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Gamma-mixed Ornstein–Uhlenbeck sheet

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Dedicated to the memory of Professor Constantin Tudor

Abstract. We construct a two-parameter variant of the Gamma-mixed Ornstein–Uhlenbeck process introduced in [8]. This process is constructed as a limit of aggregated Ornstein–Uhlenbeck sheet with common input and random coefficient. We will show that the Gamma-mixed Ornstein–Uhlenbeck sheet has various interesting properties. In particular, it approximates the Brownian sheet and its integral process approximates the Liouville fractional Brownian sheet.

1. Introduction

In [8] an interesting stochastic process has been introduced. It is called *Gamma-mixed Ornstein–Uhlenbeck process.* Let us briefly recall its construction. Consider a sequence of stationary Ornstein–Uhlenbeck processes $(X_k)_{k\geq 1}$ with random coefficients, that is, for every $k \geq 1$, X_k is the solution of the Langevin equation

$$dX_k(t) = \alpha_k X_k(t)dt + dB(t)$$

where $X_k(0) = 0$, $B = (B(t))_{t\in\mathbb{R}}$ is a standard Wiener process with time interval $\mathbb{R}$ defined on the probability space $(\Omega_B, \mathcal{F}_B, P_B)$ and α_k are independent random

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variables, also independent by B, defined on the probability space $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$. It is assumed that for every $k \in \mathbb{N}$, the random variable α_k has Gamma distribution $\Gamma(1-h, \lambda)$ with $\lambda > 0$ and $h \in \left(0, \frac{1}{2}\right)$. Denote by

$$Y_n(t) = \frac{1}{n} \sum_{k=1}^{n} X_k(t), \quad t \in \mathbb{R}$$

the so-called *aggregated process.* Then Y^n converges as $n \to \infty$ (in $L^2(\Omega_B)$ for fixed time and in the weak sense) to the stochastic process $Y^\lambda = (Y^\lambda(t))_{t \in \mathbb{R}}$ which can be represented as a Wiener integral with respect to the Brownian motion B in the following way

$$Y^\lambda(t) = \int_{-\infty}^{t} \left(\frac{\lambda}{\lambda + t - s} \right)^{1-h} dB(s). \tag{1}$$

The stochastic process Y^λ is called Gamma-mixed Ornstein–Uhlenbeck process. It has various interesting properties. First, as $\lambda \to \infty$ it converges (again in $L^2(\Omega_B)$ for fixed time and in the weak sense) to the Wiener process while its integrated renormalized process

$$Z^\lambda(t) = \lambda^{\frac{3}{2} - H} \int_0^t Y^\lambda(s) ds \tag{2}$$

converges to (modulo a constant) the fractional Brownian motion with Hurst parameter $H = h + \frac{1}{2} \in \left(\frac{1}{2}, 1\right)$. It is stationary, it exhibits long-range dependence, it is asymptotically self-similar and it is a semimartingale. As explained in [8], it is a good candidate to be a model for various applications, such as heart rate variability. Other results related to the Gamma-mixed Ornstein–Uhlenbeck process can be found in [13]. Also we mention some related works on fractional sheets and mixed Ornstein–Uhlenbeck processes: [2], [7], [6], [10].

The purpose of this paper is to study the two-parameter counterpart of the Gamma-mixed Ornstein–Uhlenbeck process (1). We will define it as a limit of an aggregated process involving the solution of the two-parameter Langevin equation (3) (this solution will be called Ornstein–Uhlenbeck sheet). But in the two-parameter case, the situation is different given the more complex structure of the solution of the Langevin equation. We will prove that, in order to find after "aggregation" and limit a process that approximate the Wiener sheet and the fractional Brownian sheet, the two-parameter Ornstein–Uhlenbeck sheet has to be "mixed" by a different law, which is a product of a random variable with

Gamma distribution and of an independent random variable with exponential distribution.

Our paper is organized as follows. In Section 2 we analyze the solution of the two-parameter Langevin equation with random coefficient and we investigate the limit behavior of its aggregated process. Section 3 is devoted to the study of the properties of the Gamma-mixed Ornstein–Uhlenbeck sheet.

2. Aggregated Ornstein–Uhlenbeck sheet with random coefficient

Let us consider a Brownian sheet $(W_{t,s})_{t,s\geq 0}$ on a probability space $(\Omega_W, \mathcal{F}_W, P_W)$. Recall that W is a defined as a centered Gaussian process, null on the axis, such that

$$\mathbf{E}W_{s,t}W_{u,v} = (t\wedge u)(s\wedge v), \quad \text{for every } s,t,u,v \geq 0.$$

Also consider a sequence of independent identically distributed random variables $(\alpha_k)_{k\geq 1}$ defined on another probability space $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$. We will assume that W is independent by the sequence $(\alpha_k)_{k\geq 1}$ and we will consider W and α_k as extended versions on the product space. We will denote by $\mathbf{E}_W, \mathbf{E}_\alpha$ the expectation with respect to P_W, P_α respectively. The expectation with respect to the product probability measure with be denoted by $\mathbf{E}$. Consider the following two-parameter stochastic differential equation with additive noise W and with random coefficient α_k

$$dX^k_{t,s} = -\alpha_k X^k_{t,s}dtds + dW_{t,s} \tag{3}$$

with initial condition $X^k_{0,0} = X^k_{0,s} = X^k_{t,0} = 0$ for every $t,s\geq 0$, where W is a standard Brownian sheet and $k\geq 1$. The equation has the integral form

$$X^k_{t,s} = \int_0^t\int_0^s -\alpha_k X^k_{u,v}dudv + W_{t,s}, \quad t,s\geq 0.$$

The first step is to express the solution to (3) as a Wiener integral with respect to the Brownian sheet W.

Proposition 1. *For every $k\geq 1$ and $t,s,t_0,s_0\geq 0$ define*

$$f^k(t,s,t_0,s_0)) = 1_{(0,t)}(t_0)1_{(0,s)}(s_0)\sum_{n\geq 0}\frac{(-1)^n\alpha_k^n}{(n!)^2}(t-t_0)^n(s-s_0)^n. \tag{4}$$

Assume that for every $k\geq 1$ and for every $t,s\geq 0$

$$f^k(t,s,\cdot,\cdot)\in L^2([0,\infty)^2) \tag{5}$$

and

$$\int_0^t du \int_0^s dv \int_0^\infty dt_0 \int_0^\infty ds_0 \left(f^k(u,v,t_0,s_0)\right)^2 < \infty. \tag{6}$$

Then the unique solution to (3) *can be written as follows:*

$$X^k_{t,s} = \int_{\mathbb{R}} \int_{\mathbb{R}} f^k(t,s,t_0,s_0) dW_{t_0,s_0}. \tag{7}$$

Proof. It is standard to show that (3) admits an unique solution (it suffices for example to identify the chaos expansion of each side, see e.g. [11]). We will show that this unique solution is given by (7). For every $k \geq 1$, using Fubini's theorem (note that assumption (5), (6) and Exercise 3.2.7 in [12] imply that the hypothesis of the stochastic Fubini theorem are satisfied) we can write

$$\begin{aligned}
-&\alpha_k \int_0^t \int_0^s X^k_{u,v} dv du \\
&= -\alpha_k \int_0^t du \int_0^s dv \int_0^u \int_0^v \sum_{n\geq 0} \frac{(-1)^n \alpha_k^n}{(n!)^2} (u-t_0)^n (s-s_0)^n dW_{t_0,s_0} \\
&= \sum_{n\geq 0} \frac{(-1)^{n+1}\alpha_k^{n+1}}{(n!)^2} \int_0^t \int_0^s dW_{t_0,s_0} \int_{t_0}^t du \int_{s_0}^s dv (u-t_0)^n (v-s_0)^n \\
&= \sum_{n\geq 0} \frac{(-1)^{n+1}\alpha_k^{n+1}}{((n+1)!)^2} \int_0^t \int_0^s dW_{t_0,s_0} (t-t_0)^{n+1} (s-s_0)^{n+1} \\
&= \sum_{n\geq 1} \frac{(-1)^n \alpha_k^n}{(n!)^2} \int_0^t \int_0^s dW_{t_0,s_0} (t-t_0)^n (s-s_0)^n \\
&= \sum_{n\geq 0} \frac{(-1)^n \alpha_k^n}{(n!)^2} \int_0^t \int_0^s dW_{t_0,s_0} (t-t_0)^n (s-s_0)^n - W_{t,s} \\
&= X^k_{t,s} - W_{t,s}.
\end{aligned}$$

□

Remark 1. The stochastic integral in (7) is a Wiener integral with respect to the Brownian sheet W. Since the Brownian sheet is a Gaussian process we can define Wiener integrals with respect to it. This Wiener integral is an isometry between $L^2([0,\infty)^2)$ and the Gaussian space generated by W.

Remark 2. The kernel (4) can be also expressed in terms of the Bessel function (see [6]).

As mentioned in the introduction, we are interested in finding the limit of the "aggregated" sequence

$$Y_N(t,s) = \frac{1}{N} \sum_{k=1}^N X^k_{t,s}. \tag{8}$$

More precisely, we want to find the suitable law of the random variables α_k which will imply the convergence of Y_N (in some sense that will be explicated later) as $N \to \infty$ to a two-parameter version of the Gamma-mixed Ornstein–Uhlenbeck process. At this point, let us make some heuristic considerations in order to find the candidate for the limit of Y_N. We have, for every $t, s \geq 0$

$$\begin{aligned} Y_N(t,s) &= \frac{1}{N}\sum_{k=1}^{N}\sum_{n\geq 0}\frac{(-1)^n\alpha_k^n}{(n!)^2}\int_0^t\int_0^s dW_{t_0,s_0}(t-t_0)^n(s-s_0)^n \\ &= \int_0^t\int_0^s\sum_{n\geq 0}\frac{(-1)^n}{(n!)^2}(t-t_0)^n(s-s_0)^n\left(\frac{1}{N}\sum_{k=1}^{N}\alpha_k^n\right)dW_{t_0,s_0}. \end{aligned}$$

By the law of large numbers, for every n,

$$\frac{1}{N}\sum_{k=1}^{N}\alpha_k^n \to_{N\to\infty} \mathbf{E}_\alpha \alpha_1^n \tag{9}$$

almost surely with respect to P_α. Therefore, for every s, t

$$Y_N(t,s) \to_{N\to\infty} \int_0^t\int_0^s dW_{t_0,s_0}\sum_{n\geq 0}\frac{(-1)^n}{(n!)^2}(t-t_0)^n(s-s_0)^n\mathbf{E}\alpha_1^n$$

(at this point, the symbol "$\to$" is only heuristic and we can use either the notation $\mathbf{E}$ or $\mathbf{E}_\alpha$). Now, we will introduce the common distributions of α_k. Take $H \in \left(\frac{1}{2}, 1\right)$ and $\lambda > 0$. We assume that for every $k \geq 1$ we have

$$\alpha_k = a_k b_k \tag{10}$$

where a_k, b_k are independent random variables defined on the space Ω_α such that

$$a_k \sim \Gamma\left(\frac{3}{2} - H, \lambda\right) \quad \text{and} \quad b_k \sim \operatorname{Exp}(1) \tag{11}$$

(the notation above means that a_k has Gamma distribution with parameters $\frac{3}{2}-H$ and $\lambda > 0$ and b_k has exponential law with parameter 1). Then, with $h = H - \frac{1}{2}$

(this notation will be used often throughout the paper)

$$\mathbf{E}(b_1^n) = \Gamma(n+1) = n! \quad \text{and}$$

$$\mathbf{E}(a_1^n) = \frac{\lambda^{1-h}}{\Gamma(1-h)} \int_0^\infty e^{-\lambda x} x^{-h+n} dx = \frac{\lambda^{-n}}{\Gamma(1-h)} \Gamma(n-h+1).$$

This implies that

$$\mathbf{E}\alpha_1^n = \frac{\lambda^{-n}\Gamma(n-h+1)n!}{\Gamma(1-h)}. \tag{12}$$

Consequently, the natural limit of the aggregated sequence Y_N as $N \to \infty$ would be

$$Y^\lambda(t,s) = \int_0^t \int_0^s dW_{t_0,s_0} \sum_{n\geq 0} \frac{(-1)^n}{n!} \frac{(t-t_0)^n (s-s_0)^n}{\lambda^n} \frac{\Gamma(n-h+1)}{\Gamma(1-h)}. \tag{13}$$

Recall that the power series expansion of the function $(1-x)^{b-1}$ with $b > 0$ is

$$(1-x)^{b-1} = \sum_{n\geq 0} \frac{\Gamma(n-b+1)}{\Gamma(1-b)n!} x^n, \quad x \in (-1,1). \tag{14}$$

Using this fact, we get that for λ large enough, (13) equals

$$\begin{aligned} Y^\lambda(t,s) &= \int_0^t \int_0^s \left(1 + \frac{(t-t_0)(s-s_0)}{\lambda}\right)^{H-\frac{3}{2}} dW_{t_0,s_0} \\ &= \int_0^t \int_0^s \left(\frac{\lambda}{\lambda + (t-t_0)(s-s_0)}\right)^{\frac{3}{2}-H} dW_{t_0,s_0}. \end{aligned} \tag{15}$$

Definition 1. The process $(Y^\lambda(t,s))_{t,s\geq 0}$ given by (15) will be called Gamma-mixed Ornstein–Uhlenbeck sheet with parameter H and λ. Note that the process Y^λ is well-defined for every $\lambda > 0$ as a Wiener integral in $L^2(\Omega)$, although the expansion (14) is valid only for $x \in (-1,1)$.

We now prove the convergence of Y_N (8) to Y^λ (15). We start with the following lemma.

Lemma 1. *Assume* (10) *and* (11). *Then for every* $t,s \geq 0$, $\lambda > 0$ *such that* $\lambda > 4ts$ *and for every* $k \geq 1$

$$\mathbf{E}\left(X_{t,s}^k\right)^2 \leq ts.$$

PROOF. Fix $t, s \geq 0$, $\lambda > 0, k \geq 1$ with $\lambda > 4ts$. Using the isometry of the Wiener integral and relation (12)

$$
\begin{aligned}
\mathbf{E}\left(X^k_{t,s}\right)^2 &= \mathbf{E}_\alpha \int_0^t dt_0 \int_0^s ds_0 \left(\sum_{n\geq 0} \frac{(-1)^n \alpha_k^n}{(n!)^2} (t-t_0)^n (s-s_0)^n \right)^2 \\
&= \int_0^t dt_0 \int_0^s ds_0 \sum_{n,m\geq 0} \frac{(-1)^{n+m} \mathbf{E}_\alpha \alpha_k^{n+m}}{(n!)^2 (m!)^2} (t-t_0)^{n+m} (s-s_0)^{n+m} \\
&= \int_0^t dt_0 \int_0^s ds_0 \sum_{n,m\geq 0} \frac{(-1)^{n+m} \lambda^{-(n+m)} (m+n)!}{(n!)^2 (m!)^2} \\
&\quad \times (t-t_0)^{n+m} (s-s_0)^{n+m} \frac{\Gamma(n+m+\frac{3}{2}-H)}{\Gamma(\frac{3}{2}-H)} \\
&= \sum_{n,m\geq 0} \frac{(-1)^{n+m} \lambda^{-(n+m)} (m+n)!}{(n!)^2 (m!)^2} \frac{\Gamma(n+m+\frac{3}{2}-H)}{\Gamma(\frac{3}{2}-H)} \frac{1}{(m+n+1)^2} (ts)^{n+m+1} \\
&= \sum_{n,m\geq 0} \frac{(-1)^{n+m} \lambda^{-(n+m)} C^n_{m+n}}{n! m! (m+n+1)^2} \frac{\Gamma(n+m+\frac{3}{2}-H)}{\Gamma(\frac{3}{2}-H)} (ts)^{n+m+1} \\
&= \sum_{q\geq 0} \frac{(-1)^q \lambda^{-q}}{(q+1)^2} \frac{\Gamma(q+\frac{3}{2}-H)}{\Gamma(\frac{3}{2}-H)} (ts)^{q+1} \sum_{n=0}^q C_q^n \frac{1}{n!(q-n)!}
\end{aligned}
$$

where we used the change of summation $m+n=q$. Thus

$$
\begin{aligned}
\mathbf{E}\left(X^k_{t,s}\right)^2 &= \sum_{q\geq 0} \frac{(-1)^q \lambda^{-q}}{(q+1)^2 q!} \frac{\Gamma(q+\frac{3}{2}-H)}{\Gamma(\frac{3}{2}-H)} (ts)^{q+1} \sum_{n=0}^q (C_q^n)^2 \\
&= \sum_{q\geq 0} \frac{(-1)^q \lambda^{-q}}{(q+1)^2 q!} \frac{\Gamma(q+\frac{3}{2}-H)}{\Gamma(\frac{3}{2}-H)} (ts)^{q+1} C^q_{2q}
\end{aligned}
$$

because $\sum_{n=0}^q (C_q^n)^2 = C^q_{2q}$ (this identity is well-known: it can be proven by developing $(1+x)^{2n} = (1+x)^n(1+x)^n$ and identifying the coefficient of x^n). The above series converges absolutely if $\lambda > 4ts$. Indeed, denoting by

$$
A_q := \left| \frac{(-1)^q \lambda^{-q}}{(q+1)^2 q!} \frac{\Gamma(q+\frac{3}{2}-H)}{\Gamma(\frac{3}{2}-H)} (ts)^{q+1} C^q_{2q} \right|
$$

we have

$$
\frac{A_{q+1}}{A_q} = \frac{(2q+1)(2q+2)(q+\frac{3}{2}-H)ts}{(q+1)(q+2)^2\lambda} \to_{q\to\infty} \frac{4ts}{\lambda}
$$

and this is strictly less than 1 if $\lambda > 4ts$. Moreover, denoting by $B_q = \left(\frac{4ts}{\lambda}\right)^q$ we can see that $\frac{A_{q+1}}{A_q} \leq \frac{B_{q+1}}{B_q}$ for every $q \geq 0$ and a classical argument implies that $A_q \leq \frac{A_0}{B_0} = ts$. □

Theorem 1. *Let Y_N be given by* (8) *and assume* (10) *and* (11). *Then for every $s, t \in [0,T]$ such that $4ts < \lambda$*

$$Y_N(t,s) \to_{N\to\infty} Y^\lambda(t,s) \text{ in } L^2(\Omega).$$

Also, as $N \to \infty$, the family of stochastic processes $(Y_N)_{N\geq 1}$ converges weakly in the space $C([a,b]\times[c,d])$ (with $a<b, c<d$, $a,b,c,d \geq 0$) to the process Y^λ.

Proof. Throughout this proof, we will use the following notation:

$$f^\lambda(t,s,t_0,s_0) = 1_{(0,t)}(t_0)1_{(0,s)}(s_0)\sum_{n\geq 0}\frac{(-1)^n}{n!}\frac{(t-t_0)^n(s-s_0)^n}{\lambda^n}\frac{\Gamma(n-h+1)}{\Gamma(1-h)}$$

and

$$f^N(t,s,t_0,s_0) = 1_{(0,t)}(t_0)1_{(0,s)}(s_0)\sum_{n\geq 0}\frac{(-1)^n}{(n!)^2}(t-t_0)^n(s-s_0)^n\left(\frac{1}{N}\sum_{k=1}^N \alpha_k^n\right)$$

First, note that

$$\mathbf{E}_\alpha\mathbf{E}_B\left|Y_N(t,s)-Y^\lambda(t,s)\right|^2 = \mathbf{E}_\alpha\int_0^t\int_0^s\left(f^N(t,s,t_0,s_0)-f^\lambda(t,s,t_0,s_0)\right)^2 dt_0 ds_0.$$

Lemma 1 (which allows to interchange the order of integration before (9)) and relation (9) show that, under the assumptions in the statement, for fixed t,s,t_0,s_0 the difference $f^N(t,s,t_0,s_0)-f^\lambda(t,s,t_0,s_0)$ converges to zero almost surely with respect to P_α. In order to apply the dominated convergence theorem, we need to show that for every t,s,t_0,s_0,

$$\mathbf{E}_\alpha\left(f^N(t,s,t_0,s_0)-f^\lambda(t,s,t_0,s_0)\right)^2 \leq g(t,s,t_0,s_0)$$

where the function g that not depend on N (it may depend on λ) and satisfies for every t,s

$$\int_0^t dt_0\int_0^s ds_0 g(t,s,t_0,s_0) \leq C$$

with C a strictly positive constant.

First obviously $\left|f^\lambda(t,s,t_0,s_0)\right| \le 1$ and for every t, s

$$\mathbf{E}_\alpha f^N(t,s,t_0,s_0)^2 = \mathbf{E}_\alpha \left(f^N(t,s,t_0,s_0)\right)^2$$
$$= \mathbf{E}_\alpha \left(\sum_{n\ge 0} \frac{(-1)^n}{(n!)^2}(t-t_0)^n(s-s_0)^n \frac{1}{N}\sum_{k=1}^N \alpha_k^n \right)^2 .$$

We follow the lines of the proof of Lemma 1. We have

$$\mathbf{E}_\alpha f^N(t,s,t_0,s_0)^2$$
$$= \sum_{n,m\ge 0} \frac{(-1)^{n+m}}{(n!)^2(m!)^2}(t-t_0)^{n+m}(s-s_0)^{n+m}\frac{1}{N^2}\sum_{k,l=1}^N \mathbf{E}_\alpha \alpha_k^n \alpha_l^m$$
$$= \sum_{n,m\ge 0} \frac{(-1)^{n+m}}{(n!)^2(m!)^2}(t-t_0)^{n+m}(s-s_0)^{n+m}\frac{1}{N^2}\left(\sum_{k=1}^N \mathbf{E}_\alpha \alpha_k^{m+n} + \sum_{k\ne=l;k,l=1}^N \mathbf{E}_\alpha \alpha_k^n \mathbf{E}_\alpha \alpha_l^n\right).$$

By (12), using again the notation $h = H - \frac{1}{2}$

$$\mathbf{E}_\alpha f^N(t,s,t_0,s_0) = \sum_{n,m\ge 0}\frac{(-1)^{n+m}}{n!m!}(t-t_0)^{n+m}(s-s_0)^{n+m}$$
$$\times \frac{N(N-1)}{N^2}\,\frac{\lambda^{-(n+m)}\Gamma(n-h+1)\Gamma(m-h+1)}{\Gamma(1-h)^2}$$
$$+ \sum_{n,m\ge 0}\frac{(-1)^{n+m}}{(n!)^2(m!)^2}(t-t_0)^{n+m}(s-s_0)^{n+m}\frac{1}{N}\,\frac{(n+m)!\lambda^{-(n+m)}\Gamma(n+m-h+1)}{\Gamma(1-h)^2}$$
$$:= T^{nd}(t,s,t_0,s_0) + T^d(t,s,t_0,s_0).$$

The first sum correspond to the non-diagonal term while the first sum corresponds to the diagonal term. Let us compute the non diagonal term.

$$T^{nd} = \frac{N(N-1)}{N^2}\left(\sum_{n\ge 0}\frac{(-1)^n\lambda^{-n}}{n!}(t-t_0)^n(s-s_0)^n\frac{\Gamma(n-h+1)}{\Gamma(1-h)}\right)^2$$
$$= \frac{N(N-1)}{N^2}\left(1+\frac{(t-t_0)(s-s_0)}{\lambda}\right)^{2h-2}$$

by (14). Obviously

$$\int_0^t dt_0 \int_0^s ds_0 T^{nd}(t,s,t_0,s_0) \le ts.$$

The diagonal term can be expressed as

$$
\begin{aligned}
T^d(t,s,t_0,s_0) &= \frac{1}{N}\sum_{n,m\geq 0}\frac{(-1)^{n+m}\lambda^{-(n+m)}}{n!m!}C^n_{n+m}(t-t_0)^{n+m}(s-s_0)^{n+m} \\
&\quad\times\frac{\Gamma(n-h+1)\Gamma(m-h+1)}{\Gamma(1-h)} \\
&= \frac{1}{N}\sum_{n\geq 0}\frac{1}{n!}\sum_{k=n}^{\infty}\frac{(-1)^k\lambda^{-k}}{(n-k)!}(t-t_0)^k(s-s_0)^k C^n_k\frac{\Gamma(k-h+1)}{\Gamma(1-h)} \\
&= \frac{1}{N}\sum_{k\geq 0}(-1)^k\lambda^{-k}(t-t_0)^k(s-s_0)^k\frac{\Gamma(k-h+1)}{\Gamma(1-h)}\sum_{n=0}^{k}\frac{1}{n!(n-k)!}C^n_k.
\end{aligned}
$$

and as in the proof of Lemma 1,

$$
\int_0^t dt_0\int_0^s ds_0 T^d(t,s,t_0,s_0)
$$

is less than ts.

Next, we will prove the weak convergence of Y_N to Y^λ in the space of continuous functions $C([a,b]\times[c,d])$ where a, b, c, d are as in the statement. Clearly the L^2 convergence proved above implies the convergence of finite dimensional distributions of Y_N to those of Y^λ as $N\to\infty$. We need to show that Y_N and Y^λ have continuous paths and the family $(Y_N)_N$ is tight. Let us notice that the process Y^λ has continuous paths. Indeed, for every $u,v\geq 0$

$$
\begin{aligned}
&\mathbf{E}\left|Y^\lambda(t+u,s+v)-Y^\lambda(t+u,s)-Y^\lambda(t,s+v)+Y^\lambda(t,s)\right|^2 \\
&= \int_t^{t+u}\int_s^{s+v}\left(\frac{\lambda}{\lambda+(t-t_0)(s-s_0)}\right)^{3-2H}dt_0 ds_0\leq uv
\end{aligned}
$$

and the continuity of Y^λ follows from the Kolmogorov criterium for two-parameter processes (see e.g. [1]).

It remains to show that Y_N has continuous paths for every $N\geq 1$ and the family $(Y_N)_{N\geq 1}$ is tight in $C([a,b]\times[c,d])$. Both claims will follow from the following calculations. Let $x,y>0$. We will estimate the L^p norm of the rectangular increment

$$
\begin{aligned}
&Y_N(t+x,s+y)-Y_N(t+x,s)-Y_N(t,s+y)+Y_N(t,s) \\
&\quad= \frac{1}{N}\sum_{k=1}^{N}X^k_{t+x,s+y}-X^k_{t+x,s}-X^k_{t,s+x}+X^k_{t,s} \\
&\quad= -\frac{1}{N}\sum_{k=1}^{N}\alpha_k\int_t^{t+x}\int_s^{s+y}X^k_{u,v}dvdu+W_{t+x,s+y}-W_{t+x,s}-W_{t,s+y}+W_{t,s}.
\end{aligned}
$$

Then

$$\begin{aligned}
&\mathbf{E}\left|Y_N(t+x,s+y)-Y_N(t+x,s)-Y_N(t,s+y)+Y_N(t,s)\right|\\
&\quad\le \frac{1}{N}\left(\mathbf{E}_\alpha|\alpha_k|^2\right)^{\frac12}\int_t^{t+x}\int_s^{s+y}\left(\mathbf{E}|X^k_{u,v}|^2\right)^{\frac12}dvdu\\
&\quad+\mathbf{E}|W_{t+x,s+y}-W_{t+x,s}-W_{t,s+y}+W_{t,s}|\le C(xy+(xy)^{\frac12})
\end{aligned}$$

using Lemma 1 and the Hölder continuity of the Brownian sheet W, where the constant C may depend on a, b, c, d. Since for every Gaussian random variable X one has $\|X\|_{L^2}=\sqrt{\frac{\pi}{2}}\|X\|_{L^1}$ and since Y_N is Gaussian, we obtain that

$$\begin{aligned}
&\mathbf{E}\left|Y_N(t+x,s+y)-Y_N(t+x,s)-Y_N(t,s+y)+Y_N(t,s)\right|^p\\
&\le C_p\mathbf{E}|Y_N(t+x,s+y)-Y_N(t+x,s)-Y_N(t,s+y)+Y_N(t,s)|\le C_p(xy)^{\frac{p}{2}}.
\end{aligned}$$

The above inequality, together with a tightness criterium for the two-parameter processes (see e.g. [3]) will give the conclusion. □

Remark 3. We notice that, in contrast to the one-parameter case, the convergence above holds in $L^2(\Omega)=L^2(\Omega_W\times\Omega_\alpha)$. In the one-parameter case, $\mathbf{E}_W\mathbf{E}_\alpha Y_N(t)^2=\infty$ for every $t\ge 0$ (see Remark 2 in [8]).

3. Properties of the Gamma-mixed Ornstein–Uhlenbeck sheet

Recall (see [8]) that the (one-parameter) Gamma-mixed Ornstein–Uhlenbeck process (1) converges to the Brownian motion as $\lambda\to\infty$ and its integrated renormalized process (2) converges to the fractional Brownian motion. Moreover, the process (1) is stationary and almost self-similar. We will investigate these properties in the two-parameter case. Note that the Wiener integral in (15) is defined only on square of the positive real line while in (1) the negative line is also included. These will lead to some new situations in the two-parameter case. One losses the stationarity but we will have the self-similarity property. Also, the integrated process will convergence, in the two-parameter case, to a variant of the fractional Brownian sheet called Liouville fractional Brownian sheet.

We first remark the following.

Proposition 2. *Let Y^λ be given by (15). Then for every $a,b>0$*

$$\left(Y^\lambda(at,bs)\right)_{t,s\ge 0}=^{(d)}\left((ab)^{\frac12}Y^{\frac{\lambda}{ab}}(t,s)\right)_{t,s\ge 0}$$

Proof. Fix $a, b > 0$ and recall that the Wiener sheet W satisfies the following self-similarity property: $(W_{at,bs})_{t,s} =^{(d)} ((ab)^{\frac{1}{2}} W_{t,s\geq 0})_{t,s\geq 0}$ ("$=^{(d)}$" means the equivalence of finite dimensional distributions). Then

$$\begin{aligned} Y^{\lambda}(at, bs) &= \lambda^{\frac{3}{2}-H} \int_{-\infty}^{at} \int_{-\infty}^{bs} dW_{t_0,s_0} \left(\lambda + (at - t_0)(bs - s_0)\right)^{H-\frac{3}{2}} \\ &= \lambda^{\frac{3}{2}-H} \int_0^t \int_0^s dW_{at_0,bs_0} \left(\lambda + (at - at_0)(bs - bs_0)\right)^{H-\frac{3}{2}} \\ &=^{(d)} \lambda^{\frac{3}{2}-H} (ab)^{H-1} \int_0^t \int_0^s dW_{t_0,s_0} \left(\frac{\lambda}{ab} + (t - t_0)(s - s_0)\right)^{H-\frac{3}{2}} \\ &= (ab)^{\frac{1}{2}} Y^{\frac{\lambda}{ab}}(t, s). \end{aligned}$$

□

The result in Proposition 2 suggests that when λ is close to infinity the process (15) is self-similar of order $\frac{1}{2}$ as the Wiener sheet W. It is therefore natural to expect the convergence of Y^{λ} to the Brownian sheet when $\lambda \to \infty$.

Proposition 3. *For every $a, b, c, d \in \mathbb{R}_+$, $a < b, c < d$*

$$\sup_{(t,s)\in[a,b]\times[c,d]} \mathbf{E} \left| Y^{\lambda}(t, s) - W_{t,s} \right|^2 \to_{\lambda\to\infty} 0.$$

Proof. Indeed, for every $(t, s) \in [a, b] \times [c, d]$

$$\mathbf{E} \left| Y^{\lambda}(t, s) - W_{t,s} \right|^2 = \int_0^t \int_0^s ds_0 dt_0 \left[\left(\frac{\lambda}{\lambda + (t - t_0)(s - s_0)} \right)^{\frac{3}{2}-H} - 1 \right]^2$$

and the conclusion follows since $\left(\frac{\lambda}{\lambda+(t-t_0)(s-s_0)}\right) \to 1$ as $\lambda \to \infty$ (for fixed t, s, t_0, s_0 and

$$\left| \left(\frac{\lambda}{\lambda + (t - t_0)(s - s_0)} \right) \right| \leq 1.$$

□

Remark 4. Using the result in Proposition 3, the weak convergence of Y^{λ} to W in $C([a, b] \times [c, d])$ can be also obtained as in the proof of Theorem 1.

Remark 5. We remark first that the process Y^{λ} is not stationary. On the other hand, the process given by

$$Y^{',\lambda}(t, s) = \int_{-\infty}^t \int_{-\infty}^s \left(\frac{\lambda}{\lambda + (t - t_0)(s - s_0)} \right)^{\frac{3}{2}-H} dW_{t_0,s_0}$$

is stationary (here we allow the integration domain to contain negative values). Indeed, for every $s, t, u, v \geq 0$, $t > u, s > v$ and for every $a, b \geq 0$ we have

$$\begin{aligned}
&\mathbf{E}Y^{',\lambda}(t+a, s+b)Y^{',\lambda}(u+a, v+b) \\
&\quad = \int_{-\infty}^{u+a}\int_{-\infty}^{v+b}\left(\frac{\lambda}{\lambda+(t+a-t_0)(s+b-s_0)}\right)^{\frac{3}{2}-H} \\
&\quad \times \left(\frac{\lambda}{\lambda+(u+a-t_0)(v+b-s_0)}\right)^{\frac{3}{2}-H} dt_0 ds_0 \\
&\quad = \int_{-\infty}^{u}\int_{-\infty}^{v}\left(\frac{\lambda}{\lambda+(t-t_0)(s-s_0)}\right)^{\frac{3}{2}-H}\left(\frac{\lambda}{\lambda+(u-t_0)(v-s_0)}\right)^{\frac{3}{2}-H} dt_0 ds_0 \\
&\quad = \mathbf{E}Y^{',\lambda}(t, s)Y^{',\lambda}(u, v)
\end{aligned}$$

Next we will prove that the integrated process of Y^λ converges to a variant of the fractional Brownian sheet called Liouville fractional Brownian sheet. Several extensions of the fractional Brownian motion have been proposed in the literature. This includes for example the *fractional Brownian field* ([4]), the *Lévy's fractional Brownian field* ([5]) and the *anisotropic fractional Brownian sheet* ([9], [1]), which we consider in this paper.

We begin with the definition of the anisotropic Liouville fractional Brownian sheet.

Definition 2. Let $\alpha, \beta \in (0, 1)$. The Liouville fractional Brownian sheet with Hurst parameters α, β is defined by

$$W^{\alpha,\beta}(t, s) = \int_0^t \int_0^s (t-t_0)^{\alpha-\frac{1}{2}}(s-s_0)^{\beta-\frac{1}{2}} dW_{t_0,s_0}, \quad t, s \geq 0. \tag{16}$$

Let us also consider the integrated process

$$Z^\lambda(t, s) = \lambda^{H-\frac{3}{2}} \int_0^t \int_0^s Y^\lambda(u, v) dv du, \quad t, s \geq 0. \tag{17}$$

Proposition 4. *For every $s, t \geq 0$, $Z^\lambda(t, s)$ converges in $L^2(\Omega)$ as $\lambda \to 0$ to the random variable $W^{H,H}$. Also, for every $0 \leq a < b, 0 \leq c < d$ the family of stochastic processes $(Z^\lambda, \lambda > 0)$ converges weakly in $C([a, b] \times [c, d])$ $(a, b, c, d > 0$, $a < b, c < d)$ to the Liouville fractional Brownian sheet $W^{H,H}$ multiplied by the constant $\left(H - \frac{1}{2}\right)^{-2}$.*

PROOF. By interchanging the order of integration (this can be argued as in the proof of Proposition 1, since Y^λ is in L^2 and $\int_0^t\int_0^s (Y^\lambda_{u,v})^2 dudv < \infty$ for every t, s, which can be seen from the below computations), we can write

$$Z^\lambda(t,s) = \int_0^t\int_0^s dW_{t_0,s_0}\int_{t_0}^t du \int_{s_0}^s dv\,(\lambda + (u-t_0)(v-s_0))^{H-\frac{3}{2}}$$

and the L^2 follows from the trivial convergence as $\lambda \to 0$ (for fixed t, s, t_0, s_0) of

$$\int_{t_0}^t du \int_{s_0}^s dv\,(\lambda + (u-t_0)(v-s_0))^{H-\frac{3}{2}}$$

to $(H-\frac{1}{2})^{-2}(t-t_0)^{H-\frac{1}{2}}(s-s_0)^{H-\frac{1}{2}}$ (then we can easily apply the dominated convergence theorem).

The convergence of the finite dimensional distributions of Z^λ to the finite dimensional distributions of $W^{H,H}$ is a consequence of the L^2 convergence of $Z^\lambda(t,s)$ to $W^{H,H}(t,s)$ for every s, t. It suffices to show that the family $(Z^\lambda, \lambda > 0)$ is tight in $C([a,b]\times[c,d])$ and this follows since for every $x, y > 0$

$$\begin{aligned}
&\mathbf{E}\left|Z^\lambda(t+x,s+y) - Z^\lambda(t+x,s) - Z^\lambda(t,s+y) + Z^\lambda(t,s)\right|^2\\
&\mathbf{E}\left|\int_t^{t+x} du\int_s^{s+y} dv \int_0^u\int_0^v dW_{t_0,s_0}(\lambda + (u-t_0)(v-s_0))^{H-\frac{3}{2}}\right|^2\\
= &\int_0^{t+x} dt_0 \int_0^{s+y} ds_0 \left(\int_{t_0\vee t}^{t+x} du \int_{s_0\vee s}^{s+y} dv(\lambda + (u-t_0)(u-s_0))^{H-\frac{3}{2}}\right)^2\\
\leq &\int_0^{t+x} dt_0 \int_0^{s+y} ds_0 \left(\int_{t_0\vee t}^{t+x} du \int_{s_0\vee s}^{s+y} dv((u-t_0)(v-s_0))^{H-\frac{3}{2}}\right)^2\\
= &K\int_0^{t+x} dt_0 \int_0^{s+y} ds_0\\
&\times\left(\left[(t+x-t_0)^{H-\frac{1}{2}} - (t\vee t_0 - t_0)^{H-\frac{1}{2}}\right]\left[(s+y-s_0)^{H-\frac{1}{2}} - (s\vee s_0 - s_0)^{H-\frac{1}{2}}\right]\right)^2\\
\leq &K(xy)^{2H-1}
\end{aligned}$$

where the constant K depends on H, a, b, c, d. Thus

$$\mathbf{E}\left|Z^\lambda(t+x,s+y) - Z^\lambda(t+x,s) - Z^\lambda(t,s+y) + Z^\lambda(t,s)\right|^2 \leq C_p(xy)^{p(H-\frac{1}{2})}$$

and the conclusion is obtained by using the fact that Z^λ is Gaussian, $H > \frac{1}{2}$ and by using the criterium in [3] for the tightness of multiparameter stochastic processes. $\square$

Remark 6. It is also possible to construct variants of the Gamma-mixed Ornstein–Uhlenbeck sheet (15) that converges as $\lambda \to \infty$ to the Wiener sheet and as $\lambda \to 0$ to the Liouville fractional Brownian sheet with Hurst parameters $H_1, H_2 \in \left(\frac{1}{2}, 1\right)$ as defined in (16) . Indeed, for $t, s \geq 0$, consider

$$Y^{\lambda,H_1,H_2}(t,s) = \int_0^t \int_0^s \left(\frac{\lambda}{\lambda + t - u}\right)^{\frac{3}{2}-H_1} \left(\frac{\lambda}{\lambda + s - v}\right)^{\frac{3}{2}-H_2} dW_{u,v} \tag{18}$$

where $(W_{u,v})_{u,v\in\mathbb{R}}$ is a Brownian sheet.

We can show that, for every t, s the sequence $Y^{\lambda,H_1,H_2}(t,s)$ converges in $L^2(\Omega)$ as $\lambda \to \infty$ to the Brownian sheet $W_{t,s}$ and the integral process

$$Z^{\lambda,H_1,H_2}(t,s) = \lambda^{H_1-\frac{3}{2}}\lambda^{H_2-\frac{3}{2}} \int_0^t \int_0^s Y^{\lambda,H_1,H_2}(u,v)dudv$$

converges in $L^2(\Omega)$ for every t, s to $\left(H_1 - \frac{1}{2}\right)^{-1}\left(H_2 - \frac{1}{2}\right)^{-1} W^{H_1,H_2}(t,s)$ with W^{H_1,H_2} given by (16). On the other hand, the process (18) cannot be easily interpreted as limit of aggregated Ornstein–Uhlenbeck processes.

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