Publ. Math. Debrecen
45 / 1-2 (1994), 153–158

The equation $SL(n,K) = C^4 \cup Z$ for $n = 2,3,4;\ K = \mathbb{R},\mathbb{C}$

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Abstract. In the paper it has been proved that $SL(n,K) = C^4 \cup Z$ for $n = 2,3,4;\ K = \mathbb{R},\mathbb{C};\ PSL(2,\mathbb{R}) = C^4$ and $PSL(n,K) = C^8$ $(K = \mathbb{R},\mathbb{C});\ n = 3,4$, where C denotes noncentral conjugacy class.

In the paper [3, IV] it has been proved that $G = PSL(2,R) = C^s$ for some $s \geq 3$, for all conjugacy classes $C \neq \{1\}$ of G.

In this paper we shall prove that $SL(n,K) = C^4 \cup Z$ for $n = 2,3,4;$ $K = \mathbb{R},\mathbb{C}$.

By above results it follows that $PSL(2,\mathbb{R}) = C^4$ and $PSL(n,K) = C^8$ for $n = 3,4;\ K = \mathbb{R},\mathbb{C}$.

The following notations will be used.
C_V denotes the conjugacy class of the matrix V, $\mathbb{R}$ — the field of real numbers, $\mathbb{C}$ — the field of complex numbers and Z denotes the center of a group G. The remaing notations are standard.

The following lemmas will be used.

Lemma 1 (see [1])**.** *If $V = \operatorname{diag}(v_1,\dots,v_n)$, $W = \operatorname{diag}(w_1,\dots,w_n)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$ and $V, W \in SL(n,K)$, then $SL(n,K) = C_V C_W \cup Z$.*

From Lemma 1 it follows that in the special case $W = V$ and V, V^{-1} are similar, $PSL(n,K) = C_V^2$ (see [1]).

If all eigenvalues of V are not distinct, the equality $PSL(n,K) = C_V^2$ does not necessarily hold. This problem was investigated by a few authors, chiefly by J. L. Brenner (see his papers cited in the book [4] and his reviews: Math. Rev. 1987h, 20001 and Zbl. 561, 20004, 1985).

Lemma 2. *If $A \in GL(2,K)$ $(K = \mathbb{R}, \mathbb{C})$, $A \notin Z$, then there exists $S, T \in SL(2,K)$ such that the eigenvalues λ_1, λ_2 of $A^S A^T$ are distinct and $\lambda_i \neq k$, where k is an arbitrary number $\neq 0$.*

Proof. Let $N = A^P = P^{-1}AP$ denotes the rational canonical form of A in the group $GL(n,K)$.

Note that if all eigenvalues of NN^X are distinct, where $X \in SL(n,K)$, then all eigenvalues of $A^{PU}A^{PXU}$ also are distinct and we can choose the matrix U such that $\det(PU) = \det(PXU) = 1$.

Therefore it suffices to prove that there exists $X \in SL(n,K)$ such that all eigenvalues of NN^X are distinct.

Any noncentral matrix $A \in GL(2,K)$ is similar to the matrix

$$N = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}.$$

For N and $X = \begin{bmatrix} 0 & 1 \\ -1 & x \end{bmatrix}$,

$$\left|NN^X - \lambda E\right| = \lambda^2 - s\lambda + t, \tag{1}$$

where

$$s = ax^2 + b(1-a)x - a^2 - 1, \quad t = a^2.$$

It is clear that there are infinitely many $x \in K$ such that $\Delta = s^2 - 4t > 0$ or $\Delta \neq 0$, respectively to the case $K = \mathbb{R}$ or $K = \mathbb{C}$. Hence there are infinitely many matrices X such that all eigenvalues of NN^X are distinct.

The second assertion follows from the fact that the equation $(1) = 0$ in unkown x has at most two solution for given λ.

Lemma 3. *If $A \in GL(3,K)$ $(K = \mathbb{R}, \mathbb{C})$, $A \notin Z$, then there exists $X \in SL(3,K)$ such that the eigenvalue $\lambda_1, \lambda_2, \lambda_3$ of AA^X are mutualy distinct and $\lambda_i \neq k$, where k is an arbitrary number $\neq 0$.*

Proof. Any noncentral matrix $A \in GL(3,K)$ is similar by a matrix of determinant equal to an arbitrary number $\neq 0$ to the one of the following matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ a & b & 0 \\ 0 & 0 & c \end{bmatrix}. \tag{2}$$

For A_1 and $X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & x & x\alpha \end{bmatrix}$ (α–parameter),

$$\left|A_1 A_1^X - \lambda E\right| = -\lambda^3 + s\lambda^2 - t\lambda + r, \tag{3}$$

where

$$s=-\,a\alpha^2x^2+x(c\alpha+ac\alpha+1-a)+b+ab\,,$$
$$t=-\,ax^2+x(b+ab-a^2\alpha+a\alpha)-c-ac\,,\quad r=a^2\,.$$

It is clear that there are infinitely many $x\in K$ such that for $|x|$ large enough,

$$D=27q^2+4p^3=-s^2t^2+4t^3+4s^3r-18str+27r^2<0$$

or $D\neq 0$, respectively to the case $K=\mathbb{R}$ or $K=\mathbb{C}$. Hence there are infinitely many matrices X such that all eigenvalues of $A_1A_1^X$ are different.

For A_2 and

$$X=\begin{bmatrix}0&1&0\\0&\beta x&1\\1&x&0\end{bmatrix}\quad(\beta-\text{parameter})\,,$$

(4) $$\left|A_2A_2^X-\lambda E\right|=-\lambda^3+s\lambda^2-t\lambda+r\,,$$

where

$$s=-\,a\beta x^2+(b\beta+ac-\beta c)x+cb\,,$$
$$t=-\,ac^2\beta x^2+ac(-\beta+bc+a)x-abc\,,\quad r^2=a^2c^2\,.$$

It is clear that there are infinitely many $x\in K$ such that $D<0$ or $D\neq 0$ for $|x|$ large enough, respectively to the case $K=\mathbb{R}$ or $K=\mathbb{C}$. Thus there exist infinitely many matrices X such that all eigenvalues of $A_2A_2^X$ are distinct.

A simple calculation shows that we can choose the parameters α,β such that each equation $(3)=0$, $(4)=0$ has at most two solutions for given λ. This proves the second assertion of Lemma 3.

Lemma 4. *If $A\in SL(4,K)$ $(K=\mathbb{R},\mathbb{C})$ $A\notin Z$, then there exists $S,T\in SL(4,K)$ such that all eigenvalues of A^SA^T are distinct.*

Proof. As in the proof of Lemma 2 it suffices to prove that there exists $X\in SL(4,K)$ such that all eigvalues of NN^X are distinct, where N denotes the rational canonical form of A in the group $GL(4,K)$.

Any noncentral matrix $A \in SL(4,K)$ is similar to the one of following matrices:

$$N_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & a & b & c \end{bmatrix}; \quad N_2 = \begin{bmatrix} A_i & 0 \\ 0 & d \end{bmatrix}, \quad A_i \in (2),\ i = 1,2;$$

$$N_3 = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 0 & 1 \\ a_i & b_i \end{bmatrix}.$$

For N_1 and $X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & y & 0 \end{bmatrix}$,

$$\left| N_1 N_1^X - \lambda E \right| = (\lambda^2 - y\lambda + 1)[\lambda^2 - (2b - y)\lambda + 1]\,. \tag{6}$$

It is clear that there exists $y \in K$ such that the polynomial (6) has four distinct roots.

From Lemma 3 it follows that for A_i there exists $Y \in SL(3,K)$ such all eigenvalues of $A_i A_i^Y$ are distinct which can be chosen different from d^2. Hence there exists the matrix

$$X = \begin{bmatrix} Y & 0 \\ 0 & 1 \end{bmatrix}$$

such that all eigenvalues of $N_2 N_2^X$ are distinct.

By Lemma 2 it follows that for C_i $(i = 1,2)$ there exist $Y, U \in SL(2,K)$ such that the eigenvalues v_1, v_2 and w_1, w_2 of $C_1 C_1^Y$ and $C_2 C_2^U$, respectively are different and that Y and U can be chosen such that $v_i \neq w_j$. Hence there exists the matrix

$$X = \begin{bmatrix} Y & 0 \\ 0 & U \end{bmatrix},$$

such that all eigenvalues of $N_3 N_3^X$ are distinct.

Theorem 1. *If C is any noncentral conjugacy class of $SL(2,\mathbb{R})$, then $SL(2,\mathbb{R}) = C^4 \cup Z$.*

Proof. In the set $A^S A^T$, where $A, S, T \in SL(2,\mathbb{R})$, there exists the matrix B with two distinct eigenvalues v, v^{-1}, by Lemma 2.

The matrix B is similar in the group $SL(2,\mathbb{R})$ to the matrix $V = \operatorname{diag}(v, v^{-1})$. Hence

$$C_V = C_B \subseteq C_A^2 \quad \text{and} \quad C_V^2 \subseteq C_A^4\,.$$

By Lemma 1 it follows that $SL(2,\mathbb{R}) - Z \subseteq C_V C_V \subseteq C_A^4$ for any $A \notin Z$. Hence $SL(2,\mathbb{R}) \subseteq C_A^4 \cup Z$. The inverse inclusion is obvious.

Corollary 1.1. *If C is any nonidentity conjugacy class of $PSL(2,\mathbb{R})$, then $PSL(2,\mathbb{R}) = C^4$.*

Proof. The matrices $V = \mathrm{diag}(v, v^{-1})$ and V^{-1} are similar in the group $PSL(2,\mathbb{R})$, so $Z \subseteq C_V C_V \subseteq C^4$.

Note that $PSL(2,\mathbb{C}) = C^2$. It follows by ([2] or Lemma 1) and the fact that the equation $A = V^X V^Y$ in unknowns $X, Y \in SL(2,\mathbb{C})$, where $V = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $A = \mathrm{diag}(a, a^{-1})$ $(a \neq 1)$ or $A = V$, has a solution. The proof given in [4] is wrong (see p. 242, row 2).

Theorem 2. *If $K = \mathbb{R},\mathbb{C}$ and C – noncentral conjugacy class of $SL(3,K)$, then $SL(3,K) = C^4 \cup Z$.*

Proof. In the set AA^X, where $A, X \in SL(3,K)$, $A \notin Z$, there exists the matrix B with all distinct eigenvalues v_1, v_2, v_3, by Lemma 3. The matrix B is similar to the diagonal matrix $V = \mathrm{diag}(v_1, v_2, v_3)$ in $SL(3,K)$ i. e. $V^T = B$. Hence

$$C_V = C_B \subseteq C_A^2 \quad \text{and} \quad C_V^2 \subseteq C_A^4\,.$$

From Lemma 1 it follows that

$$SL(3,K) - Z \subseteq C_V^2 \subseteq C_A^4, \ \text{ so } \ SL(3,K) \subseteq C^4 \cup Z\,.$$

Thus $SL(3,K) = C_A^4 \cup Z$ for any $A \notin Z$.

Theorem 3. *If C is any noncentral conjugacy class of $SL(4,K)$ and $K = \mathbb{R},\mathbb{C}$, then $SL(4,K) = C^4 \cup Z$.*

The proof is the same as in Theorem 2 but instead of Lemma 3, Lemma 4 is used.

Corollary 2.1. *If C is any noncentral conjugacy class of $PSL(3,K)$, then $PSL(3,K) = C^8$.*

Proof. From our proof of Lemma 3 it follows that the matrix E does not necessarily belongs to the set C^4. But it is clear that the diagonal matrices $V = \mathrm{diag}(1, v, v^{-1})$ $(v \neq \pm 1)$ and V^{-1} belong to the set C^4, so $Z \subseteq C_V C_V \subseteq C^8$. Hence $PSL(3,K) = C^8$.

In the same way we can prove the following

Corollary 3.1. *If C is any nonidentity conjugacy class of $PSL(4,K)$, then $PSL(4,K) = C^8$.*

Note that our method do not permit to state whether the exponent of C is minimal.

The method of proofs of Lemmas 2–4 suggests the conjecture: if $K = \mathbb{R},\mathbb{C}$, then $SL(n,K) = C^4 \cup Z$ for any $n \geq 2$.

Remark. One of reviewers of my paper informed the editor that recently he received a paper (which probably did not apper yet) of Mr. Arieh Lev: "The covering number of the group $PSL_n(F)$" in which it has been proved the following two theorems:

Theorem I. Let G be the group $PSL_n(F)$, where $n \geq 3,\ F$ is a field and $|F| \geq 4$. Denote by $\mathrm{cn}(G)$ the minimal value of k such that $C^k = G$ for every nontrivial conjugacy class C of G. Then $\mathrm{cn}(G) = n$.

Theorem II. Let C be a nonscalar SL_n — conjugacy class of $GL_n(F)$, where $n \geq 3,\ F$ is a field and $|F| \geq 4$. Let $\boldsymbol{M} = \{M \in GL_n(F) - Z(GL_n(F)) | \det M = (\det C)^n\}$. Then $\boldsymbol{M} \subseteq C^n$. In particular, if $\det C$=1, then $C^n \supseteq SL_n(F) - Z(SL_n(F))$.

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(Received June 1, 1993; revised September 10, 1993)