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Coleman automorphisms of standard wreath products of finite abelian groups by 2-closed groups

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Abstract. Let G be the standard wreath product of a finite abelian group by a 2-closed group (a group having a normal Sylow 2-subgroup). It is shown that every Coleman automorphism of G is an inner automorphism. As an immediate consequence of this result, we obtain that the normalizer property holds for such G.

1. Introduction

Let G be a finite group. Recall that an automorphism ρ of G is called a Coleman automorphism, provided that $\rho^2 \in \mathrm{Inn}(G)$, ρ preserves the conjugacy classes of G and the restriction of ρ to any Sylow subgroup of G equals the restriction of some inner automorphism of G. This notion was initially introduced by Marciniak and Roggenkamp [10]. It should be noted that the same notion was also used by Hertweck and Kimmerle in [5], but which has different meanings with that mentioned above. Coleman automorphisms discussed in this paper are in the sense of that introduced by Marciniak and Roggenkamp. All Coleman automorphisms of G form a group, denoted by $\mathrm{Aut}_C(G)$; obviously, $\mathrm{Inn}(G) \leq \mathrm{Aut}_C(G)$. Recently, many results on Coleman automorphisms of finite groups have appeared in the literature, see [3], [4], [7], [8], [10].

Our interest in Coleman automorphisms arises from the fact that they play an important role in the study of the normalizer problem of integral group rings.

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Denote by $\mathbb{Z}G$ the integral group ring of G over $\mathbb{Z}$. Denote by $\mathcal{U}(\mathbb{Z}G)$ the group of all the units of the ring $\mathbb{Z}G$. A longstanding problem (see [15, Problem 43]) asks whether $\mathrm{N}_{\mathcal{U}(\mathbb{Z}G)}(G) = G\,\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ for any finite group G, where $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ denotes the center of $\mathcal{U}(\mathbb{Z}G)$ and $\mathrm{N}_{\mathcal{U}(\mathbb{Z}G)}(G)$ is the normalizer of G in $\mathcal{U}(\mathbb{Z}G)$. We often refer to this problem as the normalizer problem. If this equality holds, then we say that the normalizer property holds for G. This equality was first shown to be true for finite nilpotent groups by Coleman [1], and later this result was extended to any finite group having a normal Sylow 2-subgroup by Jackowski and Marciniak [6]. It was Mazur [11] who first noticed that there are close connections between the normalizer problem and the isomorphism problem. Based on Mazur's observations, among other things, Hertweck [2] constructed the first counterexample to the normalizer problem and then the first counterexample to the isomorphism problem. Nevertheless, it is still of interest to determine for which groups the normalizer property holds. Recently, many positive results on the normalizer problem have been obtained, see [3], [9], [13], [14]. To see how Coleman automorphisms occur naturally in the study of the normalizer problem, we should recall the equivalent form of the normalizer problem. For any $u \in \mathrm{N}_{\mathcal{U}(\mathbb{Z}G)}(G)$, denote by φ_u the automorphism of G induced by u via conjugation, i.e., $\varphi_u(g) = u^{-1}gu$ for all $g \in G$. Denote by $\mathrm{Aut}_{\mathcal{U}}(G)$ the group formed by all such automorphisms of G. Obviously, $\mathrm{Inn}(G) \leq \mathrm{Aut}_{\mathcal{U}}(G)$. A question raised by Jackowski and Marciniak in [6] asks whether $\mathrm{Aut}_{\mathcal{U}}(G) = \mathrm{Inn}(G)$ for any finite group G. It is easy to see that this question is equivalent to the normalizer problem mentioned above. So we can investigate the normalizer problem by using this equivalent form, which is more convenient than not often. It is known that $\mathrm{Aut}_{\mathcal{U}}(G) \leq \mathrm{Aut}_C(G)$. Thus, if one can show that Coleman automorphisms of G are inner then the normalizer property holds for G.

The aim of this paper is to study Coleman automorphisms of standard wreath products of finite abelian groups by finite 2-closed groups. For general information on standard wreath products, refer to [12]. Before stating our main result, we would like to mention that Marciniak and Roggenkamp in [10] established a finite metabelian group isomorphic to $(C_2^4 \times C_3) \rtimes C_2^3$ which has a non-inner Coleman automorphism. This example demonstrates that if G is a finite abelian-by-2-closed group, then in general it is not the case that $\mathrm{Aut}_C(G) = \mathrm{Inn}(G)$. However, in this paper, we shall prove the following main result:

Main Theorem. *Let $G = A\mathit{wr}N$ be the standard wreath product of A by N, where A is a finite abelian group and N is a finite 2-closed group. Then every Coleman automorphism of G is an inner automorphism, i.e.,* $\mathrm{Aut}_C(G) = \mathrm{Inn}(G)$.

2. Preliminaries

In this section, we recall several results which will be used in the proof of Main Theorem. First we fix some notation. Let G be a finite group and let ρ be an automorphism of G. For any subset U of G, write $\rho|_U$ for the restriction of ρ to U. Let N be a normal subgroup of G which is fixed by ρ. We write $\rho|_{G/N}$ for the automorphism of G/N induced by ρ. For a fixed element $x \in G$, $\mathrm{conj}(x)$ is used to denote the inner automorphism of G induced by x via conjugation, i.e., $\mathrm{conj}(x)(g) = g^x$ for all $g \in G$. Denote by $\pi(G)$ the set of all prime divisors of the order of G. Other notation is mostly standard.

Definition 2.1. Let N^m be the direct product of m copies of a finite group N. A subgroup H of N^m is said to be extensive in N^m if the intersection of H with $(1, \dots, 1, \underbrace{N}_{ith}, 1 \dots, 1)$ is non-trivial for any $i \in \{1, 2, \dots, m\}$.

Lemma 2.2. *Suppose that N^m is the direct product of m copies of a non-trivial finite nilpotent group N. Then, for any $p \in \pi(N)$, the following hold:*

(1) *any Sylow p-subgroup of N^m is extensive in N^m;*

(2) *the center of any Sylow p-subgroup of N^m is extensive in N^m;*

(3) *N^m is extensive in itself.*

Proof. These assertions follow directly from Definition 2.1. □

Lemma 2.3 (Lemma 2 in [3])**.** *Let p be a prime and let φ be a p-power order automorphism of a finite group G. Suppose that there is a normal subgroup N of G such that φ fixes all elements of N, and that φ induces the identity on the quotient group G/N. Then φ induces the identity on $G/\mathrm{O}_p(\mathrm{Z}(N))$. Further, if φ fixes elementwise a Sylow p-subgroup of G, then φ is an inner automorphism of G.*

3. Proof of Main Theorem

In this section, we will present a proof of the Main Theorem. We begin by proving the following result, which generalizes a result (see Proposition 2.3 in [8]) due to Li:

Proposition 3.1. *Let G be a finite group and let B be a normal subgroup of G for which the quotient group G/B has a normal Sylow 2-subgroup. Let ρ be a Coleman automorphism of G. Then ρ is an inner automorphism of G if and only if $\rho|_{B \cup P} = \mathrm{conj}(g)|_{B \cup P}$ for some $g \in G$ and some Sylow 2-subgroup P of G.*

Proof. The necessity of Proposition 3.1 is evident. Conversely, let ρ be a Coleman automorphism of G satisfying $\rho|_{B\cup P} = \mathrm{conj}(g)|_{B\cup P}$ for some $g \in G$ and some Sylow 2-subgroup P of G. Then we have $\mathrm{conj}(g^{-1})\rho|_{B\cup P} = \mathrm{id}\,|_{B\cup P}$. It follows that $\mathrm{conj}(g^{-1})\rho|_B = \mathrm{id}\,|_B$ and $\mathrm{conj}(g^{-1})\rho|_P = \mathrm{id}\,|_P$. Replacing ρ by $\mathrm{conj}(g^{-1})\rho$, we may assume that $\rho|_B = \mathrm{id}\,|_B$ and $\rho|_P = \mathrm{id}\,|_P$. Write $N := BP$. Then we have $\rho|_N = \mathrm{id}\,|_N$. Since by assumption ρ^2 is an inner automorphism, it follows that ρ is inner if some odd power of ρ is inner. Thus, by taking some suitable odd power of ρ, we may assume that ρ is of 2-power order. Note that N/B is exactly the Sylow 2-subgroup of G/B, so by assumption $N/B \trianglelefteq G/B$, which implies that $N \trianglelefteq G$. Now take the quotient G/N. Since ρ is a Coleman automorphism of G of 2-power order, it follows that $\rho|_{G/N}$ is a Coleman automorphism of G/N of 2-power order. But note that G/N is of odd order, which forces $\rho|_{G/N} = \mathrm{id}\,|_{G/N}$. Then, by Lemma 2.3, ρ is an inner automorphism of G. $\square$

Now we are in position to prove the Main Theorem. For the reader's convenience, we rewrite it here as

Theorem 3.2. *Let $G = AwrN$ be the standard wreath product of A by N, where A is a finite abelian group and N is a finite 2-closed group. Then every Coleman automorphism of G is an inner automorphism, i.e., $\mathrm{Aut}_C(G) = \mathrm{Inn}(G)$.*

Proof. Let $|N| = m$. Then $G = AwrN = A^m \rtimes N$. If A is trivial, then G itself is a finite 2-closed group and it is easy to see that assertion holds in this case. Hereafter we assume that A is non-trivial. Let ρ be an arbitrary Coleman automorphism of G. We have to show that ρ is an inner automorphism of G. Let $P = A_2 \rtimes N_2$ be a fixed Sylow 2-subgroup of G, where A_2 and N_2 are the Sylow 2-subgroups of A^m and N respectively. Since ρ is a Coleman automorphism, there exists $g \in G$ such that $\rho|_P = \mathrm{conj}(g)|_P$, or equivalently, $\mathrm{conj}(g^{-1})\rho|_P = \mathrm{id}\,|_P$. Replacing ρ by $\mathrm{conj}(g^{-1})\rho$, we may assume that

$$\rho|_P = \mathrm{id}\,|_P. \tag{3.1}$$

Since $\rho^2 \in \mathrm{Inn}(G)$, it follows that $\rho \in \mathrm{Inn}(G)$ if some odd power of ρ is an inner automorphism of G. Without loss of generality, we may assume that ρ is of 2-power order.

Next we check the action of ρ on the base group A^m. Note that A^m is abelian, so we may set that $A^m = P_1 \times \cdots \times P_r$, where P_i is the Sylow p_i-subgroup of A^m for each $i \in \{1, 2, \ldots, r\}$. Since ρ is a Coleman automorphism, for each P_i and all $x_i \in P_i$, there exist $h_i \in N$ and $a_i \in A^m$ such that

$$\rho(x_i) = {a_i}^{-1}{h_i}^{-1}x_ih_ia_i = {h_i}^{-1}x_ih_i. \tag{3.2}$$

Thus, for any $x_i \in P_i$ and any $x_j \in P_j$ with $i \neq j$, by (3.2), we obtain that

$$\rho(x_i x_j) = \rho(x_i)\rho(x_j) = ({h_i}^{-1} x_i h_i)\,({h_j}^{-1} x_j h_j). \tag{3.3}$$

On the other hand, since ρ preserves the conjugacy classes of G, there exist $a_0 \in A^m$ and $h \in N$ such that

$$\rho(x_i x_j) = {a_0}^{-1} h^{-1}(x_i x_j) h a_0 = (h^{-1} x_i h)\,(h^{-1} x_j h). \tag{3.4}$$

Then, by (3.3) and (3.4), we have

$${h_i}^{-1} x_i h_i = h^{-1} x_i h$$

and

$${h_j}^{-1} x_j h_j = h^{-1} x_j h,$$

namely,

$$h {h_i}^{-1} x_i h_i h^{-1} = x_i \tag{3.5}$$

and

$$h {h_j}^{-1} x_j h_j h^{-1} = x_j. \tag{3.6}$$

By Lemma 2.2, P_i and P_j are extensive in A^m, so (3.5) and (3.6) imply that $h = h_i = h_j$. Thus we must have $h = h_1 = h_2 = \cdots = h_r$. Hence, for any $a \in A^m$, say $a = y_1 y_2 \cdots y_r$ with $y_i \in P_i$, by (3.2), we have

$$\rho(a) = \rho(y_1 y_2 \cdots y_r) = y_1^{h_1} y_2^{h_2} \cdots y_r^{h_r} = a^h. \tag{3.7}$$

Remember that we have assumed that ρ is of 2-power order, say $|\rho| = 2^s$ with $s \in \mathbb{N}$. Then, on the one hand, for all $a \in A^m$, we have

$$\rho^{2^s}(a) = a. \tag{3.8}$$

On the other hand, by (3.7), we have

$$\rho^{2^s}(a) = a^{h^{2^s}}. \tag{3.9}$$

Consequently, by (3.8) and (3.9), for all $a \in A^m$, we have $a^{h^{2^s}} = a$, which implies that $h^{2^s} = 1$ since by Lemma 2.2 A^m is extensive in itself. This shows that h is a 2-element of N. But by assumption N is 2-closed, it follows that $h \in N_2$. Next we will show that h is actually in $\mathrm{Z}(N_2)$, the center of N_2. In fact, for any $y \in N_2$

and any $a \in A^m$, we have $a^y \in A^m$ since A^m is normal in G. Then, on the one hand, by (3.7), we obtain that

$$\rho(a^y) = (a^y)^h = a^{yh}. \tag{3.10}$$

On the other hand, by (3.1) and (3.7), we have

$$\rho(a^y) = \rho(a)^{\rho(y)} = (a^h)^y = a^{hy}. \tag{3.11}$$

Hence, by (3.10) and (3.11), for all $a \in A^m$, we have $a^{yh} = a^{hy}$, which implies that $yh = hy$ since by Lemma 2.2 A^m is extensive in itself. As $y \in N_2$ is arbitrary, $h \in \mathrm{Z}(N_2)$. Note that $N_2 \leq P$. Then, by (3.1), we have

$$\rho|_{N_2} = \mathrm{id}\,|_{N_2} = \mathrm{conj}(h)|_{N_2}. \tag{3.12}$$

On the other hand, note that $A_2 \leq A^m$, so by (3.7) one gets that

$$\rho|_{A_2} = \mathrm{conj}(h)|_{A_2}. \tag{3.13}$$

Recall that $P = A_2 \rtimes N_2$. So (3.12) and (3.13) yield that

$$\rho|_P = \mathrm{conj}(h)|_P. \tag{3.14}$$

Hence, by (3.7) and (3.14), we obtain that $\rho|_{A^m \cup P} = \mathrm{conj}(h)|_{A^m \cup P}$. It follows from Proposition 3.1 that $\rho \in \mathrm{Inn}(G)$. As ρ is arbitrary, we have $\mathrm{Aut}_C(G) \subseteq \mathrm{Inn}(G)$. On the other hand, it is clear that $\mathrm{Inn}(G) \subseteq \mathrm{Aut}_C(G)$. Hence $\mathrm{Aut}_C(G) = \mathrm{Inn}(G)$. This completes the proof of Theorem 3.2. $\square$

As immediate consequences of Theorem 3.2, we have the following results.

Corollary 3.3. *Let $G = AwrN$ be the standard wreath product of A by N, where A is a finite abelian group and N is a finite 2-closed group. Then the normalizer property holds for G.*

Corollary 3.4. *Let $G = AwrN$ be the standard wreath product of A by N, where A is a finite abelian group and N is a finite nilpotent group. Then every Coleman automorphism of G is an inner automorphism, i.e., $\mathrm{Aut}_C(G) = \mathrm{Inn}(G)$. In particular, the normalizer property holds for G.*

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