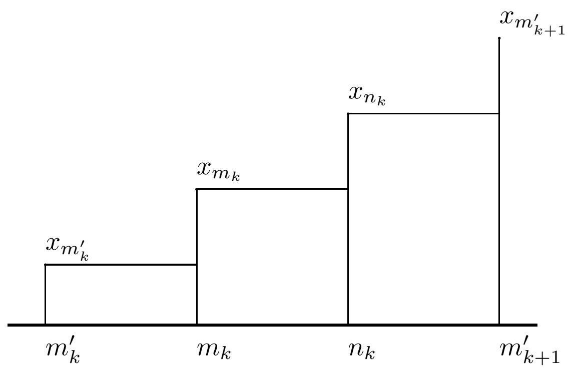

Figure 3: The (v)–(viii) properties.

b) Before solving (i)-(ix) we must capture a role of $\underline{d}$ and $\overline{d}$. By (i) and, (ii) we have the limit

$$\frac{n_k}{m_k}\frac{x_{m_k}}{x_{n_k}} \to \frac{\underline{d}}{\overline{d}}.$$

Selecting a subsequence of (m_k, n_k), $k = 1, 2, \dots$, we can assume the existence of the limits $\frac{n_k}{x_{n_k}} \to d_{h_2}$, and $\frac{m_k}{x_{m_k}} \to d_g$ (for simplicity, also assume $F(X_{m_k}, x) \to g(x)$). Then

$$\frac{n_k}{m_k}\frac{x_{m_k}}{x_{n_k}} = \frac{\frac{x_{m_k}}{m_k}}{\frac{x_{n_k}}{n_k}} \to \frac{\frac{1}{d_g}}{\frac{1}{d_{h_2}}} = \frac{\underline{d}}{\overline{d}},$$

and since

$$\frac{\underline{d}}{\overline{d}} = \min_{d_1,d_2 \in [\underline{d},\overline{d}\,]} \frac{d_1}{d_2},$$

we have $d_{h_2} = \underline{d}$, and $d_g = \overline{d}$. This yields the additional conditions

(x) $\frac{n_k}{x_{n_k}} \to \underline{d}$,

(xi) $\frac{m_k}{x_{m_k}} \to \overline{d}$.

c) In what follows, we assume $\underline{d} < \overline{d}$, because from Theorem 2, by $0 < \underline{d} = \overline{d}$, we get $G(X_n) = \{x\}$, and also $h_2(x) = x$.

d) To find a sequence x_n which satisfies (i)–(xi), we define x_{n_k}, x_{m_k}, m_k, $x_{m'_k}$, m'_k by using n_k (for a simplifying the definitions, the integer part will be omitted):

$$\begin{aligned}
x_{n_k} &= \frac{n_k}{\underline{d}}, \\
x_{m_k} &= x_{n_k}\frac{\underline{d}}{\overline{d}} = \frac{n_k}{\overline{d}}, \\
m_k &= x_{m_k}\overline{d} - o(n_k) = n_k - \sqrt{n_k}, \\
x_{m'_k} &= \sqrt{x_{m_k}} = \sqrt{\frac{n_k}{\overline{d}}}, \\
m'_k &= d' x_{m'_k} = d'\sqrt{\frac{n_k}{\overline{d}}},
\end{aligned}$$

for some $d' \in (\underline{d}, \overline{d})$.

These $x_{n_k}, x_{m_k}, x_{m'_k}, m_k, m'_k$ satisfy (i)–(vii), (x), (xi). For (viii) we need

$$n_{k+1} > \frac{\overline{d}}{d'^2}\frac{1}{\underline{d}^2} n_k^2$$

for $k = 2, 3, \dots$, and for (ix) the n_1 must be large.

e) For linearity of $h_2(x)$ in $\left[0, \frac{\underline{d}}{\overline{d}}\right]$, and to guarantee the asymptotic densities $\underline{d}, \overline{d}$, define

(xii) $x_n = x_a + (n-a)\frac{x_b - x_a}{b-a}$ for $n \in (a, b)$, where (a, b) coincides successively with (m'_k, m_k), (m_k, n_k), or (n_k, m'_{k+1}).

Then

$$\frac{n}{x_n} = \frac{a + (n-a)}{x_a + (n-a)\frac{x_b - x_a}{b-a}},$$

and because the derivative $\left(\frac{Ax+B}{Cx+D}\right)' = \frac{AD-BC}{(Cx+D)^2}$, the minimum and maximum of $\frac{n}{x_n}$ for $n \in (a,b)$ are attained at the endpoints $n = a$, and $n = b$, i.e., for $n = m'_k, m_k, n_k$. Since the limits of $\frac{n_k}{x_{n_k}}$, $\frac{m_k}{x_{m_k}}$, $\frac{m_k}{x_{m'_k}}$ are from $[\underline{d}, \overline{d}]$, and the boundary points are attained, $\liminf_{n\to\infty} \frac{n}{x_n} = \underline{d}$, and $\limsup_{n\to\infty} \frac{n}{x_n} = \overline{d}$.

f) For such $x_1 < x_2 < \dots$ we have $\underline{d}, \overline{d}$, and $F(X_{n_k}, x) \to h_2(x)$ for $x \in [0,1]$; hence, the proof of Theorem 6 is finished. □

Remark 1. In a sharp contrast to $h_2(x) \in G(X_n)$ in 6^0 we note that for every sequence of integers $x_1 < x_2 < \dots$, $0 < \underline{d} < \overline{d}$, we have $h_1(x) \notin G(X_n)$, because for every $g(x) \in G(X_n)$, $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$, and $h_{1,g}(x) \ne h_1(x)$.

Theorem 6 implies the following best possible boundaries of the sum

$$\frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}.$$

Theorem 7. *For every increasing sequence $x_1 < x_2 < \dots$ of positive integers with $0 < \underline{d} \le \overline{d}$ we have*

$$\frac{1}{2}\frac{\underline{d}}{\overline{d}} \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n}, \tag{17}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \frac{1}{2} + \frac{1}{2}\left(\frac{1-\min(\sqrt{\underline{d}}, \overline{d})}{1-\underline{d}}\right)\left(1 - \frac{\underline{d}}{\min(\sqrt{\underline{d}}, \overline{d})}\right). \tag{18}$$

Here the equality in both (17) *and* (18)[3] *can be attained.*

Proof. By the Helly theorem, $F(X_{n_k}, x) \to g(x)$ forces

$$\int_0^1 x \mathrm{d}F(X_{n_k}, x) = \frac{1}{n_k}\sum_{i=1}^{n_k} \frac{x_i}{x_{n_k}} \to \int_0^1 x \mathrm{d}g(x) = 1 - \int_0^1 g(x)\mathrm{d}x; \tag{19}$$

thus,

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \max_{g\in G(X_n)} \int_0^1 g(x)\mathrm{d}x, \tag{20}$$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \min_{g\in G(X_n)} \int_0^1 g(x)\mathrm{d}x. \tag{21}$$

[3]If $\sqrt{\underline{d}} \le \overline{d}$ then the right-hand side in (18) is $\frac{1}{1+\sqrt{\underline{d}}}$.

If $\underline{d} > 0$, then by Theorem 6, $h_1(x) \le g(x) \le h_2(x)$, which implies

$$1 - \int_0^1 h_2(x)\mathrm{d}x \le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le 1 - \int_0^1 h_1(x)\mathrm{d}x. \quad (22)$$

For $x_1 < x_2 < \dots$ in step 6^0, where $h_2(x) \in G(X_n)$, we have equality on the left hand side of (22). On the other hand, Remark 1 implies a sharp inequality on the right hand side, therefore,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} < 1 - \frac{1}{2}\frac{\underline{d}}{\overline{d}}\left(\frac{1-\overline{d}}{1-\underline{d}}\right)^2 - \frac{\underline{d}}{(1-\underline{d})^2}\left(\log\frac{\underline{d}}{\overline{d}} - (\overline{d}-\underline{d})\right) \quad (23)$$

holds for an arbitrary sequence of integers $x_1 < x_2 < \dots$ with $0 < \underline{d} < \overline{d}$.

Applying the inequality $h_{1,g}(x) \le g(x) \le h_{2,g}(x)$ for every $g \in G(X_n)$ from step 3^0 to (19), we obtain

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} \le \max_{g(x)\in G(X_n)} \left(1 - \int_0^1 h_{1,g}(x)\mathrm{d}x\right). \quad (24)$$

If the maximum in (24) is attained for $g_0(x) \in G(X_n)$, and $h_{1,g_0}(x) \in G(X_n)$, then $g_0(x) = h_{1,g_0}(x)$, and

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{x_n} = 1 - \int_0^1 h_{1,g_0}(x)\mathrm{d}x. \quad (25)$$

Using (14) we get

$$\int_0^1 h_{1,g}(x)\mathrm{d}x = \frac{1}{2}\left(1 + \frac{1-d_g}{1-\underline{d}}\left(\frac{\underline{d}}{d_g} - 1\right)\right),$$

and taking derivative with respect to $d_g \in [\underline{d}, \overline{d}\,]$

$$\left(\int_0^1 h_{1,g}(x)\mathrm{d}x\right)' = \frac{1}{2(1-\underline{d})}\left(1 - \frac{\underline{d}}{(d_g)^2}\right)$$

shows that $\min \int_0^1 h_{1,g}(x)\mathrm{d}x$ is attained for $d_{g_0} = \min(\sqrt{\underline{d}}, \overline{d})$.

Now, to prove (25) we shall construct integers $x_1 < x_2 < \dots$ with $0 < \underline{d} \le \overline{d}$ such that $h_{1,g_0}(x) \in G(X_n)$. We start with the sequence of indices n_k, and then by (14) we shall find indices $m'_k < m_k < n_k$, and integers $x_{m'_k} < x_{m_k} < x_{n_k}$ such that

(i) $\frac{n_k}{x_{n_k}} \to d_{g_0}$,

(ii) $\frac{m_k}{n_k} \to \frac{\underline{d}}{d_{g_0}} \frac{1-d_{g_0}}{1-\underline{d}}$,

(iii) $\frac{x_{m_k}}{x_{n_k}} \to \frac{1-d_{g_0}}{1-\underline{d}}$,

(iv) $\frac{x_{m'_k}}{x_{n_k}} \to 0$,

(v) $\frac{m'_k}{n'_k} \to 0$,

(vi) $\frac{m'_k}{x_{m'_k}} \to \overline{d}$.

Then from (i), (ii), and (iii) it follows that $\frac{m_k}{x_{m_k}} \to \underline{d}$. Furthermore, assume

(v) $x_{m_k} - x_{m'_k} \geq m_k - m'_k$,

(vi) $x_{n_k} - x_{m_k} \geq n_k - m_k$,

(vii) $x_{m'_{k+1}} - x_{n_k} \geq m'_{k+1} - n_k$,

(viii) $n_k < m'_{k+1}$,

(ix) $m'_1 \leq x_{m'_1}$.

For these (i)–(ix) a sequence of integers x_n can be found similarly to 6^0d). The rest of the terms of x_n define linearly as in e). □

4. Examples

Example 1. a) If $0 < \underline{d} = \overline{d}$, then the bounds in both (17), and (18) equal to $\frac{1}{2}$, which implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} = \frac{1}{2}.$$

This also follows from the fact that $G(X_n) = \{x\}$, see Theorem 2.

b) If $\underline{d} = \frac{1}{2}$, and $\overline{d} = 1$, then by (23), $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} < 2 - \log 4 < 1$. Using (18) we have an even better estimate $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_n} \leq 2 - \sqrt{2}$.

Example 2. Omitting $\underline{d} > 0$, we can find a sequence of positive integers $x_1 < x_2 < \dots$ such that $c_0(x), c_1(x) \in G(X_n)$, where $c_0(x), c_1(x)$ are one-steps distribution functions defined by (3) in the Introduction. In this case

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}=0=1-\int_0^1 c_0(x)\mathrm{d}x,$$
$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}=1=1-\int_0^1 c_1(x)\mathrm{d}x.$$

We shall construct such x_n by applying [GS, Theorem 5]. For the index sequences $m_k' < m_k < n_k' < n_k$ we shall find sequences of positive integers $x_{m_k'} < x_{m_k} < x_{n_k'} < x_{n_k}$ such that

(i) $\frac{n_k'}{n_k} \to 0$,

(ii) $\frac{m_k'}{m_k} \to 1$,

(iii) $\frac{x_{n_k'}}{x_{n_k}} \to 1$,

(iv) $\frac{x_{m_k'}}{x_{m_k}} \to 0$.

Furthermore,

(v) $x_{n_k} - x_{n_k'} \geq n_k - n_k'$,

(vi) $x_{n_k'} - x_{m_k} \geq n_k' - m_k$,

(vii) $x_{m_k} - x_{m_k'} \geq m_k - m_k'$,

(viii) $m_{k+1}' > n_k$,

(ix) $x_{m_1'} \geq m_1'$,

(x) $x_{m_{k+1}'} - x_{n_k} \geq m_{k+1}' - n_k$.

Then (i)–(x) will be satisfied, if for a given n_k we put $x_{n_k} = n_k^2$, $x_{n_k'} = n_k^2 - n_k$, $x_{m_k} = n_k^2 - 2n_k$, $x_{m_k'} = n_k$, $n_k' = \sqrt{n_k}$, $m_k = \sqrt{n_k} - \sqrt[4]{n_k}$, $m_k' = \sqrt{n_k} - 2\sqrt[4]{n_k}$; further, (viii) holds if $n_{k+1} \geq n_k^4$. For the other n's in the intervals (m_k', m_k), (m_k, n_k'), (n_k', n_k), and (n_k, m_{k+1}') define x_n linearly.

Now, by (i), and (iii) we have $F(X_{n_k}, x) \to c_1(x)$, and (ii), (iv) imply $F(X_{m_k}, x) \to c_0(x)$.

Example 3. In this example we extend a characterization of distribution functions of the sequence $x_1 < x_2 < \ldots$ in [ST, Example 11.2]. This sequence was used in the proof of Theorem 6, part 5^0.

Let x_n, $n = 1, 2, \dots$, be the increasing sequence of all integer points in the sequence of intervals $(\gamma a^k, \delta a^k)$ (in short $a^k(\gamma, \delta)$), $k = 0, 1, 2, \dots$, where $1 \le \gamma < \delta \le a$ are real numbers.

It is proved in [ST, Ex. 11.2] that

1^0. The set of all distribution functions can be expressed in parametric form as $G(X_n) = \{g_t(x); t \in [0, 1]\}$, where

$$F(X_{n_k}, x) \to g_t(x) \text{ for } n_k \quad \text{such that } x_{n_k} = [a^k\gamma + ta^k(\delta - \gamma)] \tag{26}$$

2^0. The function $g_t(x)$ has constant values $g_t(x) = \frac{1}{a^i(1+t(a-1))}$ for $x \in \frac{(\delta, a\gamma)}{a^{i+1}(\gamma+t(\delta-\gamma))}$, $i = 0, 1, 2, \dots$, and in the component intervals it has a constant derivative $g_t'(x) = \frac{(a-1)(\gamma+t(\delta-\gamma))}{(\delta-\gamma)(1+t(a-1))}$ for $x \in \frac{(\gamma,\delta)}{a^{i+1}(\gamma+t(\delta-\gamma))}$, $i = 0, 1, 2, \dots$, and $x \in \left(\frac{\gamma}{\gamma+t(\delta-\gamma)}, 1\right)$.[4]

3^0. The graph of every $g \in G(X_n)$ lies in the intervals

$$[1/a, 1] \times [1/a, 1] \cup \left[1/a^2, 1/a\right] \times \left[1/a^2, 1a/\right] \cup \dots,$$

and the graph of g in $\left[1/a^k, 1/a^{k-1}\right] \times \left[1/a^k, 1/a^{k-1}\right]$ is similar to the graph of g in $\left[1/a^{k+1}, 1/a^k\right] \times \left[1/a^{k+1}, 1/a^k\right]$ with coefficient a.

4^0. We have $g_0(x) = \overline{g}(x)$, $\underline{g}(x) \notin G(X_n)$, and the asymptotic densities $\underline{d}$, $\overline{d}$ are

$$\underline{d} = \frac{(\delta - \gamma)}{\gamma(a-1)}, \qquad \overline{d} = \frac{(\delta - \gamma)a}{\delta(a-1)}.$$

We can add the following new properties 5^0–8^0:

5^0. By definition (8) of the local asymptotic density d_g, along with (26) for $g(x) = g_t(x)$ we get

$$\begin{aligned} d_{g_t} &= \lim_{k\to\infty} \frac{n_k}{x_{n_k}} = \lim_{k\to\infty} \frac{\sum_{i=0}^{k-1} a^i(\delta - \gamma) + ta^k(\delta - \gamma)}{a^k\gamma + ta^k(\delta - \gamma)} \\ &= \frac{(\delta - \gamma)(1 + t(a-1))}{(a-1)(\gamma + t(\delta - \gamma))}, \end{aligned} \tag{27}$$

for $t = 0$, $d_{g_0} = \underline{d}$, for $t = 1$, $d_{g_1} = \overline{d}$, and we have

$$g_t'(x) = \frac{1}{d_{g_t}} \tag{28}$$

for x in intervals where the derivative of $g_t(x)$ is constant.

[4] Here, as above, we write $(xz, yz) = (x, y)z$, and $(x/z, y/z) = (x, y)/z$.

6^0. For the function $h_{1,g}(x)$ defined in (14), putting $g(x) = g_t(x)$, we have

$$\frac{\underline{d}}{d_{g_t}} = \frac{\gamma + t(\delta-\gamma)}{\gamma(1+t(a-1))}, \quad \frac{1-d_{g_t}}{1-\underline{d}} = \frac{\gamma}{\gamma+t(\delta-\gamma)}, \quad \frac{\underline{d}}{d_{g_t}}\frac{1-d_{g_t}}{1-\underline{d}} = \frac{1}{1+t(a-1)}.$$

Then

$$h_{1,g_t}(x) = \begin{cases} g_t(x) = x\frac{1}{d_{g_t}} + 1 - \frac{1}{d_{g_t}}, & \text{for } x \in \left(\frac{\gamma}{\gamma+t(\delta-\gamma}, 1\right); \\ g_t(x) = \frac{1}{a^i(1+t(a-1))}, & \text{for } x = \frac{\gamma}{a^i(\gamma+t(\delta-\gamma))}, \\ & \qquad i = 0,1,2,\dots, \end{cases} \tag{29}$$

see Figure 4.

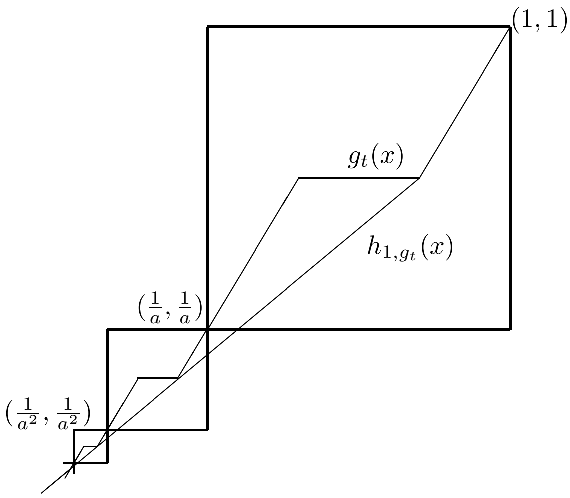

Figure 4: $g_t(x)$ and $h_{1,g_t}(x)$.

7^0. In the proof of the upper bound (18) we have proved that $1-\int_0^1 h_{1,g}(x)\mathrm{d}x$ is maximal for $d_g = \min(\sqrt{\underline{d}}, \overline{d})$. Let $t_0 \in [0,1]$ be such that $d_{g_{t_0}} = \min(\sqrt{\underline{d}}, \overline{d})$. This $t = t_0$ we shall find from (27) as

$$t = \frac{d_{g_t}(a-1)\gamma - (\delta-\gamma)}{(\delta-\gamma)(a-1)(1-d_{g_t})}. \tag{30}$$

8^0. Let $P(t)$ be the area in $\left[\frac{1}{a}, 1\right] \times \left[\frac{1}{a}, 1\right]$ bounded by the graph of $g_t(x)$. Then

$$\int_0^1 g_t(x)\mathrm{d}x = P(t)\frac{1}{1-\frac{1}{a^2}} + \frac{1}{a+1} = \frac{1}{2} + \frac{1}{2}.\frac{1}{(a+1)}.\frac{(\gamma a - \delta)}{(1+t(a-1))(\gamma+t(\delta-\gamma))}$$

$$+\frac{1}{2}.\frac{t(\delta-\gamma a)}{(1+t(a-1))(\gamma+t(\delta-\gamma))}, \tag{31}$$

and, since $g_0(x)=\overline{g}(x)$, $\max_{t\in[0,1]}\int_0^1 g_t(x)\mathrm{d}x$ is attained at $t=0$. Putting $P'(t)=0$ it follows that $\min_{t\in[0,1]}\int_0^1 g_t(x)\mathrm{d}x$ is attained at $t=1$. This can be derived also from the fact that for $x_{n+1}=x_n+1$,

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{x_i}{x_{n+1}}-\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}$$
$$=\frac{1}{n+1}-\left(\frac{1}{x_n+1}+\frac{1}{n+1}.\frac{1}{1+\frac{1}{x_n}}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}\right)>0,$$

and, because $c_1(x)\notin G(X_n)$, $\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}<1$. Denoting the index n_k for $x_{n_k}=[a^k\delta]$, the lim sup of $\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}$ is attained over $n=n_k$, $k=0,1,2,\dots$, and for such n_k (see (26)) we have $F(X_{n_k},x)\to g_1(x)$ for $x\in[0,1]$.

It follows, by (20), and (21) that

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}=1-\int_0^1 g_0(x)\mathrm{d}x=\frac{1}{2}-\frac{1}{2}.\frac{1}{(a+1)}\left(\frac{\gamma a-\delta}{\gamma}\right), \tag{32}$$
$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_n}=1-\int_0^1 g_1(x)\mathrm{d}x=\frac{1}{2}+\frac{1}{2}.\frac{1}{(a+1)}\left(\frac{\gamma a-\delta}{\delta}\right). \tag{33}$$

The upper bound in (18) coincides with the maximal value of $1-\int_0^1 h_{1,g}(x)\mathrm{d}x$ attained for $d_g=\min(\sqrt{\underline{d}},\overline{d})$. Since $1-\int_0^1 g_1(x)\mathrm{d}x$ is maximal for all $1-\int_0^1 g_t(x)\mathrm{d}x$, $t\in[0,1]$, and $1-\int_0^1 g_1(x)\mathrm{d}x\le 1-\int_0^1 h_{1,g_1}(x)\mathrm{d}x$, the upper bound (33) satisfies (18).

Using the explicit formulas (16) for asymptotic densities, we see again that (32), and (33) satisfy (17), and (18), respectively, in Theorem 7.

References

[DT] M. Drmota and R.F. Tichy, Sequences, Discrepancies and Applications, Lecture Notes in Mathematics 1651, *Springer-Verlag, Berlin, Heidelberg*, 1997.

[GS] G. Grekos and O. Strauch, Distribution functions of ratio sequences, II, *Unif. Distrib. Theory* **2**, no. 1 (2007), 53–77.

[KN] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, *John Wiley & Sons*, *New York*, 1974, reprint: Dover Publications, Inc. Mineola, New York, 2006.

[Sa] T. Šalát, On ratio sets of sets of natural numbers, *Acta Arith.* **XV** (1969), 273–278.

[SN] O. Strauch and R. Nair, Unsolved Problems, 2008, `http://udt.mat.savba.sk/`.

[SP] O. Strauch and Š. Porubský, Distribution of Sequences: A Sampler, *Peter Lang*, *Frankfurt am Main*, 2005.

[ST] O. Strauch and J.T. Tóth, Distribution functions of ratio sequences, *Publ. Math. Debrecen* **58** (2001), 751–778.

VLADIMÍR BALÁŽ
INSTITUTE OF INFORMATION, ENGINEERING
AUTOMATION AND MATHEMATICS
FACULTY OF CHEMICAL
AND FOOD TECHNOLOGY
SLOVAK UNIVERSITY OF TECHNOLOGY
IN BRATISLAVA
SLOVAKIA

E-mail: vladimir.balaz@stuba.sk

OTO STRAUCH
MATHEMATICAL INSTITUTE
SLOVAK ACADEMY OF SCIENCES
ŠTEFÁNIKOVA 49
SK-814 73 BRATISLAVA
SLOVAKIA

E-mail: strauch@mat.savba.sk

LADISLAV MIŠÍK
DEPARTMENT OF MATHEMATICS AND
CENTRE OF EXCELLENCE IT4INNOVATIONS
DIVISION UO - IRAFM
UNIVERSITY OF OSTRAVA
30. DUBNA 22
701 03 OSTRAVA 1
CZECH REPUBLIC

E-mail: ladislav.misik@osu.cz

JÁNOS T. TÓTH
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF J. SELYE
BRATISLAVSKÁ 3322
SK-945 01 KOMÁRNO
SLOVAKIA

E-mail: tothj@selyeuni.sk

(Received November 16, 2009; revised January 28, 2013)