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An η-Einstein Kenmotsu metric as a Ricci soliton

By AMALENDU GHOSH (Krishnanagar)

Abstract. We prove that, if the metric of an η-Einstein Kenmotsu manifold (of dimension > 3) is a Ricci soliton, then it is Einstein and the soliton is expanding.

1. Introduction

A Ricci soliton is a Riemannian manifold (M, g) together with a vector field V and a constant λ such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

where $\pounds_V$ denotes the Lie derivative operator along the vector field V and S is the Ricci tensor of g. Actually, it is a fixed point of the Hamilton's [7] Ricci flow: $\frac{\partial}{\partial t} g = -2S$, up to diffeomorphisms and scalings. A Ricci soliton with V zero or Killing is known as a trivial soliton. Thus, the Ricci soliton may be considered as an apt generalization of Einstein metric. The Ricci soliton is said to be *shrinking* when $\lambda < 0$, *steady* when $\lambda = 0$, and *expanding* when $\lambda > 0$. If the vector field V is the gradient of a potential function $-f$, then g is called a *gradient Ricci soliton.* We remark that on compact manifold Ricci solitons are always gradient solitons (see Perelman [9]). For details about Ricci solitons and their connection to the Ricci flow, we refer to Chow–Knopf [3].

In [8], a new class of non-compact almost contact metric manifolds was introduced and studied, which are known as Kenmotsu manifolds. This kind of manifold is characterized through the warped product. Actually, the warped product

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space $R \times_f V$ with the warping function $f(t) = ce^t$ on the real line R and V is a Kähler manifold admits such a structure. Moreover, every point of a Kenmotsu manifold has a neighbourhood which is locally a warped product $(-\epsilon, \epsilon) \times_f V$, where $f(t) = ce^t$ is a function on the open interval. Recently, in [5], the author proved that *if the metric of a 3-dimensional Kenmotsu manifold is a Ricci soliton, then it is of constant curvature -1 and the soliton is expanding.* Such metric also exists on the warped product of a Riemann surface N of constant negative curvature (a Kähler manifold) with the real line. It may be mentioned in this connection that any 3-dimensional Kenmotsu manifold is η-Einstein (i.e. the Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a, b are known as associated functions). However, in higher dimensions this is not true. We also know [8] that for dimension > 3, the associated functions of an η-Einstein Kenmotsu manifold are not constant, like K-contact manifolds [12]. In the literature, the case of compact Ricci solitons has been studied widely and extensively by several authors (e.g. see [3]). Thus, in view of recent results on Sasakian manifold [10] and η-Einstein K-contact manifold [6], a natural question to consider is whether there exist non-compact non-Sasakian almost contact metric manifolds whose metric is a Ricci soliton. For this, we consider an η-Einstein Kenmotsu manifold; such a manifold is not compact and in general not K-contact. Here we prove:

Theorem 1. *If the metric of an η-Einstein Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, $n > 1$ is a Ricci soliton then it is Einstein and the soliton is expanding.*

Since the warped product $R \times_f V(k)$, where $V(k)$ is a Kähler manifold of constant holomorphic sectional curvature of dimension $2n$ and $f(t) = ce^t$ is the warping function, naturally admits Kenmotsu structure, we have the following:

Corollary 1. *If the metric of the warped product $R \times_f V(k)$, $(n > 1)$ is a Ricci soliton then it is of constant curvature -1 and the soliton is expanding.*

2. Preliminaries

A $(2n+1)$-dimensional manifold (M, g) is said to have an almost contact metric structure if there exists a $(1,1)$ tensor field φ, a unit vector field ξ (called the Reeb vector field), and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi,$$

where I is the identity transformation. A Riemannian metric g is said to be the associated metric if it satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M. Then the following formulas also hold

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(.) = g(., \xi).$$

The manifold M equipped with the structure (φ, ξ, η, g) is called an almost contact metric manifold. On such a manifold, one can always define a 2-form ϕ by $\phi(., .) = g(., \varphi.)$, known as the fundamental 2-form. An almost contact metric manifold with $\phi = d\eta$ is known as contact metric manifold. If, in addition ξ is Killing, then M is said to be K-contact. Also, an almost contact metric manifold is said to be Sasakian if and only if [2]:

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any vector field X, Y on M. On the other hand, an almost contact metric manifold is said to be Kenmotsu [8], if it satisfies

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2}$$

for any vector field X, Y on M. An almost contact metric structure (φ, ξ, η, g) is said to be a Kenmotsu structure if it satisfies the condition (2). The following formulas are also valid for a Kenmotsu manifold (see [8])

$$\nabla_X \xi = X - \eta(X)\xi. \tag{3}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{4}$$

$$Q\xi = -2n\xi, \tag{5}$$

where R denotes the curvature tensor and Q denotes the Ricci operator associated with the S, i.e. $S(X, Y) = g(QX, Y)$. An almost contact metric manifold is said to be η-Einstein if the Ricci tensor S satisfies

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z), \tag{6}$$

for any vector field Y, Z on M and a, b are arbitrary functions on M. For a K-contact manifold of dimension > 3, the functions a, b are constant (see [12]), but for a Kenmotsu manifold this need not be true (see [8]).

3. Proof of the results

Proof of Theorem 1. Since M is η-Einstein, equation (6) shows that the scalar curvature r takes the form

$$r = (2n+1)a + b. \tag{7}$$

Also, making use of (5) in (6) we see that $a + b = -2n$. Combining this with (7) gives $a = 1 + \frac{r}{2n}$ and $b = -\{(2n+1) + \frac{r}{2n}\}$. Therefore, equation (6) can be written as

$$S(Y,Z) = \Big(1 + \frac{r}{2n}\Big)\, g(Y,Z) - \Big\{(2n+1) + \frac{r}{2n}\Big\}\, \eta(X)\eta(Y). \tag{8}$$

By virtue of this, the soliton equation transforms into

$$(\pounds_V g)(Y,Z) = -\Big(2 + \frac{r}{n} + 2\lambda\Big)\, g(Y,Z) + \Big\{2(2n+1) + \frac{r}{n}\Big\}\, \eta(Y)\eta(Z). \tag{9}$$

Now, from the well known commutation formula (see p. 23 of [11]):

$$\begin{aligned}(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) \\ = -g((\pounds_V\nabla)(X,Y),Z) - g((\pounds_V\nabla)(X,Z),Y),\end{aligned}$$

we obtain

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V\nabla)(X,Y),Z) + g((\pounds_V\nabla)(X,Z),Y). \tag{10}$$

Thus, differentiating (1), using it in (10), and through the straightforward combinatorial computation, we easily derive

$$g((\pounds_V\nabla)(X,Y),Z) = (\nabla_Z S)(X,Y) - (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z). \tag{11}$$

Taking $X = Y = e_i$ (where $\{\ e_i : i = 1,2,...,2n+1\}$) is an orthonormal frame) in (11) and summing over i, we find

$$(\pounds_V\nabla)(e_i,e_i) = 0, \tag{12}$$

for all vector fields Z. Differentiating (9) along an arbitrary vector field X and using equations (3) and (10), we have

$$\begin{aligned}g((\pounds_V\nabla)(X,Y),Z) + g((\pounds_V\nabla)(X,Z),Y) = -\frac{(Xr)}{n} g(Y,Z) + \frac{(Xr)}{n}\eta(Y)\eta(Z) \\ + \Big\{2(2n+1) + \frac{r}{n}\Big\}\, \{g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)\}.\end{aligned}$$

By a straightforward combinatorial computation, and since $(\pounds_V \nabla)$ is a symmetric operator, the foregoing equation yields

$$\begin{aligned} 2n(\pounds_V \nabla)(X,Y) &= g(Y,Z)Dr - (Xr)Y - (Yr)X + (Xr)\eta(Y) + (Yr)\eta(X) \\ &-\eta(X)\eta(Y)Dr + 2\{2n(2n+1)+r\}\{g(X,Y)\xi - \eta(X)\eta(Y)\xi\}, \end{aligned} \tag{13}$$

for all vector fields Z and D is the gradient operator of g. Setting $X = Y = e_i$ in (13), we at once obtain

$$(n-1)Dr + (\xi r)\xi + 2n\{2n(2n+1)+r\}\xi = 0. \tag{14}$$

Inner product of (14) with ξ gives $\xi r + 2\{2n(2n+1)+r\} = 0$. Applying this in (14) provides $Dr = (\xi r)\xi$, as $n > 1$. Next, taking $X = \xi$ in (13) it follows that

$$2n(\pounds_V \nabla)(Y,\xi) = (\xi r)\varphi^2 Y. \tag{15}$$

Differentiating (15) along an arbitrary vector field X and making use of (3) and (15), we find

$$\begin{aligned} 2n(\nabla_X \pounds_V \nabla)(Y,\xi) + 2n(\pounds_V \nabla)(Y,X) &= (X(\xi r))\varphi^2 Y \\ &+ (\xi r)\{g(X,Y)\xi + \eta(Y)X - \eta(X)Y - \eta(X)\eta(Y)\xi\}. \end{aligned}$$

Interchanging X, Y of this equation and applying the identity (see p. 23 of [11]):

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z),$$

it follows that

$$2n(\pounds_V R)(X,Y)\xi = (X(\xi r))\varphi^2 Y - (Y(\xi r))\varphi^2 X + 2(\xi r)\{\eta(Y)X - \eta(X)Y\}.$$

Contracting this equation over X and since $Dr = (\xi r)\xi$, we have $(\pounds_V S)(Y,\xi) = 0$. Next, taking the Lie derivative of (5) along V, using the last equation and (8), we obtain

$$\begin{aligned} \left(1+\frac{r}{2n}\right) g(Y,\pounds_V \xi) - \left\{(2n+1)+\frac{r}{2n}\right\}\eta(\pounds_V \xi)\eta(Y) \\ = -4n(2n-\lambda)\eta(Y) - 2ng(Y,\pounds_V \xi). \end{aligned} \tag{16}$$

Setting $Y = \xi$ in (16) we see that $\lambda = 2n$ and hence the soliton is expanding. On the other hand, substituting ξ for Y and Z in (9) yields $\eta(\pounds_V \xi) = 0$. Consequently, equation (16) implies that

$$[r + 2n(2n+1)]\pounds_V \xi = 0.$$

Now if $r=-2n(2n+1)$, then from (8) we see that M is Einstein. So we suppose that $r\neq -2n(2n+1)$ in some open set N of M. Then on N, $\pounds_V\xi=0$. This together with (3) provides

$$\nabla_\xi V = V-\eta(V)\xi. \tag{17}$$

Finally, taking $Y=\xi$ in the well-known formula (see p. 39 of [4]):

$$(\pounds_V\nabla)(X,Y)=\nabla_X\nabla_Y V-\nabla_{\nabla_X Y}V+R(V,X)Y,$$

and making use of (3), (15), (17) and (4), we have $\xi r=0$. Since $Dr=(\xi r)\xi$, we see that r is constant. Therefore, (14) implies that $r=-2n(2n+1)$ on N. Thus, we arrive at a contradiction on N. This completes the proof. □

Proof of Corollary 1. By the result mentioned in the introduction, it is obvious that the warped product under consideration is a Kenmotsu manifold. Moreover, the curvature tensor of such a warped product space is given by (see [8], [1])

$$\begin{aligned} R(X,Y)Z=H(t)\{g(Y,Z)X-g(X,Z)Y\}+(H(t)+1)\{g(X,Z)\eta(Y)\xi \\ -g(Y,Z)\eta(X)\xi+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X \\ +g(X,\varphi Z)\varphi Y-g(Y,\varphi Z)\varphi X+2g(X,\varphi Y)\varphi Z\}. \end{aligned} \tag{18}$$

From (18) it is easy to see that the Ricci tensor S takes the form

$$S(X,Y)=2\{(n-1)H(t)-1\}g(X,Y)-2(n-1)(H(t)+1)\eta(X)\eta(Y).$$

This is clearly η-Einstein. Hence applying Theorem 1 we see that the warped product is Einstein and since $n>1$, the last equation implies that $H(t)=-1$. Finally, using this in (18) we complete the proof. □

4. Example

We shall now exhibit an example of a Kenmotsu manifold which satisfies the Theorem 1. Let M be an η-Einstein Kenmotsu manifold (any Kenmotsu space form provides such example). For this class of space it is well known that (see [8]) $a+b=-2n$ and $Xb+2b\eta(X)=0$, if $n>1$, for any vector field X on M. We choose the vector field V of the Ricci soliton as a constant multiple of the

Reeb vector field, i.e. $V = f\xi$, for some constant f. Differentiating this along an arbitrary vector field X and using (3) we get

$$\nabla_X V = (Xf)\xi + f(X - \eta(X)\xi). \tag{19}$$

Making use of this and (6) it is easy to see that

$$(\pounds_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = (Xf)\eta(Y) + (Yf)\eta(X) \\ + 2(a + f + \lambda)g(X,Y) + 2(b - f)\eta(X)\eta(Y). \tag{20}$$

Since f is constant, the left hand side of this equation will vanish if and only if $f = b$ and $\lambda = -(a + b) = 2n$. Hence the soliton is expanding. By this choice of f, it remains to show that the manifold is Einstein. This easily follows from the formula $Xb + 2b\eta(X) = 0$.

In particular, the metric of the warped product space $R \times_f V(k)$ is a Ricci soliton whose potential vector field V is given by $-2\{(n-1)(H(t)+1)\}\xi$, for $n > 1$.

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AMALENDU GHOSH
DEPARTMENT OF MATHEMATICS
KRISHNAGAR GOVERNMENT COLLEGE
KRISHNANAGAR: 741 101, W.B.
INDIA

E-mail: aghosh_70@yahoo.com

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