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Browder spectra of upper triangular matrix linear relations

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Abstract. In this paper, we define a matrix linear relation and present some properties of this one. When $A \in \mathcal{BCR}(H)$ and $B \in \mathcal{BCR}(K)$ are given, we denote by M_C the matrix linear relation acting on the infinite dimensional separable Hilbert space $H \oplus K$, of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. It is shown that M_C is Browder relation for some operator $C \in \mathcal{B}(K, H)$ if and only if A is upper semi Fredholm relation with finite ascent, B is lower semi Fredholm relation with finite descent and $n(A) + n(B) = d(A) + d(B)$.

1. Introduction

Let H and K be two infinite dimensional normed spaces. A linear relation $A : H \longrightarrow K$ is a mapping from a subspace $D(A) \subset H$ called the domain of A, into the collection of nonempty subsets of K such that $A(\alpha x_1 + \beta x_2) = \alpha A(x_1) + \beta A(x_2)$ for all nonzero α, β scalars and $x_1, x_2 \in D(A)$. If A maps the points of its domain to singletons, then A is said to be a single valued linear operator or simply an operator. We denote the class of linear relation from H into K by $\mathcal{LR}(H, K)$ and abbreviate $\mathcal{LR}(H, H)$ to $\mathcal{LR}(H)$. $A \in \mathcal{LR}(H, K)$ is uniquely determined by its graph $G(A)$, which is defined by :

$$G(A) = \{(x, y) \in H \oplus K \text{ such that } x \in D(A) \text{ and } y \in Ax\}.$$

Let $A \in \mathcal{LR}(H)$. The inverse of A is a linear relation A^{-1} given by :

$$G(A^{-1}) = \{(y, x) \in K \oplus H \text{ such that } (x, y) \in G(A)\}.$$

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For A and $B \in \mathcal{LR}(H)$, the notation $A \subset B$ means that $G(A) \subset G(B)$. The linear relations $A+B$ and AB are defined respectively by :

$$G(A+B) = \{(x, y+z) \in H \oplus H \text{ such that } (x,y) \in G(A) \text{ and } (x,z) \in G(B)\}$$

and

$$G(AB) = \{(x,y) \in H \oplus H : \exists z \in H \text{ such that } (x,z) \in G(B) \text{ and } (z,y) \in G(A)\}.$$

The subspace $A^{-1}(0)$ is denoted by $N(A)$ and A is called injective if $N(A) = \{0\}$, that is, if A^{-1} is a single valued linear operator. The range of A is the subspace $R(A) := A(D(A))$ and A is called surjective if $R(A) = H$. We write $n(A) = \dim N(A)$, $d(A) = \dim H/R(A)$ and the index of A, $ind(A)$ is defined by $ind(A) = n(A) - d(A)$ provided $n(A)$ and $d(A)$ are not both infinite. The ascent, $asc(A)$ and the descent, $des(A)$ of A are given respectively by $asc(A) = \inf\{n \geq 0$ such that $N(A^n) = N(A^{n+1})\}$ and $des(A) = \inf\{n \geq 0$ such that $R(A^n) = R(A^{n+1})\}$.

The singular chain manifold noted $R_c(A)$ is defined by $R_c(A) = R_0(A) \cap R_\infty(A)$, where $R_0(A) = \bigcup_{i=1}^{\infty} N(A^i)$ and $R_\infty(A) = \bigcup_{i=1}^{\infty} A^i(0)$. The linear space $R_c(A)$ is non trivial if and only if there exists a number $s \in \mathbb{N}$ and elements $x_i \in H$, $1 \leq i \leq s$, not all equal to zero, such that

$$(0, x_1), (x_1, x_2), \ldots (x_{s-1}, x_s), (x_s, 0) \in G(A).$$

Let Q_A denote the quotient map from H onto $H/\overline{A(0)}$. Clearly $Q_A A$ is a single valued operator and the norm of A is defined by $\|A\| := \|Q_A A\|$. We say that A is closed if its graph is a closed subspace, continuous if for each neighbourhood V in $R(A)$, $A^{-1}(V)$ is a neighbourhood in $D(A)$ (equivalently $\|A\| < \infty$), open if its inverse is continuous. The resolvent set of a linear relation A is the set given by :

$$\rho(A) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \text{ is injective, open and has dense range}\}.$$

We denote the set of all closed linear relations on H by $\mathcal{CR}(H)$. Continuous defined everywhere linear relation on H are referred to as bounded linear relation. The class of such relation is denoted by $\mathcal{BR}(H)$.

Let $A \in \mathcal{CR}(H)$. We say that A is upper semi Fredholm linear relation if it has finite dimensional null space and closed range, A is lower semi Fredholm linear relation if its range is closed and has a finite codimensional and A is Fredholm linear relation if it is both upper and lower semi Fredholm linear relation. The set of upper and lower semi Fredholm linear relations is denoted respectively by :

$$\phi_+(H) = \{A \in \mathcal{CR}(H) \text{ such that } R(A) \text{ is closed and } n(A) \text{ is finite}\},$$

$$\phi_-(H) = \{A \in \mathcal{CR}(H) \text{ such that } R(A) \text{ is closed and } d(A) \text{ is finite}\}.$$

A closed linear relation $A \in \mathcal{CR}(H)$ is called Weyl if it is Fredholm of index zero and is called Browder if it is Fredholm of index zero and has finite ascent and descent [1]. Let $\phi_+^-(H)$ be the class of all $A \in \phi_+(H)$ with $ind(A) \leq 0$. If $A \in \mathcal{CR}(H)$, then the Browder spectrum $\sigma_b(A)$ and the Browder essential approximate point spectrum $\sigma_{ab}(A)$ of A are defined respectively by:

$$\sigma_b(A) = \{\lambda \in \mathbb{C} \text{ such that } A - \lambda I \text{ is not Browder}\}$$

and

$$\sigma_{ab}(A) = \{\lambda \in \mathbb{C} \text{ such that } A - \lambda I \notin \phi_+^-(H) \text{ or } asc(A - \lambda I) = \infty\}.$$

It is well known that $\lambda \notin \sigma_{ab}(A)$ if and only if $A - \lambda I$ is upper semi-Fredholm of finite ascent. The set of all bounded and closed linear relations on H is denoted by $\mathcal{BCR}(H)$.
Recently, many authors have paid much attention to 2×2 upper triangular operators matrices [4], [9], [19]. For $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$ and $C \in \mathcal{B}(K, H)$, let M_C denote the upper triangular operator matrix

$$M_C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

A study of the spectrum, the Browder and Weyl spectra and the Browder and Weyl's theorems for the operators M_C, and the related diagonal $M_0 = A \oplus B$, has been carried by a number of authors in the recent past (see [2], [5], [13], [19]). This motivated us to consider the linear relations instead of operators and extend some of the extant results.

The purpose of the present paper is to extend some results given by X. Cao [2] in the context of linear relations. For a given pair (A, B) of operators, X. Cao gives necessary and sufficient conditions for which M_C is Browder for some linear operator C. Throughout this note, we will extend this result for entries linear relations. For this we must first define the notion of matrix relation, and review the basic rules of the product of matrix linear relations that we need to know to define the adjoint and the index matrices. In order to give a similar characterization of X. Cao, we need an analogue of the results of Taylor [18] and Kaashoek [11], [12] given for linear operators, in the case of linear relations.

Our paper is organized as follows: Section 2 contains technical results needed in the proof of Theorem 22 and Theorem 24. In section 3, we give a definition of matrix linear relation, then we collect some useful calculus formulas that often appear next. The second part of this section deals with triangular matrix linear relation. When A and B are two linear relations, we introduce the matrix linear relation M_C where C is a linear operator and we investigate some properties concerning index, singular chain manifold and adjoint of M_C. In section 4, we provide the main results. We recall that for A and B two linear operators, M_C is Browder for some linear operator C if and only if A is upper semi Fredholm with finite ascent, B is lower semi Fredholm with finite descent and $n(A)+n(B) = d(A)+d(B)$. We will show that this result, given by [2], can be extended to bounded and closed linear relations which have a trivial singular chain manifold, see for instance Theorem 27. This theorem will be used next to obtain a characterization of Browder spectrum of a matrix M_C according to the Browder essential approximate point spectrum of A and B^*, where B^* is the adjoint of B.

2. Auxiliary results

The goal of this section is to establish some results which we use in the next section. We begin by giving some auxiliary results from the theory of linear relations in Banach space.

Lemma 1. *Let H be a Banach space, $A \in \mathcal{LR}(H)$ and $B \in \mathcal{LR}(H)$.*

$$\text{If } A \in \phi(H) \text{ and } B \in \phi(H), \text{ then } AB \in \phi(H).$$

Proof. From [6], we know that if $A \in \phi_+(H)$ and $B \in \phi_+(H)$, then $AB \in \phi_+(H)$. On the other hand, from [17, Lemma 5.1], we have $d(AB) \le d(A) + d(B) < \infty$. Thus, $AB \in \phi(H)$. □

For $n \in \mathbb{N}^*$ the identity $A^{n*} = A^{*n}$, has already been proved in the densely defined operators case (see [14, Theorem 4.2]), where the statement made sense. Next we will extend this result to the multivalued case. This suggest the following definitions.

Definition 2. Let H be a Banach space and $A \in \mathcal{LR}(H)$. The adjoint of the linear relation A, noted A^* is defined by its graph given by :

$$G(A^*) = G(-A^{-1})^{\perp},$$

that is, $(y', x') \in G(A^*)$ if and only if, for all $(x, y) \in G(A)$, $y'y - x'x = 0$.

Definition 3. Let H be a Banach space, M a subspace of H and $A \in \mathcal{LR}(H)$. We say that M is a core of A if $G(A) \subset \overline{G(A_{|M})}$, where $G(A_{|M}) = \{(x, y) \in G(A)$ such that $x \in M\}$.

Definition 4. Let X, Y be two Banach spaces, T and $S \in \mathcal{LR}(X, Y)$. Then, T is said to be S-co-continuous if there exist constants $\alpha, \beta > 0$ such that

$$TB_X \subset \alpha SB_X + \beta B_Y + T(0),$$

where B_X, and B_Y designate respectively the closed unit ball in X and Y.

Remark 5. If T is a continuous linear relation, we note that there is a $\beta > 0$ with

$$TB_X \subset \beta B_Y + T(0).$$

(see for instance [4]). From this we can prove that : if $T \in \mathcal{LR}(X, Y)$ is continuous, then T is S-co-continuous for all $S \in \mathcal{LR}(X, Y)$.

Definition 6. Let X, Y be two Banach spaces, T and $S \in \mathcal{LR}(X, Y)$. Then, T is said to be S-bounded if $D(S) \subset D(T)$ and there exist constants $\alpha, \beta > 0$ such that for all $x \in D(S)$ we have

$$\|Tx\| \leq \alpha\|Sx\| + \beta\|x\|$$

where $\|Tx\| = d(Tx, T(0)) = d(Tx, 0)$ (see [4, Proposition II.1.4]).

Lemma 7 ([10, Theorem 2.10])**.** *Let X, Y be two Banach spaces, T and $S \in \mathcal{LR}(X, Y)$.*

(i) *If T is S-co-continuous, then there are $\alpha, \beta > 0$ with*

$$\|y'T\| \leq \alpha\|y'S\| + \beta\|y'\| \qquad \text{for all } y' \in D(S^*).$$

(ii) *If furthermore $T(0) \subset S(0)$, then T^* is S^*-bounded.*

Lemma 8 ([10, Theorem 3.1])**.** *Let X, Y be two Banach spaces, T and $S \in \mathcal{LR}(X, Y)$. Suppose that $D(TS)$ is a core of S. Then, $(TS)^* = S^*T^*$ if and only if T^* is $(TS)^*$-bounded.*

We are now ready to state our result.

Lemma 9. *Let H be a Banach space and $A \in \mathcal{LR}(H)$.*

$$\textit{If } A \textit{ is bounded then, for all } n \in \mathbb{N}^*,\ A^{n*} = A^{*n}.$$

Proof. Let A be a bounded linear relation. Since A is continuous, then A is A^2-co-continuous. Furthermore $A(0) \subset A^2(0)$, then using Lemma 7, A^* is A^{2^*}-bounded. On the other hand, since $\overline{D(A)} = H$, then by [15, Corollary 3.6], we have $D(A^2) = H$. Then, $G(A) \subseteq \overline{G(A_{|D(A^2)})}$. This implies that $D(A^2)$ is a core of A. From Lemma 8, we have $A^{2*} = A^{*2}$. By induction we get the result for all $n \in \mathbb{N}^*$. $\square$

Proposition 10. *Let H be a Banach space and $A \in \mathcal{BCR}(H)$.*

(i) *If $A \in \phi_+(H)$, then $asc(A) = des(A^*)$ and $des(A) = asc(A^*)$.*

(ii) *If $A \in \phi_-(H)$ and $\rho(A) \neq \emptyset$, then $asc(A) = des(A^*)$ and $des(A) = asc(A^*)$.*

Proof. (i) Suppose that $A \in \phi_+(H)$. In [7], we have shown that if $A \in \phi_+(H)$, then for all $n \in \mathbb{N}$, $A^n \in \phi_+(H)$. Using this result, we get that A^n and $R(A^n)$ are closed and $n(A^n) < \infty$. Let $asc(A) = p$. Then $N(A^p) = N(A^{p+1})$. Suppose that $p < \infty$. Since $A \in \phi_+(H)$, then $A^p \in \phi_+(H)$. This implies that A^p and $R(A^p)$ are closed. We get $R(A^{*p})^\perp = N(A^p) = N(A^{p+1}) = R(A^{*p+1})^\perp$. This induces that $R(A^{*p}) = R(A^{*p+1})$. Thus, $des(A^*) \leq asc(A)$. If $des(A^*) = q < p$. Then, $N(A^q) \subsetneq N(A^p)$. Since $p > q$, then $R(A^{*q}) = R(A^{*p})$. It follows that $R(A^{*q})^\perp = R(A^{*p})^\perp$. And so, $N(A^q) = N(A^p)$, which is absurd. Hence, $des(A^*) = p$. Suppose that $p = \infty$. Then for all $n \in \mathbb{N}$, $N(A^n) \subsetneq N(A^{n+1})$. If $des(A^*) = q < \infty$. Then $R(A^{*q}) = R(A^{*q+1})$. Thus, $R(A^{*q})^\perp = R(A^{*q+1})^\perp$. It follows that $N(A^q) = N(A^{q+1})$. This is in contradiction with the fact that $p = \infty$. Thus, $des(A^*) = \infty$. The remaining statements can be proved similarly.

(ii) Suppose that $A \in \phi_-(H)$ and $\rho(A) \neq \emptyset$. From [6, Proposition 3.1], we know that for all $n \in \mathbb{N}$, $A^n \in \phi_-(H)$. Thus, from the definition, A^n and $R(A^n)$ are closed and $d(A^n) < \infty$. Using the same technique in (i), we get the result. $\square$

Proposition 11. *Let H be a Banach space and $A \in \mathcal{BCR}(H)$ with $R_c(A) = \{0\}$.*

If A is a Weyl linear relation, such that $asc(A) < \infty$ or $des(A) < \infty$, then A is Browder and $asc(A) = des(A)$.

Proof. Using the definition of Weyl linear relation, we know that $A \in \phi(H)$ and $ind(A) = 0$. Suppose that $asc(A) < \infty$. We need only to prove that $des(A) < \infty$. Let $asc(A) = p$. Then $n(A^p) = n(A^{p+1})$. Since A is Weyl, then from [1, Proposition 8], for all $n \in \mathbb{N}$, A^n is Weyl. Hence $ind(A^p) = ind(A^{p+1}) = 0$. This implies that $n(A^p) = d(A^p)$ and $n(A^{p+1}) = d(A^{p+1})$. Hence $d(A^p) = d(A^{p+1})$, which means that $dim\ R(A^p)^\perp = dim\ R(A^{p+1})^\perp$. Therefore, $R(A^p)^\perp = R(A^{p+1})^\perp$. Moving to the orthogonal, we get that $R(A^p) = R(A^{p+1})$. So, $des(A) \leq p$. Thus, A is Browder and from [15, Theorem 6.11], we conclude that $asc(A) = des(A)$.

For the case $des(A) < \infty$, by Proposition 10, we have $asc(A^*) < \infty$. The same proof gives the result. $\square$

Lemma 12 ([15, Lemma 5.5])**.** *Let H be a Banach space, $A \in \mathcal{LR}(H)$ and $m \in \mathbb{N}^*$.*

1- *If $R_c(A) = \{0\}$ and $asc(A) \leq m$, then for all $n \in \mathbb{N}$ $N(A^n) \cap R(A^m) = \{0\}$.*

2- *If $N(A) \cap R(A^m) = \{0\}$, then $R_c(A) = \{0\}$ and $asc(A) \leq m$.*

3. Properties of matrix linear relations

For H and K two Banach spaces, consider $A \in \mathcal{LR}(H)$, $B \in \mathcal{LR}(K)$, $C \in \mathcal{LR}(K,H)$ and $D \in \mathcal{LR}(H,K)$. We define the matrix linear relation acting on $H \oplus K$ of the form $\begin{pmatrix} A & C \\ D & B \end{pmatrix}$ by :

$$G\left(\begin{pmatrix} A & C \\ D & B \end{pmatrix}\right) = \left\{ \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in (H \oplus K)^2 \text{such that } x_1 \in D(A) \cap D(D), \right.$$
$$\left. x_2 \in D(C) \cap D(B), y_1 \in Ax_1 + Cx_2 \text{and } y_2 \in Dx_1 + Bx_2 \right\}.$$

3.1. Matrix Product. In this subsection, we investigate some properties of the product of matrix linear relations.

We begin by the simple case, the product of diagonal matrix linear relations.

Lemma 13. *Let H and K be two Banach spaces, $A, A' \in \mathcal{LR}(H)$ and $B, B' \in \mathcal{LR}(K)$. Then*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ 0 & BB' \end{pmatrix}.$$

Proof. Let

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}\right).$$

This means that there exists $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in H \oplus K$, such that

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}\right) \quad \text{and} \quad \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right).$$

We get $z_1 \in A'x_1$, $z_2 \in B'x_2$, $y_1 \in Az_1$ and $y_2 \in Bz_2$. Thus, $y_1 \in AA'x_1$ and $y_2 \in BB'x_2$. This is equivalent to :

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} AA' & 0 \\ 0 & BB' \end{pmatrix}\right).$$

For the converse inclusion, we use the same technique and we get the result. $\square$

Proposition 14. *Let H and K be two Banach spaces, $A, A' \in \mathcal{LR}(H)$, $B, B' \in \mathcal{LR}(K)$, $C, C' \in \mathcal{LR}(K, H)$ and $D, D' \in \mathcal{LR}(H, K)$. Then*

(i)

$$\begin{pmatrix} A & C \\ D & B \end{pmatrix}\begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix} \subseteq \begin{pmatrix} AA' + CD' & AC' + CB' \\ DA' + BD' & DC' + BB' \end{pmatrix}.$$

(ii) *Moreover, if A', B', C' and D' are single valued or $D = D' = 0$ and B' is single valued, then we have the equality.*

Proof. (i) Let

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A & C \\ D & B \end{pmatrix}\begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix}\right).$$

Then there exists $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in H \oplus K$, such that

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix}\right) \text{ and } \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A & C \\ D & B \end{pmatrix}\right).$$

This implies that $z_1 \in A'x_1 + C'x_2$, $z_2 \in D'x_1 + B'x_2$, $y_1 \in Az_1 + Cz_2$ and $y_2 \in Dz_1 + Bz_2$. Thus, $y_1 \in A(A'x_1 + C'x_2) + C(D'x_1 + B'x_2) = (AA' + CD')x_1 + (AC' + CB')x_2$ and $y_2 \in D(A'x_1 + C'x_2) + B(D'x_1 + B'x_2) = (DA' + BD')x_1 + (DC' + BB')x_2$. It follows that

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} AA' + CD' & AC' + CB' \\ DA' + BD' & DC' + BB' \end{pmatrix}\right).$$

(ii) Suppose that A', B', C' and D' are single valued. Let

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} AA' + CD' & AC' + CB' \\ DA' + BD' & DC' + BB' \end{pmatrix}\right).$$

Then $y_1 \in (AA'+CD')x_1+(AC'+CB')x_2 = A(A'x_1+C'x_2)+C(D'x_1+B'x_2)$ and $y_2 \in (DA'+BD')x_1+(DC'+BB')x_2 = D(A'x_1+C'x_2)+B(D'x_1+B'x_2)$. Let $z_1 = A'x_1 + C'x_2$ and $z_2 = D'x_1 + B'x_2$. We get $y_1 \in Az_1 + Cz_2$ and $y_2 \in Dz_1 + Bz_2$. Thus,

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A & C \\ D & B \end{pmatrix}\begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix}\right).$$

Now, suppose that $D = D' = 0$ and B' is single valued. Let

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} AA' & AC'+CB' \\ 0 & BB' \end{pmatrix}\right).$$

Then $y_1 \in AA'x_1 + (AC'+CB')x_2 = A(A'x_1+C'x_2)+CB'x_2$ and $y_2 \in BB'x_2$. It follows that there exists $z_1 \in A'x_1 + C'x_2$, such that $y_1 \in Az_1 + CB'x_2$. Let $z_2 = B'x_2$, then

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A' & C' \\ 0 & B' \end{pmatrix}\right).$$

This completes the proof. $\square$

3.2. Some properties of triangular linear relations. Let H and K be two Banach spaces. When $A \in \mathcal{LR}(H)$, $B \in \mathcal{LR}(K)$ and $C \in \mathcal{LR}(K,H)$ are given, we denote by M_C the matrix linear relation acting on $H \oplus K$ of the form :

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Further, as an application of Proposition 14, we have :

1- For all $p \in \mathbb{N}^*$,

$$M_C^p \subseteq \begin{pmatrix} A^p & \sum_{k=0}^{p-1} A^{p-1-k}CB^k \\ 0 & B^p \end{pmatrix}.$$

2- The matrix linear relation M_C, admits the following decomposition :

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{3.1}$$

3- Using the decomposition of M_C and Lemma 1, we can prove that : if $A \in \phi(H)$ and $B \in \phi(K)$, then for all $C \in \mathcal{B}(K,H)$, $M_C \in \phi(H \oplus K)$.

Index of M_C. It is well known that if A, B and C are linear operators with finite indices, we have $ind(M_C) = ind(A) + ind(B)$. We will show that this property remains valid in the case of linear relations. For this, we recall the index theorem for the product of linear relation needed below.

Lemma 15 ([3, Corollary 3.2]). *Let X, Y, Z be three vector spaces, $T \in \mathcal{LR}(X,Y)$, $S \in \mathcal{LR}(Y,Z)$, $D(S) = Y$ and suppose that T and S have finite indices. Then, ST has a finite index and :*

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T) - \dim(T(0) \cap S^{-1}(0)).$$

Theorem 16. *Let H and K be two Banach spaces, $A \in \phi(H)$ and $B \in \phi(K)$. Then, for all $C \in \mathcal{B}(K,H)$, $ind(M_C) = ind(A) + ind(B)$.*

Proof. Since $A \in \phi(H)$ and $B \in \phi(K)$, then $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \in \phi(H \oplus K)$ and $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in \phi(H \oplus K)$. The use of Lemma 15 leads to :

$$\begin{aligned}\operatorname{ind}(M_C) = {} & \operatorname{ind}\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right) + \operatorname{ind}\left(\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right) \\ & - \dim\left(\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cap \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).\end{aligned}$$

We claim that $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ 0 & B^{-1} \end{pmatrix}$. Let

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix}\right) \in G\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^{-1}\right).$$

This is equivalent to

$$\left(\begin{pmatrix} z \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \in G\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right).$$

This means that $x = z$ and $y \in Bt$. It follows that $x = z$ and $t \in B^{-1}y$. Thus,

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ t \end{pmatrix}\right) \in G\left(\begin{pmatrix} I & 0 \\ 0 & B^{-1} \end{pmatrix}\right).$$

We note that

$$\begin{aligned}
&\dim\left(\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cap \begin{pmatrix} I & 0 \\ 0 & B^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \\
&\quad = \dim\left(\begin{pmatrix} A(0) \\ 0 \end{pmatrix} \cap \begin{pmatrix} 0 \\ B^{-1}(0) \end{pmatrix}\right) = 0.
\end{aligned}$$

It follows that

$$\operatorname{ind}(M_C) = \operatorname{ind}\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right) + \operatorname{ind}\left(\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right).$$

Applying Lemma 15 again, we get :

$$\begin{aligned}
\operatorname{ind}(M_C) &= \operatorname{ind}(B) + \operatorname{ind}\left(\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\right) + \operatorname{ind}\left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right) \\
&\quad - \dim\left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cap \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \\
&= \operatorname{ind}(B) + \operatorname{ind}(A) - \dim\left(\begin{pmatrix} A(0) \\ 0 \end{pmatrix} \cap \begin{pmatrix} -C(0) \\ 0 \end{pmatrix}\right) = \operatorname{ind}(B) + \operatorname{ind}(A).
\end{aligned}$$

This achieves the proof. □

Singular chain manifold of M_C. Let C be a bounded linear operator. In the following theorem, we give some sufficient conditions on the relations A and B to have $R_c(M_C) = \{0\}$.

Theorem 17. *Let H and K be two Banach spaces, $A \in \mathcal{LR}(H)$ and $B \in \mathcal{LR}(K)$.*
If $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$, then for all $C \in \mathcal{B}(K, H)$, $R_c(M_C) = \{0\}$.

Proof. Let $C \in \mathcal{B}(K, H)$. Suppose that $R_c(M_C) \neq \{0\}$. Then, there exists a chain $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in H \oplus K$, $1 \leq i \leq n$, such that :

$$\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right), \left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right), \ldots, \left(\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \in G(M_C).$$

Let $x_0 = y_0 = x_{n+1} = y_{n+1} = 0$. Then for all $1 \leq i \leq n$, $\left(\begin{pmatrix} x_i \\ y_i \end{pmatrix}, \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}\right) \in G(M_C)$. Thus, $x_{i+1} \in Ax_i + Cy_i$ and $(y_i, y_{i+1}) \in G(B)$. We have construct a chain y_i, $1 \leq i \leq n$, such that $(0, y_1), (y_1, y_2), ..., (y_n, 0) \in G(B)$. Since $R_c(B) = \{0\}$, then, for all $1 \leq i \leq n$, $y_i = 0$. On the other hand, C is an operator. It follows that $(0, x_1), (x_1, x_2), ..., (x_n, 0) \in G(A)$. Since $R_c(A) = \{0\}$, then for all $1 \leq i \leq n$, $x_i = 0$, which is absurd. Consequently $R_c(M_C) = \{0\}$. □

Proposition 18. *Let H, K be two Banach spaces, $A \in \mathcal{LR}(H)$ and $B \in \mathcal{LR}(K)$, with $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$.*

$$\text{If } \operatorname{asc}(M_C) < \infty \text{ for some } C \in \mathcal{B}(K, H), \text{ then } \operatorname{asc}(A) < \infty.$$

Proof. Since $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$, then by Theorem 17, $R_c(M_C) = \{0\}$. Let $asc(M_C) = p$. Then, $N(M_C) \cap R(M_C^p) = \{0\}$. Using Lemma 12, we need only to prove that $N(A) \cap R(A^p) = \{0\}$. Let $x \in N(A) \cap R(A^p)$. Then, $0 \in Ax$ and there exists $y \in H$, such that $x \in A^p y$. It is easy to see that $\begin{pmatrix} x \\ 0 \end{pmatrix} \in N(M_C)$. Since $x \in A^p y$, then

$$\begin{pmatrix} x \\ 0 \end{pmatrix} \in \begin{pmatrix} A^p & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^p \begin{pmatrix} y \\ 0 \end{pmatrix} \subseteq M_C^p \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

It follows that $\begin{pmatrix} x \\ 0 \end{pmatrix} \in R(M_C^p)$. Thus, $\begin{pmatrix} x \\ 0 \end{pmatrix} \in N(M_C) \cap R(M_C^p) = \{0\}$. Hence, $x = 0$. Consequently $asc(A) < \infty$. □

Adjoint of M_C. In general, the product of adjoint linear relations is not the adjoint of the product. Cross [4, Theorem III.1.6] and Jaftha [10, Theorem 3.1] have shown that under some conditions, we can get the above equality. In this note, we will use these characterizations to study the adjoint of triangular matrix linear relation.

Theorem 19. *Let H, K be two Banach spaces, $A \in \mathcal{BCR}(H)$, $B \in \mathcal{BCR}(K)$ and $C \in \mathcal{B}(K, H)$. Then, the adjoint of M_C is :*

$$M_C^* = \begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix}.$$

PROOF. Using equation (3.1) and moving to the adjoint, we get :

$$M_C^* = \left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right)^*.$$

Since $\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}$ is bounded, then it is continuous. Using Remark 5, we get $\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}$ is $\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$-co-continuous. However, from [16] we know that: $\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} I & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. So by Lemma 7 we have $\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}^*$ is $\left(\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right)^*$-bounded. On the other hand, it is easy to see that $D\left(\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right)$ is a core of $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$. Indeed, Proposition 3.1 (iii) leads to $D\left(\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right) = D\left(\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\right) = H \times K$ since A, B and C are bounded. Further, $D\left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\right) = H \times K$ since A is bounded. So using Definition 3 of the core, we get the result.

We obtain that the conditions of Lemma 8 are satisfied, then $M_C^* = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^*\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}^*$. Using the same technique to $\begin{pmatrix} I & C \\ 0 & B \end{pmatrix}^*$, it follows that

$$M_C^* = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^*\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}^*\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^* = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^*\begin{pmatrix} I & 0 \\ C^* & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^*.$$

We need only to prove that $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^* = \begin{pmatrix} I & 0 \\ 0 & B^* \end{pmatrix}$ and by analogy on $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^*$ we get the result. Let $\left(\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}, \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}\right) \in G\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^*\right)$. So, for all $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \in G\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$, we have $y_1'y_1 = x_1'x_1$ and $y_2'y_2 = x_2'x_2$. Since $y_1 = x_1$ and $y_2 \in Bx_2$, then $y_1' = x_1'$ and for all $(x_2, y_2) \in G(B)$, $y_2'y_2 = x_2'x_2$.

This is equivalent to $\left(\begin{pmatrix} y_1' \\ y_2' \end{pmatrix}, \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}\right) \in G\left(\begin{pmatrix} I & 0 \\ 0 & B^* \end{pmatrix}\right)$. Thus, $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}^* = \begin{pmatrix} I & 0 \\ 0 & B^* \end{pmatrix}$. This achieves the proof. $\square$

4. Main results

In this section H and K are two infinite dimensional separable Hilbert spaces. Let $A \in \mathcal{BCR}(H)$ and $B \in \mathcal{BCR}(K)$ with $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$. We recall that for $C \in \mathcal{B}(K, H)$, we denote by M_C the matrix linear relation :

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

The aim of this section is to give some necessary and sufficient conditions on A and B for which there exists an operator C such that M_C is a Browder matrix linear relation.

Lemma 20. *Let H and K be two infinite dimensional separable Hilbert spaces, $A \in \mathcal{LR}(H)$ and $B \in \mathcal{LR}(K)$. Suppose $A \in \phi_+(H)$, $B \in \phi_-(K)$ and $d(A) = n(B) = \infty$. Then for all $p \in \mathbb{N}$, there exists an isometry :*

$$T : (R(B^*) + N(B^{*p}))^{\perp} \longrightarrow (R(A) + N(A^p))^{\perp}.$$

Proof. Let $p \in \mathbb{N}$. We claim that $\dim(R(A) + N(A^p))^{\perp} = \infty$. Since $A \in \phi_+(H)$, then $A^p \in \phi_+(H)$. And so, $n(A^p) < \infty$. We also have $R(A)$ is closed, which implies that $(R(A) + N(A^p))$ is closed. Then

$$H = (R(A) + N(A^p)) \oplus (R(A) + N(A^p))^{\perp}.$$

Suppose that $\dim(R(A) + N(A^p))^{\perp} < \infty$. Then $codim(R(A)) = d(A) < \infty$, which is a contradiction. Hence, $\dim(R(A) + N(A^p))^{\perp} = \infty$. Using the same proof, we can show that $\dim(R(B^*) + N(B^{*p}))^{\perp} = \infty$. $(R(B^*) + N(B^{*p}))^{\perp}$ and $(R(A) + N(A^p))^{\perp}$ are two closed subspaces acting on H and K respectively. It follows that $(R(B^*)+N(B^{*p}))^{\perp}$ and $(R(A)+N(A^p))^{\perp}$ are two infinite dimensional separable Hilbert spaces. So, there exists an isometry :

$$T : (R(B^*) + N(B^{*p}))^{\perp} \longrightarrow (R(A) + N(A^p))^{\perp}. \qquad \square$$

Proposition 21. *Let H and K be two infinite dimensional separable Hilbert spaces, $p \in \mathbb{N}$, $A \in \mathcal{LR}(H)$ and $B \in \mathcal{BR}(K)$ with $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$. Suppose $A \in \phi_+(H)$ with $asc(A) < p$, $B \in \phi_-(K)$ and $d(A) = n(B) = \infty$. Then, there exists $C \in \mathcal{B}(K, H)$ such that $asc(M_C) \leq p$.*

Proof. Using Lemma 20, we define the single valued linear operator C by :

$$C = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} (R(B^*) + N(B^{*p}))^{\perp} \\ R(B^*) + N(B^{*p}) \end{pmatrix} \longrightarrow \begin{pmatrix} (R(A) + N(A^p))^{\perp} \\ R(A) + N(A^p) \end{pmatrix}. \tag{4.1}$$

From Lemma 12, it suffices to show that $N(M_C) \cap R(M_C^p) = \{0\}$.
Let $\begin{pmatrix} x \\ y \end{pmatrix} \in N(M_C) \cap R(M_C^p)$. Then, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in M_C \begin{pmatrix} x \\ y \end{pmatrix}$ and there exists $\begin{pmatrix} z \\ t \end{pmatrix} \in H \oplus K$ such that :

$$\begin{pmatrix} x \\ y \end{pmatrix} \in M_C^p \begin{pmatrix} z \\ t \end{pmatrix} \subseteq \begin{pmatrix} A^p & \sum_{k=0}^{p-1} A^{p-1-k} C B^k \\ 0 & B^p \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}.$$

This induces that $0 \in Ax + Cy$, $0 \in By$, $x \in A^p z + A^{p-1}Ct + \ldots + CB^{p-1}t$ and $y \in B^p t$. Thus, $y \in N(B) \cap R(B^p) \subseteq (R(B^*) + N(B^{*p}))^{\perp}$ and $-Cy \in Ax \subseteq R(A) \cap (R(A) + N(A^p))^{\perp} \subseteq R(A) + R(A)^{\perp} = \{0\}$. Since $Cy = Ty = 0$, then $y = 0$ and $x \in N(A)$. We know that $x \in A^p z + A^{p-1}Ct + \ldots + CB^{p-1}t$, then there exists $z_1 \in A^p z + A^{p-1}Ct + \ldots + ACB^{p-2}t \subseteq R(A)$ such that :

$$\begin{aligned} x - z_1 &\in CB^{p-1}t \subseteq (N(A) + R(A)) \cap (N(A^p) + R(A))^{\perp}. \\ &\in (N(A^p) + R(A)) \cap (N(A^p) + R(A))^{\perp} = \{0\}. \end{aligned}$$

Thus,

$$\begin{aligned} x = z_1 &\in A^p z + A^{p-1}Ct + \ldots + ACB^{p-2}t. \\ &\in A(A^{p-1}z + A^{p-2}Ct + \ldots + ACB^{p-3}t + CB^{p-2}t). \end{aligned}$$

It follows that there exists $z_2 \in A^{p-1}z + A^{p-2}Ct + \ldots + ACB^{p-3}t + CB^{p-2}t$, such that $x \in Az_2$. Since $0 \in Ax \subseteq A^2 z_2$, then $z_2 \in N(A^2)$. Let $z_3 \in A^{p-1}z + A^{p-2}Ct + \ldots + ACB^{p-3}t$, such that $z_2 - z_3 \in CB^{p-2}t$. Then

$$\begin{aligned} z_2 - z_3 &\in CB^{p-2}t \subseteq (N(A^2) + R(A)) \cap (N(A^p) + R(A))^{\perp}. \\ &\in (N(A^p) + R(A)) \cap (N(A^p) + R(A))^{\perp} = \{0\}. \end{aligned}$$

Thus, $z_2 = z_3 \in R(A)$. From the fact that $x \in Az_2$, we have $x \in R(A^2)$. Continue this process, we get that $z_{2p-2} \in A^2z + ACt + CBt$, with $x \in A^{p-1}z_{2p-2}$. Since $0 \in Ax \subseteq A^2z_2 \subseteq A^3z_4 \subseteq \ldots \subseteq A^pz_{2p-2}$, then $z_{2p-2} \in N(A^p)$. Let $z_{2p-1} \in A^2z + ACt$, such that $z_{2p-2} - z_{2p-1} \in CBt$. Then

$$z_{2p-2} - z_{2p-1} \in CBt \subseteq (N(A^p) + R(A)) \cap (N(A^p) + R(A))^{\perp} = \{0\}.$$

Thus, $z_{2p-2} = z_{2p-1} \in R(A)$. Since $x \in A^{p-1}z_{2p-1}$, then $x \in R(A^p)$. However, $R_c(A) = \{0\}$ and $asc(A) \leq p$, then $x \in N(A) \cap R(A^p) = \{0\}$. And so, $x = y = 0$. This induces that $N(M_C) \cap R(M_C^p) = \{0\}$. Consequently $asc(M_C) \leq p$. $\square$

Theorem 22. *Let H and K be two infinite dimensional separable Hilbert spaces, $A \in \mathcal{BR}(H)$ and $B \in \mathcal{BR}(K)$ with $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$. Suppose $A \in \phi_+(H)$ with $asc(A) < \infty$ and $B \in \phi_-(K)$ with $des(B) < \infty$. If $d(A) = n(B) = \infty$, then there exists $C \in \mathcal{B}(K, H)$ such that M_C is Browder and $asc(M_C) = des(M_C)$.*

Proof. Let $p = \max(asc(A), des(B))$. Using Lemma 20, we define C by (4.1) :$C = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} (R(B^*) + N(B^{*p}))^{\perp} \\ R(B^*) + N(B^{*p}) \end{pmatrix} \longrightarrow \begin{pmatrix} (R(A) + N(A^p))^{\perp} \\ R(A) + N(A^p) \end{pmatrix}$ Then, we claim that M_C is Browder. We divide the proof into five steps.

Step 1: $n(M_C) < \infty$.

Let $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in N(M_C)$, then $0 \in Ax_0 + Cy_0$ and $0 \in By_0$. Thus,

$$\begin{aligned} -Cy_0 \in Ax_0 \in R(A) \cap (R(A) + N(A^p))^{\perp} &\in R(A) \cap R(A)^{\perp} \cap N(A^p)^{\perp} \\ &\in R(A) \cap R(A)^{\perp} = \{0\}. \end{aligned}$$

Hence, $0 \in Ax_0$ and $Cy_0 = 0$. Let $y_0 = y_1 + y_2$, where $y_1 \in (R(B^*) + N(B^{*p}))^{\perp}$ and $y_2 \in (R(B^*) + N(B^{*p}))$. Then, $Cy_0 = Ty_1 = 0$. This implies that $y_1 = 0$, which means that $y_0 \in (R(B^*)+N(B^{*p}))$. Thus, $y_0 \in (R(B^*)+N(B^{*p})) \cap N(B)$. We get that $N(M_C) \subseteq N(A) \oplus (R(B^*) + N(B^{*p})) \cap N(B)$. We claim that $dim(R(B^*) + N(B^{*p})) \cap N(B) < \infty$.

To the contrary, we assume that $(R(B^*)+N(B^{*p}))\cap N(B)$ is infinite dimensional. Let $(u_n)_{n=1}^{\infty}$ be the orthonormal sequence in $(R(B^*)+N(B^{*p}))\cap N(B)$ and write $u_n = w_n + v_n$, where $w_n \in R(B^*)$ and $v_n \in N(B^{*p})$. Then, $(v_n)_{n=1}^{\infty}$ is linear independent. In fact, for any $n \in \mathbb{N}$, if $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$, then $a_1(u_1 - w_1) + a_2(u_2 - w_2) + \cdots + a_n(u_n - w_n) = 0$. Hence, $a_1u_1 + a_2u_2 + ... + a_nu_n = a_1w_1 + a_2w_2 + \cdots + a_nw_n \in N(B) \cap R(B^*)$. Since $R(B^*) = N(B)^{\perp}$, it follows that $R(B^*) \cap N(B) = N(B)^\perp \cap N(B) = \{0\}$. Then, $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$. Thus, $a_1 = a_2 = \cdots = a_n = 0$, therefore $(v_n)_{i=1}^n \subseteq N(B^{*p})$ is linear independent for any $n \in \mathbb{N}$. This induces that $dimN(B^{*p}) = \infty$. It is in contradiction with the fact that $B \in \phi_-(K)$. From the preceding proof, we get that $n(M_C) \leq n(A) + dim(R(B^*) + N(B^{*p})) \cap N(B) < \infty$.

Step 2: $R(M_C)$ is closed.

Since $M_C \in \mathcal{BCR}(\mathcal{H} \oplus \mathcal{K})$, then by [8], M_C^* is a closed linear operator. Using the closed range theorem, we need only to prove that $R(M_C^*)$ is closed. Suppose that $M_C^* \begin{pmatrix} x_n \\ y_n \end{pmatrix} \longrightarrow \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$Then, $A^*x_n \longrightarrow u_0$ and $C^*x_n + B^*y_n \longrightarrow v_0$. Thus, $(A^*x_n)_{n=1}^\infty$, $(C^*x_n)_{n=1}^\infty$ and $(B^*y_n)_{n=1}^\infty$ are Cauchy sequences. Write $x_n = u_n + v_n$, where $u_n \in (R(A) + N(A^p))^\perp$ and $v_n \in (R(A) + N(A^p))$. Then, $C^*x_n = T^*u_n$, and so $(u_n)_{n=1}^\infty$ is a Cauchy sequence. Also $(A^*v_n)_{n=1}^\infty$ is a Cauchy sequence since $A^*x_n = A^*v_n$. Using the fact that $R(A)$ is closed and $dimN(A^p) < \infty$, we know that $R(A) + N(A^p)$ is closed. There is an orthogonal decomposition $(R(A) + N(A^p)) = N(A^*)^\perp \oplus M$ and $M \subseteq N(A^*)$. Let $v_n = w_n + z_n$ where $w_n \in N(A^*)^\perp$ and $z_n \in M \subseteq N(A^*)$. Then, $A^*v_n = A^*w_n$ and therefore, $(A^*w_n)_{n=1}^\infty$ is a Cauchy sequence. We claim that w_n is a Cauchy sequence. Indeed, since $A^*v_n = A^*w_n \longrightarrow u_0$ and $R(A^*)$ is closed, it follows that there exists $z_0 \in N(A^*)^\perp$ such that $A^*w_n \longrightarrow A^*z_0 = u_0$. So, $A^*(w_n - z_0) \longrightarrow 0$. Since $A^*_{|N(A^*)^\perp}$ is invertible, we get that $w_n - z_0 \longrightarrow 0$. Then, w_n is a Cauchy sequence. Let $x'_n = u_n + w_n$. Then, $(x'_n)_{n=1}^\infty$ is a Cauchy sequence. Suppose $x'_n \longrightarrow x_0$. Then, $C^*x_n = T^*u_n = C^*x'_n \longrightarrow C^*x_0$ and $A^*x_n = A^*x'_n \longrightarrow A^*x_0$. Suppose $B^*y_n \longrightarrow B^*y_0$. We get $M_C^* \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \longrightarrow \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$. This proves that $R(M_C^*)$ is closed. And so, $R(M_C)$ is closed.

Step 3: M_C has finite ascent and finite descent.
From the construction of C, we have by Proposition 21 that $asc(M_C) \leq p$. On the other hand, since $M_C \in \phi_+(H \oplus K)$, then using Proposition 10, we get $des(M_C) = asc(M_C^*)$. However, the entries A^*, B^* and C^* of M_C^* are operators, so the same technique adopting in the proof of [2, Theorem 2.1], shows that $des(M_C) \leq p$.

Step 4: $d(M_C) < \infty$.
We have $D(M_C) = H \oplus K$ and $des(M_C) \leq p$. This implies, from [15, Theorem 6.7], that $d(M_C) \leq n(M_C)$. Further $M_C \in \phi_+(H \oplus K)$, then $d(M_C) < \infty$.

Step 5: $ind(M_C) = 0$.
We have $d(M_C) \leq n(M_C)$, which means that $ind(M_C) \geq 0$.

Since $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$, then by Theorem 17, $R_c(M_C) = \{0\}$. However, since $asc(M_C) \leq p$ then from [15, Theorem 6.5], it follows that $ind(M_C) \leq 0$. This proves that $ind(M_C) = 0$. Consequently M_C is Browder. Moreover M_C is Browder, and $R_c(M_C) = \{0\}$, then using [15, Theorem 6.11], we get $asc(M_C) = des(M_C)$. □

Theorem 23. *Let H and K be two infinite dimensional separable Hilbert spaces, $A \in \mathcal{BR}(H)$ and $B \in \mathcal{BR}(K)$ with $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$. Suppose $A \in \phi(H)$ with $asc(A) < \infty$ and $B \in \phi(K)$ with $des(B) < \infty$. If $n(A) + n(B) = d(A) + d(B)$, then there exists $C \in \mathcal{B}(K, H)$ such that M_C is Browder.*

Proof. From the hypothesis, we know that M_C is Weyl for every $C \in \mathcal{B}(K, H)$. From Proposition 11, we need only to prove that there exists $C \in \mathcal{B}(K, H)$ such that $asc(M_C)$ or $des(M_C)$ is finite. Let $p = \max(asc(A), des(B))$. There are two cases to consider:

Case 1: Suppose that $\dim(R(B^*) + N(B^{*p}))^\perp \leq \dim(R(A) + N(A^p))^\perp$.
Let M be a closed subspace such that $M \subseteq (R(A) + N(A^p))^\perp$ and $\dim M = \dim(R(B^*) + N(B^{*p}))^\perp$. Then, there exists a linear operator T with domain $(R(B^*) + N(B^{*p}))^\perp$ and range M such that $\|Ty\| = \|y\|$ for every $y \in N(B) \cap R(B^p)$. Define an operator $C : K \longrightarrow H$ by :

$$C = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} (R(B^*) + N(B^{*p}))^\perp \\ R(B^*) + N(B^{*p} \end{pmatrix} \longrightarrow \begin{pmatrix} M \\ M^\perp \end{pmatrix}$$

Adopting the technique of the proof of Proposition 21, we get that M_C is Browder.
Case 2: Suppose that $\dim(R(B^*) + N(B^{*p}))^\perp \geq \dim(R(A) + N(A^p))^\perp$.
Let M be a closed subspace such that $M \subseteq (R(B^*) + N(B^{*p}))^\perp$ and $\dim M = \dim(R(A)+N(A^p))^\perp$. Then, there exists a linear operator T with domain $(R(A)+N(A^p))^\perp$ and range M such that $\|Ty\| = \|y\|$ for every $y \in (R(A) + N(A^p))^\perp$. Define an operator $C_1 : K \longrightarrow H$ by :

$$C_1 = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} (R(A) + N(A^p))^\perp \\ (R(A) + N(A^p)) \end{pmatrix} \longrightarrow \begin{pmatrix} M \\ M^\perp \end{pmatrix}.$$

Let $M_{C_1{}^*} = \begin{pmatrix} A & C_1{}^* \\ 0 & B \end{pmatrix}$. Then, the same technique used in the proof of [2, Theorem 2.1], shows that $M_{C_1{}^*}$ has finite descent. □

Theorem 24. *Let H and K be two infinite dimensional separable Hilbert spaces, $A \in \mathcal{BCR}(H)$ and $B \in \mathcal{BCR}(K)$ with $\rho(B) \neq \emptyset$, $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$. If M_C is Browder for some $C \in \mathcal{B}(K, H)$, then*

(i) *$A \in \phi_+(H)$ with $asc(A) < \infty$ and $B \in \phi_-(K)$ with $des(B) < \infty$.*

(ii) *$d(A) < \infty$ if and only if $n(B) < \infty$.*

Proof. Let $C \in \mathcal{B}(K, H)$ such that M_C is Browder.

(i) We claim that $A \in \phi_+(H)$ and $asc(A) < \infty$. Since $N(A) \oplus \{0\} \subseteq N(M_C)$, then $n(A) \leq n(M_C) < \infty$. From the decomposition of M_C^*, it follows that $R(M_C^*) \subseteq R\left(\begin{pmatrix} A^* & 0 \\ 0 & I \end{pmatrix}\right)$. Then, $d(M_C^*) \geq d\left(\begin{pmatrix} A^* & 0 \\ 0 & I \end{pmatrix}\right) = d(A^*)$. Thus $R(A)$ is closed and therefore $A \in \phi_+(H)$. Using Proposition 18, we get $asc(A) < \infty$. We claim that $B \in \phi_-(K)$ and $des(B) < \infty$. It follows from the decomposition of M_C, that $R(M_C) \subseteq R\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right)$. Thus, $d(M_C) \geq d\left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right) = d(B)$. Since M_C is Browder, then $d(B) < \infty$. And so, $B \in \phi_-(K)$. Since $\rho(B) \neq \emptyset$, then by Proposition 10, we have $des(B) = asc(B^*)$. Thus, to show that $des(B) < \infty$, we need only to prove that $asc(B^*) < \infty$. Since M_C is Browder, then M_C^* is also Browder. Then, there exists $p \in \mathbb{N}$ such that $asc(M_C^*) = p$. Suppose that $N(B^{*p}) \subsetneq N(B^{*p+1})$. This implies that there exists $y \in N(B^{*p+1})$ such that $B^{*p}y \neq 0$. However, for all $x \in H$ we have :

$$M^{*p+1}_C \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^{*p+1}x \\ C^*A^{*p+1}x + B^*C^*A^{*p}x + \ldots + B^{*p+1}y \end{pmatrix}.$$

For $x = 0$, we get $M^{*p+1}_C \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Thus, $\begin{pmatrix} 0 \\ y \end{pmatrix} \in N(M^{*p+1}_C) = N(M_C^{*p})$. It follows that $M_C^{*p} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ B^{*p}y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then, $B^{*p}y = 0$, which is absurd. Hence, $N(B^{*p}) = N(B^{*p+1})$. And so, $des(B) = asc(B^*) < \infty$.

(ii) We claim that $d(A) < \infty$ if and only if $n(B) < \infty$. We need only to prove the first implication, and by analogy we get the converse. If $n(B) = \infty$, then we must have that $d(A) = \infty$. To the contrary, we suppose that $d(A) < \infty$. There are two cases to consider.

Case 1: Suppose that $C(N(B))$ is finite dimensional. Then, $N(C)$ must contain an orthonormal sequence $y_n \in N(B)$. Thus, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in M_C \begin{pmatrix} 0 \\ y_n \end{pmatrix}$ for each

$n = 1, 2, \ldots$. This implies that $N(M_C)$ is infinite dimensional, which is a contradiction.

Case 2: Suppose that $C(N(B))$ is infinite dimensional. Since A is Fredholm, we know that $R(A)^{\perp}$ is finite dimensional. Therefore, $C(N(B)) \cap R(A)$ is infinite dimensional. Thus, we can find an orthonormal sequence $y_n \in N(B)$ for which there exists a sequence $x_n \in H$ such that $0 \in Ax_n - Cy_n$ for each $n = 1, 2, \ldots$. Then, $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in M_C \begin{pmatrix} x_n \\ -y_n \end{pmatrix}$ for each $n = 1, 2, \ldots$, which implies that $N(M_C)$ is infinite dimensional. It is in contradiction again. Therefore, $d(A) = \infty$. □

Remark 25. Suppose M_C is Browder for some $C \in \mathcal{B}(K, H)$, then

1- If $d(A) = n(B) = \infty$, then $n(A) + n(B) = d(A) + d(B)$.

2- If $d(A) < \infty$, then $A \in \phi(H)$ and $B \in \phi(K)$. It follows from Theorem 16 that $ind(M_C) = ind(A) + ind(B)$, that is $n(A) + n(B) = d(A) + d(B)$.

Now, as a corollary we get the following result.

Corollary 26. *Let H and K be two infinite dimensional separable Hilbert spaces, $A \in \mathcal{BCR}(H)$ and $B \in \mathcal{BCR}(K)$ with $\rho(B) \neq \emptyset$, $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$. If M_C is Browder for some $C \in \mathcal{B}(K, H)$, then $A \in \phi_+(H)$ with $asc(A) < \infty$, $B \in \phi_-(K)$ with $des(B) < \infty$ and $n(A) + n(B) = d(A) + d(B)$.*

Basing on the preceding theorems, we can now state the main result which gives necessary and sufficient conditions on the linear relations A and B such that there exists $C \in \mathcal{B}(K, H)$ for which M_C is Browder.

Theorem 27. *Let H and K be two infinite dimensional separable Hilbert spaces, $A \in \mathcal{BCR}(H)$ and $B \in \mathcal{BCR}(K)$ with $\rho(B) \neq \emptyset$, $R_c(A) = \{0\}$ and $R_c(B) = \{0\}$. The 2×2 matrix linear relation $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is Browder for some $C \in \mathcal{B}(K, H)$ if and only if $A \in \phi_+(H)$ with $asc(A) < \infty$, $B \in \phi_-(K)$ with $des(B) < \infty$ and $n(A) + n(B) = d(A) + d(B)$.*

Proof. The first implication is given by Corollary 26. For the converse, we have either $d(A) = n(B) = \infty$ or $d(A) < \infty$ and $n(B) < \infty$. In the first case, the assertion is obvious from Theorem 22. For the second, we get that $A \in \phi(H)$ and $B \in \phi(K)$. The result follows from Theorem 23. □

The following corollary is immediate from Theorem 27.

Corollary 28. *Let H and K be two infinite dimensional separable Hilbert spaces. For a given linear relation $A \in \mathcal{BCR}(H)$ and $B \in \mathcal{BCR}(K)$ such that*

$\rho(B) \neq \emptyset$, $R_c(A) = \{0\}$ *and* $R_c(B) = \{0\}$, *we have :*

$$\bigcap_{C\in\mathcal{B}(K,H)} \sigma_b(M_C) = \sigma_{ab}(A) \cup \sigma_{ab}(B^*) \cup \{\lambda \in \mathbb{C} : n(A-\lambda I) + n(B-\lambda I) \neq d(A-\lambda I) + d(B-\lambda I)\}.$$

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