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On a conjecture about repdigits in k-generalized Fibonacci sequences

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Abstract. The k-generalized Fibonacci sequence $(F_n^{(k)})_n$ resembles the Fibonacci sequence in that it starts with $0, \dots, 0, 1$ (a total of k terms) and each term afterwards is the sum of the k preceding terms. F. Luca [4] in 2000 and recently D. Marques [5] proved that 55 and 44 are the largest numbers with only one distinct digit (so called *repdigits*) in the sequences $(F_n^{(2)})_n$ and $(F_n^{(3)})_n$, respectively. Further, Marques conjectured that there are no repdigits having at least 2 digits in a k-generalized Fibonacci sequence for any $k > 3$. In the present paper, we confirm this conjecture.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the *Fibonacci sequence* given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. In 2000, F. Luca [4] proved that $F_{10} = 55$ is the largest number with only one distinct digit (called *repdigit*) in the Fibonacci sequence. The *Tribonacci* sequence $(T_n)_{n\geq -1}$ is like the sequence of Fibonacci numbers except that it starts as $T_{-1} = 0,\ T_0 = 0,\ T_1 = 1$ and each term afterwards is the sum of the preceding three terms.

Recently, D. Marques [5] looked for repdigits in the Tribonacci sequence and proved that $T_8 = 44$ is the largest such. Given an integer $k \geq 2$, we

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look at the similar problem for the terms of the *k-generalized Fibonacci sequence* $(F_n^{(k)})_{n\geq -(k-2)}$ given by

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all } n \geq 2, \tag{1}$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$.

Clearly, for $k = 2$ we have $F_n^{(2)} = F_n$, our familiar Fibonacci numbers, while for $k = 3$, we have $F_n^{(3)} = T_n$, the Tribonacci numbers.

Below we present the values of these numbers for the first few values of k and $n \geq 1$.

k	Name	First non-zero terms
2	Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34, $\underline{55}$, 89, 144, 233, 377, 610, ...
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, $\underline{44}$, 81, 149, 274, 504, 927, 1705, ...
4	Tetranacci	1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, ...
5	Pentanacci	1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, ...
6	Hexanacci	1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, ...
7	Heptanacci	1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, ...
8	Octanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, ...
9	Nonanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 511, 1021, 2040, 4076, ...
10	Decanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1023, 2045, 4088, ...

The following conjecture was formulated in [5].

Conjecture 1. *The only solutions of the Diophantine equation*

$$F_n^{(k)} = a \cdot \left(\frac{10^\ell - 1}{9}\right) \tag{2}$$

in positive integers n, k, a, ℓ *with* $k \geq 2$, $1 \leq a \leq 9$ *and* $\ell \geq 2$, *are*

$$(n, k, a, \ell) \in \{(10, 2, 5, 2), (8, 3, 4, 2)\}.$$

Here, we confirm Conjecture 1. We record the result as follows.

Theorem 1. *Conjecture 1 holds.*

Our method is roughly as follows. We use lower bounds for linear forms in logarithms of algebraic numbers to bound n and ℓ polynomially in terms of k. When k is small, the theory of continued fractions suffices to lower such bounds and complete the calculations. When k is large, we use the fact that the dominant root of the k-generalized Fibonacci sequence is exponentially close to 2, so we can replace this root by 2 in our calculations with linear forms in logarithms and end up with an absolute bound for k; hence, an absolute bound for all k, ℓ and n, which we then reduce using again standard facts concerning continued fractions.

2. Preliminary inequalities

It is known that the characteristic polynomial of the k-generalized Fibonacci numbers $(F_n^{(k)})_n$, namely

$$\psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single root, which is located between $2(1-2^{-k})$ and 2 (see [7]). To simplify notation, in general we omit the dependence on k of α.

The following "Binet-like" formula for $F_n^{(k)}$ appears in Dresden [2]:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{3}$$

where $\alpha = \alpha_1, \ldots, \alpha_k$ are the roots of $\psi_k(x)$. It was proved in [2] that the contribution of the roots which are inside the unit circle to the formula (3) is very small, namely that the approximation

$$\left| F_n^{(k)} - \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \right| < \frac{1}{2} \quad \text{holds for all } n \geq 2 - k. \tag{4}$$

We will use the estimate (4) later.

For the Fibonacci sequence (namely, the case $k = 2$), it is well-known that

$$\alpha^{n-2} < F_n < \alpha^{n-1} \quad \text{holds for all } n \geq 3. \tag{5}$$

Here, the value of α is the golden section. The next result shows that the above inequality (5) holds for the k-generalized Fibonacci sequence as well.

Lemma 1. *The inequality*

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \tag{6}$$

holds for all $n \geq 1$.

Proof. We may assume that $k \geq 3$, since for $k = 2$ this is inequality (5). We prove the lemma by induction on n. We first prove that inequality (6) holds for the first k non-zero terms of the k-generalized Fibonacci sequence. Indeed, it is clear that the result is true for $n = 1$ because $\alpha > 1$, so we only need to show that

$$\alpha^i \leq F_{i+2}^{(k)} = 2^i \leq \alpha^{i+1}, \quad \text{for } 0 \leq i \leq k-2.$$

The left-hand side of the above inequality holds because $\alpha < 2$ while the right-hand side of it holds for $i = 0$ because $\alpha > 1$, so it suffices to prove that

$$2 < \alpha^{(i+1)/i} \quad \text{holds for } 1 \leq i \leq k-2. \tag{7}$$

Since the function $i \mapsto (i+1)/i$ is decreasing for $i \geq 1$, it suffices to prove that inequality (7) holds when $i = k-2$. Since $2(1-2^{-k}) < \alpha$, it follows that it is enough to prove that $2 < 2^{1+1/(k-2)}(1-2^{-k})^{(k-1)/(k-2)}$, which is equivalent to

$$-\frac{\log 2}{k-1} < \log(1-2^{-k}).$$

Since $\log 2 > 1/2$ and $\log(1-x) > -2x$ holds for all $x \in (0, 1/2)$, it suffices to show that

$$-\frac{1}{2(k-1)} \leq -2^{-k+1},$$

which is equivalent to $2^{k-2} \geq k-1$, which clearly holds for all $k \geq 2$. Thus, we have proved that inequality (6) holds for the first k non-zero terms of $(F_n^{(k)})_n$.

Now, suppose that (6) holds for all terms $F_m^{(k)}$ with $m \leq n-1$ for some $n > k$. It then follows from (1) that

$$\alpha^{n-3} + \alpha^{n-4} + \cdots + \alpha^{n-k-2} \leq F_n^{(k)} \leq \alpha^{n-2} + \alpha^{n-3} + \cdots + \alpha^{n-k-1},$$

therefore

$$\alpha^{n-k-2}(\alpha^{k-1} + \alpha^{k-2} + \cdots + 1) \leq F_n^{(k)} \leq \alpha^{n-k-1}(\alpha^{k-1} + \alpha^{k-2} + \cdots + 1),$$

which combined with the fact that $\alpha^k = \alpha^{k-1} + \alpha^{k-2} + \cdots + 1$ gives the desired result. Thus, inequality (6) holds for all positive integers n. $\square$

To conclude this section of preliminary inequalities, assume throughout that equation (2) holds. Since $10^{\ell-1} < F_n^{(k)} < 10^{\ell}$, we have $\ell - 1 < \log F_n^{(k)}/\log 10 < \ell$, so

$$\ell = \left\lfloor \frac{\log F_n^{(k)}}{\log 10} \right\rfloor + 1.$$

Moreover, from Lemma 1, we obtain

$$(n-2)\frac{\log \alpha}{\log 10} < \ell < (n-1)\frac{\log \alpha}{\log 10} + 1, \tag{8}$$

which is an estimate on ℓ in terms of n. We shall have some use for it later.

3. An inequality for n in terms of k

From now on, we assume that $k \geq 3$. Observe that for $k \geq 6$, the first $k-4$ terms which have at least 2 digits in the k-generalized Fibonacci sequence are powers of two, namely $F_6^{(k)} = 16, F_7^{(k)} = 32, \ldots, F_{k+1}^{(k)} = 2^{k-1}$. These numbers are not repdigits. Indeed, since $(10^\ell - 1)/9$ is odd for all $\ell \geq 2$, it follows that the exponent of 2 in $a(10^\ell - 1)/9$ is the same as the exponent of 2 in a, in particular it does not exceed 3. This shows that powers of 2 with at least two digits are not repdigits. Hence, $n > k+1$ when $k \geq 6$, and the same is true for $k = 3,\ 4$ and 5 also.

Using now (2) and (4), we get that

$$\left| \frac{a10^\ell}{9} - \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \right| < \frac{1}{2} + \frac{a}{9} \leq \frac{3}{2}. \tag{9}$$

Dividing both sides of the above inequality by the second term of the left-hand side, which is positive because $\alpha > 1$ and $2^k > k+1$, so

$$2 > (k+1)(2 - (2 - 2^{-k+1})) > (k+1)(2 - \alpha),$$

we obtain

$$\left| 10^\ell \cdot \alpha^{-(n-1)} \cdot \frac{a}{9} \left(\frac{2 + (k+1)(\alpha - 2)}{\alpha - 1} \right) - 1 \right| < \frac{6}{\alpha^{n-1}}, \tag{10}$$

where we used the facts $2 + (k+1)(\alpha - 2) < 2$ and $1/(\alpha - 1) < 2$, which are easily seen.

In order to prove Theorem 1, we shall use twice the following result of Matveev (see [6] or Theorem 9.4 in [1]).

Lemma 2. *Assume that $\gamma_1, \ldots, \gamma_t$ are positive numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right), \tag{11}$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{D h(\gamma_i), |\log \gamma_i|, 0.16\}, \quad \textit{for all } i = 1, \ldots, t.$$

In the above, for an algebraic number η we write $h(\eta)$ for its logarithmic height, given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right),$$

with d being the degree of η over $\mathbb{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X] \tag{12}$$

being the minimal primitive polynomial over the integers having positive leading coefficient a_0 and η as a root.

In a first application of Matveev's result Lemma 2, we take $t := 3$ and

$$\gamma_1 := 10, \qquad \gamma_2 := \alpha, \qquad \gamma_3 := \frac{a}{9}\left(\frac{2 + (k+1)(\alpha - 2)}{\alpha - 1}\right).$$

We also take $b_1 := \ell$, $b_2 := -(n-1)$ and $b_3 := 1$. Hence,

$$\Lambda := \gamma_1^{b_1} \cdot \gamma_2^{b_2} \cdot \gamma_3^{b_3} - 1. \tag{13}$$

The absolute value of Λ appears in the left-hand side of inequality (10). To see that $\Lambda \neq 0$, observe that imposing that $\Lambda = 0$ we get

$$\frac{a}{9} 10^{\ell} = \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1}.$$

Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\psi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i > 1$, we have

$$\frac{a}{9} 10^{\ell} = \left|\frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}\right|. \tag{14}$$

But the last equality above is not possible for $i \geq 2$ because

$$|2 + (k+1)(\alpha_i - 2)| \geq (k+1)|\alpha_i - 2| - 2 \geq k - 1 \geq 2 \quad \text{and } |\alpha_i - 1| < 2, \tag{15}$$

because $|\alpha_i| < 1$. Hence, we get that the right-hand side of (14) is at most 1, whereas its left-hand side is $\geq 100/9$, which is a contradiction. Thus, $\Lambda \neq 0$.

The algebraic number field containing $\gamma_1, \gamma_2, \gamma_3$ is $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := k$. Since $h(\gamma_1) = \log 10 = 2.302585\ldots$, we can take $A_1 := 2.31k > kh(\gamma_1)$. Further, since $h(\gamma_2) = (\log\alpha)/k < (\log 2)/k = (0.693147\ldots)/k$, we can take $A_2 := 0.7$.

We now need to estimate $h(\gamma_3)$. Observe that

$$h(\gamma_3) \leq \log 9 + h\left(\frac{2+(k+1)(\alpha-2)}{\alpha-1}\right) = \log 9 + h\left(\frac{\alpha-1}{2+(k+1)(\alpha-2)}\right). \quad (16)$$

Put

$$f_k(x) = \prod_{i=1}^{k}\left(x - \frac{\alpha_i - 1}{2+(k+1)(\alpha_i-2)}\right) \in \mathbb{Q}[x].$$

Then the leading coefficient a_0 of the minimal polynomial of

$$\frac{\alpha-1}{2+(k+1)(\alpha-2)}$$

over the integers (see definition (12)) divides $\prod_{i=1}^{k}(2+(k+1)(\alpha_i-2))$. But,

$$\begin{aligned}\left|\prod_{i=1}^{k}(2+(k+1)(\alpha_i-2))\right| &= (k+1)^k\left|\prod_{i=1}^{k}\left(2-\frac{2}{k+1}-\alpha_i\right)\right| \\ &= (k+1)^k\left|\psi_k\left(2-\frac{2}{k+1}\right)\right|.\end{aligned}$$

Since

$$|\psi_k(y)| < \max\{y^k, 1+y+\cdots+y^{k-1}\} < 2^k \quad \text{for all } 0<y<2,$$

it follows that

$$a_0 \leq (k+1)^k\left|\psi_k\left(2-\frac{2}{k+1}\right)\right| < 2^k\,(k+1)^k.$$

Hence,

$$\begin{aligned}h\left(\frac{\alpha-1}{2+(k+1)(\alpha-2)}\right) &= \frac{1}{k}\left(\log a_0 + \sum_{i=1}^{k}\log\max\left\{\left|\frac{\alpha_i-1}{2+(k+1)(\alpha_i-2)}\right|, 1\right\}\right) \\ &< \frac{1}{k}\left(k\log 2 + k\log(k+1)\right) = \log(k+1)+\log 2,\end{aligned}$$

where we used the facts

$$\left|\frac{\alpha_i-1}{2+(k+1)(\alpha_i-2)}\right| < 1 \quad \text{for all } i>1 \quad \text{and} \quad \left|\frac{\alpha-1}{2+(k+1)(\alpha-2)}\right| < 1,$$

which hold because $|2+(k+1)(\alpha_i-2)| \geq 2$ for $i=2,\ldots,k$ (see (15)), and $2+(k+1)(\alpha-2) \geq 1$, which is a straightforward exercise to check using the fact that $2(1-2^{-k}) < \alpha < 2$ and $k \geq 3$. Thus, from (16), we get that

$$h(\gamma_3) < \log(k+1) + \log 18.$$

So, we can take $A_3 := k\log(k+1)+3k$, because $\log 18 = 2.89037\ldots$. By recalling (8), we deduce $\ell < n$, so we can take $B := n-1$. Applying inequality (11) to get a lower bound for $|\Lambda|$ and comparing this with inequality (10), we get

$$\exp\left(-C_1(k) \times (1+\log(n-1))\,(2.31k)\,(0.7)\,(k\log(k+1)+3k)\right) < \frac{6}{\alpha^{n-1}},$$

where $C_1(k) := 1.4 \times 30^6 \times 3^{4.5} \times k^2 \times (1+\log k) < 1.5 \times 10^{11}\, k^2\,(1+\log k)$.

Taking logarithms in the above inequality, we have that

$$(n-1)\log\alpha - \log 6 < 2.43 \times 10^{11}\, k^4\,(1+\log k)\,(1+\log(n-1))\,(\log(k+1)+3),$$

which leads to

$$n-1 < 8 \times 10^{12}\, k^4\, \log^2 k\, \log(n-1),$$

where we used the facts $1+\log k \leq 2\log k$ for all $k \geq 3$, $1+\log(n-1) \leq 2\log(n-1)$ for all $n \geq 4$, $\log(k+1)+3 \leq 4\log k$ for all $k \geq 3$ and $1/\log\alpha < 2$.

Thus,

$$\frac{n-1}{\log(n-1)} < 8 \times 10^{12}\, k^4\, \log^2 k. \tag{17}$$

Since the function $x \mapsto x/\log x$ is increasing for all $x > e$, it is easy to check that the inequality

$$\frac{x}{\log x} < A \quad \text{yields} \quad x < 2A\log A,$$

whenever $A \geq 3$. Thus, taking $A := 8 \times 10^{12}\, k^4\, \log^2 k$, inequality (17) yields

$$\begin{aligned} n-1 &< 2\,(8 \times 10^{12}\, k^4\, \log^2 k)\, \log(8 \times 10^{12}\, k^4\, \log^2 k) \\ &< (1.6 \times 10^{13} k^4 \log^2 k)\,(30 + 4\log k + 2\log\log k) \\ &< 5.12 \times 10^{14} k^4 \log^3 k. \end{aligned}$$

In the last chain of inequalities, we have used that $30+4\log k+2\log\log k < 32\log k$ holds for all $k \geq 3$. Now, inserting the above upper bound for $n-1$ in the upper bound for ℓ from inequality (8), we get that $\ell < 2 \times 10^{14}\, k^4\, \log^3 k$, where we used the fact that $\log\alpha/\log 10 < \log 2/\log 10 < 1/3$. Let us record this calculation for future use.

Lemma 3. *If (n,k,a,ℓ) is a solution in positive integers of equation (2) with $k \geq 3$, then $n > k+1$ and both inequalities*

$$n < 6 \times 10^{14}\, k^4\, \log^3 k \quad \text{and} \quad \ell < 2 \times 10^{14}\, k^4\, \log^3 k$$

hold.

4. The case of small k

We next treat the cases when $k \in [3, 250]$. After finding an upper bound on n the next step is to reduce it. To do this, we use several times the following lemma, which is a variation of a result of Dujella and Pethő from [3].

Lemma 4. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\gamma q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-k},$$

in positive integers m, n and k with

$$m \le M \quad \text{and} \quad k \ge \frac{\log(Aq/\epsilon)}{\log B}.$$

Proof. The proof is completely analogous to that of Lemma 5 in [3]. We omit the details. □

In order to apply Lemma 4, we let

$$z := \ell \log 10 - (n-1)\log\alpha + \log\mu_a, \tag{18}$$

where $\mu_a := \gamma_3$. Then $e^z - 1 = \Lambda$, where Λ is given by (13). Therefore, (10) can be rewritten as

$$|e^z - 1| < \frac{6}{\alpha^{n-1}}. \tag{19}$$

Note that $z \ne 0$ since $\Lambda \ne 0$. Thus, we distinguish the following cases. If $z > 0$, then $e^z - 1 > 0$, so from (19) we obtain

$$0 < z < \frac{6}{\alpha^{n-1}},$$

where we used the fact that $x \le e^x - 1$ for all $x \in \mathbb{R}$. Replacing z in the above inequality by its formula (18) and dividing both sides of the resulting inequality by $\log\alpha$, we get

$$0 < \ell\left(\frac{\log 10}{\log\alpha}\right) - n + \left(1 + \frac{\log\mu_a}{\log\alpha}\right) < 12\cdot\alpha^{-(n-1)}, \tag{20}$$

where we have used again the fact that $1/\log\alpha < 2$. With

$$\hat{\gamma}_k := \frac{\log 10}{\log\alpha}, \quad \hat{\mu}_a := 1 + \frac{\log\mu_a}{\log\alpha}, \quad A := 12, \quad \text{and} \quad B := \alpha,$$

the above inequality (20) yields

$$0 < \ell\hat{\gamma}_k - n + \hat{\mu}_a < AB^{-(n-1)}. \tag{21}$$

It is clear that $\hat{\gamma}_k$ is an irrational number because $\alpha > 1$ is a unit in $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K}$. So α and 10 are multiplicatively independent.

For each $k \in [3, 250]$, we find a good approximation of α and a convergent p_k/q_k of the continued fraction of $\hat{\gamma}_k$ such that $q_k > 6M_k$, where $M_k := \lfloor 2 \times 10^{14}\, k^4 \log^3 k \rfloor$, which is an upper bound on ℓ from Lemma 3. After doing this, we use Lemma 4 on (21) in order to reduce our bound on n. Indeed, a computer search with Mathematica revealed that if $k \in [3, 250]$, then the maximum value of $\log(Aq_k/\epsilon_k)/\log B$, where $\epsilon_k = \|\hat{\mu}_a q_k\| - M_k\|\hat{\gamma}_k q_k\|$, is $251.095\ldots$, which, according to Lemma 4, is an upper bound on $n-1$. Hence, we deduce that the possible solutions (n, k, a, ℓ) of the equation (2) for which k is in the range $[3, 250]$ and $z > 0$ all have $n \in [2, 252]$.

Next we treat the case $z < 0$. It is a straightforward exercise to check that $6/\alpha^{n-1} < 1/2$ for all $k \geq 3$ and all $n \geq 6$. Then, from (19), we have that $|e^z - 1| < 1/2$ and therefore $e^{|z|} < 2$.

Since $z < 0$, we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{12}{\alpha^{n-1}}.$$

In a similar way as in the case when $z > 0$, we obtain

$$0 < (n-1)\hat{\gamma}_k - \ell + \hat{\mu}_a < AB^{-(n-1)}, \tag{22}$$

where now

$$\hat{\gamma}_k := \frac{\log \alpha}{\log 10}, \quad \hat{\mu}_a := -\frac{\log \mu_a}{\log 10}, \quad A := 6 \quad \text{and} \quad B := \alpha.$$

Here, we take $M_k = \lfloor 6 \times 10^{14}\, k^4 \log^3 k \rfloor$, which is an upper bound on $n-1$ by Lemma 3, and, as we have explained before, we apply Lemma 4 to inequality (22) for each $k \in [3, 250]$. In this case, with the help of Mathematica, we find that the maximum value of $\log(Aq_k/\epsilon_k)/\log B$ is $251.817\ldots$. Thus, the possible solutions (n, k, a, ℓ) of the equation (2) with k in the range $[3, 250]$ and $z < 0$ all have $n \in [2, 252]$.

Finally, we use Mathematica to display the values $F_n^{(k)} \pmod{10^{10}}$ for $1 \leq n \leq 260$, $3 \leq k \leq 250$, and check that the only one solution of the equation (2) in this range is $(n, k, a, \ell) = (8, 3, 4, 2)$, namely $F_8^{(3)} = T_8 = 44$. This completes the analysis in the case $k \in [3, 250]$.

5. An absolute upper bound on k

From now on, we assume that $k > 250$. For such k we have

$$n < 6 \times 10^{14}\, k^4\, \log^3 k < 2^{k/2}.$$

Let $\lambda > 0$ be such that $\alpha + \lambda = 2$. Since α is located between $2(1 - 2^{-k})$ and 2, we get that $\lambda < 2 - 2(1 - 2^{-k}) = 1/2^{k-1}$, i.e., $\lambda \in (0, 1/2^{k-1})$. Besides,

$$\alpha^{n-1} = (2 - \lambda)^{n-1} = 2^{n-1} \left(1 - \frac{\lambda}{2}\right)^{n-1} = 2^{n-1} e^{(n-1)\log(1-\lambda/2)} \geq 2^{n-1} e^{-\lambda(n-1)},$$

where we used the fact that $\log(1 - x) \geq -2x$ for all $x < 1/2$. But we also have that $e^{-x} \geq 1 - x$ for all $x \in \mathbb{R}$, so, $\alpha^{n-1} \geq 2^{n-1}(1 - \lambda(n - 1))$.

Moreover, $\lambda(n - 1) < (n - 1)/2^{k-1} < 2^{k/2}/2^{k-1} = 2/2^{k/2}$. Hence,

$$\alpha^{n-1} > 2^{n-1}(1 - 2/2^{k/2}).$$

It then follows that the following inequalities hold

$$2^{n-1} - \frac{2^n}{2^{k/2}} < \alpha^{n-1} < 2^{n-1},$$

or

$$\left|\alpha^{n-1} - 2^{n-1}\right| < \frac{2^n}{2^{k/2}}. \tag{23}$$

We now consider the function

$$f(x) = \frac{x - 1}{2 + (k + 1)(x - 2)} \quad \text{for } x > 2(1 - 2^{-k}).$$

Using the Mean–Value Theorem, we get that there exists some $\beta \in (\alpha, 2)$ such that $f(\alpha) = f(2) + (\alpha - 2)f'(\beta)$. Thus,

$$|f(\alpha) - f(2)| = |\alpha - 2||f'(\beta)| < \frac{2k}{2^k}, \tag{24}$$

where we used the facts that

$$|\alpha - 2| < \frac{1}{2^{k-1}} \quad \text{and } f'(\beta) = \frac{1 - k}{(2 + (k + 1)(\beta - 2))^2},$$

together with $2 + (k + 1)(\beta - 2) \geq 1$. If we write

$$\alpha^{n-1} = 2^{n-1} + \delta \quad \text{and } f(\alpha) = f(2) + \eta,$$

then inequalities (23) and (24) yield

$$|\delta| < \frac{2^n}{2^{k/2}} \quad \text{and } |\eta| < \frac{2k}{2^k}.$$

Besides, since $f(2) = 1/2$, we have

$$f(\alpha)\,\alpha^{n-1} = 2^{n-2} + \frac{\delta}{2} + 2^{n-1}\eta + \eta\,\delta.$$

So, from (9) and the above equality, we get

$$\begin{aligned}\left|2^{n-2} - \frac{a\,10^{\ell}}{9}\right| &= \left|\left(f(\alpha)\,\alpha^{n-1} - \frac{a\,10^{\ell}}{9}\right) - \frac{\delta}{2} - 2^{n-1}\eta - \eta\,\delta\right| \\ &< \frac{3}{2} + \frac{2^{n-1}}{2^{k/2}} + \frac{2^n k}{2^k} + \frac{2^{n+1}k}{2^{3k/2}}.\end{aligned}$$

Factoring out 2^{n-2} in the right-hand side of the above inequality and taking into account that $3/2^{n-1} < 1/2^{k/2}$ (because $n > k+1$ by Lemma 3), $4k/2^k < 1/2^{k/2}$ and $8k/2^{3k/2} < 1/2^{k/2}$ all valid for $k > 250$, we get that

$$\left|2^{n-2} - \frac{a\,10^{\ell}}{9}\right| < 5 \cdot \frac{2^{n-2}}{2^{k/2}}.$$

Consequently,

$$\left|1 - \frac{a}{9} \cdot 10^{\ell} \cdot 2^{-(n-2)}\right| < \frac{5}{2^{k/2}}. \tag{25}$$

We now set

$$\Lambda_1 := \frac{a}{9} \cdot 10^{\ell} \cdot 2^{-(n-2)} - 1. \tag{26}$$

The fact that Λ_1 is nonzero follows from the fact that $\ell \geq 2$, by looking at the exponent of 5 in the factorization of $\Lambda_1 + 1$. We lower bound the left-hand side of inequality (25) using again Matveev's result Lemma 2. We take $t := 3$, $\gamma_1 := a/9$, $\gamma_2 := 10$ and $\gamma_3 := 2$. We also take the exponents $b_1 := 1$, $b_2 := \ell$ and $b_3 := -(n-2)$. In this application of Matveev's result, we take $D := 1$, $A_1 := \log 9$, $A_2 := \log 10$ and $A_3 := \log 2$. Also, we can take $B := n$. We thus get that

$$\exp\left(-C_2\,(1 + \log n)\,(\log 9)\,(\log 10)\,(\log 2)\right) < \frac{5}{2^{k/2}},$$

where $C_2 := 1.4 \times 30^6 \times 3^{4.5}$.

Taking logarithms in the above inequality, we have that

$$\frac{k}{2}\log 2 - \log 5 < 5.1 \times 10^{11}\,(1 + \log n).$$

This leads to

$$\begin{aligned} k &< \frac{5.1 \times 10^{11}}{\log 2} \cdot 2\,(1 + \log n) + \frac{2 \log 5}{\log 2} \\ &< 2.21 \times 10^{12}\ \log n + 4.65 \\ &< 2.3 \times 10^{12}\ \log n. \end{aligned}$$

In the above, we used the inequalities $2\,(1 + \log n) < 3 \log n$ (valid for all $n \geq 8$) and $2.21 \times 10^{12}\ \log n + 4.65 < 2.3 \times 10^{12}\ \log n$ (valid for all $n \geq 2$). But, recall that by Lemma 3 we have $n < 6 \times 10^{14}\, k^4\ \log^3 k$. Thus,

$$\begin{aligned} k &< 2.3 \times 10^{12}\ \log(6 \times 10^{14}\, k^4\ \log^3 k) \\ &< 2.3 \times 10^{12}\,(35 + 7 \log k) \\ &< 3.22 \times 10^{13} \log k, \end{aligned}$$

where we used the fact that the inequality $35 + 7 \log k < 14 \log k$ holds for all $k \geq 149$. Mathematica gives $k < 2 \times 10^{15}$. Actually, the upper bound on k is smaller than the one shown here, but we decided to work with this bound for simplicity. By Lemma 3 once again, we obtain $n < 5 \times 10^{80}$ and $\ell < 2 \times 10^{80}$. We record our conclusion as follows.

Lemma 5. *If (n, k, a, ℓ) is a solution in positive integers of equation (2) with $k > 250$, then all inequalities*

$$n < 5 \times 10^{80}, \quad k < 2 \times 10^{15} \quad \text{and} \quad \ell < 2 \times 10^{80}$$

hold.

6. Reducing the bound on k

6.1. The case $a \neq 9$. We now want to reduce our bound on k by using again Lemma 4. Let $z := \ell \log 10 - (n-2) \log 2 + \log(a/9)$. Thus $e^z - 1 = \Lambda_1$, where Λ_1 is given by (26). So, from estimate (25), we deduce that

$$|e^z - 1| < \frac{5}{2^{k/2}}. \tag{27}$$

In what follows, we distinguish again two cases. First, if $\Lambda_1 < 0$, then $z < 0$; besides, $|e^z - 1| < 1/2$ implies that $e^{|z|} < 2$. Hence, from (27), we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{10}{2^{k/2}}.$$

Replacing z by its expression in the above inequality, we get

$$0 < (n-2)\gamma - \ell + \hat{\mu}_a < AB^{-k}, \tag{28}$$

where

$$\gamma := \frac{\log 2}{\log 10}, \quad \hat{\mu}_a := -\frac{\log(a/9)}{\log 10}, \quad A := 5 \quad \text{and} \quad B := 2^{1/2}.$$

Clearly, γ is an irrational number. Let p_n/q_n be the nth convergent of the continued fraction of γ. In order to reduce the bound on k, we take $M := 5 \times 10^{80}$, which is an upper bound on n from Lemma 5. Now, we want to find a convergent of γ whose denominator is greater than $6M = 3 \times 10^{81}$.

A quick inspection using Mathematica reveals that our desired convergent is p_{167}/q_{167}. Moreover, we get

$$M \, \|q_{167}\gamma\| = 0.02688\ldots < 0.027.$$

The minimal value of $\|q_{167}\hat{\mu}_a\|$ computed for $a \in \{1,2,3,4,5,6,7,8\}$ is > 0.128 and occurs when $a = 7$. Thus, we can take $\epsilon := \|q_{167}\hat{\mu}_a\| - M\,\|q_{167}\gamma\| > 0.128 - 0.027 = 0.101$.

It then follows from Lemma 4 that there is no solution of the inequality in (28) (and therefore for the equation (2)) with

$$k \geq \left\lfloor \frac{\log(Aq_{167}/\epsilon)}{\log B} \right\rfloor + 1 = 557 \quad \text{and} \quad a \in \{1,2,3,4,5,6,7,8\}.$$

Thus, $k \leq 556$ and then Lemma 3 tells us that $n < 2 \times 10^{28}$.

With this new upper bound for n we repeated the process, i.e., we applied again Lemma 4 with $M := 2 \times 10^{28}$. Now, our desired convergent is p_{64}/q_{64}. We also get

$$M \, \|q_{64}\gamma\| = 0.001434\ldots < 0.0015.$$

We computed the values of $\|q_{64}\hat{\mu}_a\|$ for $a \in \{1,2,3,4,5,6,7,8\}$ and we found that the minimal value of $\|q_{64}\hat{\mu}_a\|$ is > 0.0479 and it occurs when $a = 6$. Thus, we can now take $\epsilon := \|q_{64}\hat{\mu}_a\| - M\,\|q_{64}\gamma\| > 0.0479 - 0.0015 = 0.0464$.

It follows from Lemma 4 that there is no solution of the inequality in (28) for

$$k \geq \left\lfloor \frac{\log(Aq_{64}/\epsilon)}{\log B} \right\rfloor + 1 = 212 \quad \text{and} \quad a \in \{1,2,3,4,5,6,7,8\}.$$

Therefore, $k \leq 211$, which is a case already treated.

In the same way, if $\Lambda_1 > 0$, we then have $z > 0$. It follows from (27) that

$$0 < z \leq e^z - 1 < \frac{5}{2^{k/2}}.$$

Thus,

$$0 < \ell\gamma - n + \hat{\mu}_a < AB^{-k}, \tag{29}$$

with

$$\gamma := \frac{\log 10}{\log 2}, \quad \hat{\mu}_a := 2 + \frac{\log(a/9)}{\log 2}, \quad A := 8 \quad \text{and} \quad B := 2^{1/2}.$$

In order to use Lemma 4, we take $M := 2 \times 10^{80}$, which is an upper bound on ℓ by Lemma 5, so $6M = 1.2 \times 10^{81}$. Here, the convergent is p_{166}/q_{166}. Hence,

$$M \, \|q_{166}\gamma\| = 0.03572\ldots < 0.036.$$

The minimal value of $\|q_{166}\hat{\mu}_a\|$ computed for $a \in \{1,2,3,4,5,6,7,8\}$ is > 0.128 and occurs when $a = 7$. Thus, we can take $\epsilon := 0.128 - 0.036 = 0.092$.

In view of Lemma 4, we deduce that there is no solution of the inequality in (29) (and therefore for the equation (2)) for

$$k \geq \left\lfloor \frac{\log(Aq_{166}/\epsilon)}{\log B} \right\rfloor + 1 = 555 \quad \text{and} \quad a \in \{1,2,3,4,5,6,7,8\}.$$

Thus, $k \leq 554$ and then from Lemma 3 we get $\ell < 5 \times 10^{27}$.

As before we may apply Lemma 4 with $M := 5 \times 10^{27}$. Now, our desired convergent is p_{63}/q_{63}. Here, we find

$$M \, \|q_{63}\gamma\| = 0.0011910\ldots < 0.0012.$$

The minimal value of $\|q_{63}\hat{\mu}_a\|$ computed for $a \in \{1,2,3,4,5,6,7,8\}$ is > 0.0479 and occurs when $a = 3$. Thus, we take $\epsilon := 0.0479 - 0.0012 = 0.0467$.

Finally, Lemma 4 tells us that there is no solution of the inequality in (29) for

$$k \geq \left\lfloor \frac{\log(Aq_{63}/\epsilon)}{\log B} \right\rfloor + 1 = 210 \quad \text{and} \quad a \in \{1,2,3,4,5,6,7,8\}.$$

Hence, $k \leq 209$, which is a case already treated.

6.2. The case $a = 9$. We cannot study this case as before because $\hat{\mu}_9$ is always an integer. For this reason, we need to treat this case differently.

Again we distinguish two cases. When $\Lambda_1 < 0$, then $\hat{\mu}_9 = 0$, so from (28), we get

$$0 < (n-2)\gamma - \ell < 5 \cdot 2^{-k/2}, \quad \text{where} \quad \gamma := \frac{\log 2}{\log 10}. \tag{30}$$

Let $[a_0, a_1, a_2, a_3, a_4, \ldots] = [0, 3, 3, 9, 2, 2, \ldots]$ be the continued fraction of γ, and recall that we denoted by p_k/q_k its kth convergent. Recall also that $n-2 < 5 \times 10^{80}$ by Lemma 5.

We have $q_{162} = 4.36\ldots \times 10^{79} < 5 \times 10^{80}$, $q_{163} = 7.55\ldots \times 10^{80} > 5 \times 10^{80}$. Furthermore, $a_M := \max\{a_i : i = 0, 1, \ldots, 163\} = a_{136} = 5393$. From the known properties of continued fractions, we obtain that

$$|(n-2)\gamma - \ell| > \frac{1}{(a_M + 2)(n-2)}. \tag{31}$$

Comparing estimates (30) and (31), we get right away that

$$2^{k/2} < 26975(n-2) < 1.7 \times 10^{19}\, k^4\, \log^3 k,$$

where we used the fact that $n < 6 \times 10^{14}\, k^4\, \log^3 k$ from Lemma 3. Taking logarithms in the above inequality, we have that

$$k < \frac{2 \cdot \log(1.7 \times 10^{19})}{\log 2} + \frac{8 \log k}{\log 2} + \frac{6 \log(\log k)}{\log 2} < 128 + 21 \log k,$$

implying that $k \le 243$, which is a case already treated.

If on the other hand we have that $\Lambda_1 > 0$, then, from (29), we get

$$0 < \ell\gamma - (n-2) < 8 \cdot 2^{-k/2}, \quad \text{where} \quad \gamma := \frac{\log 10}{\log 2}.$$

Clearly, the present γ is the reciprocal of the previous one, so the continued fraction of it is the same up to a shift of 1. Hence, $a_M = a_{135} = 5393$, and

$$\frac{1}{(a_M + 2)\ell} < |\ell\gamma - (n-2)| < 8 \cdot 2^{-k/2}.$$

After some algebra and taking into account that $\ell < 2 \times 10^{14}\, k^4\, \log^3 k$ from Lemma 3, we finally get $k \le 241$, which is also a case already treated.

Hence, we confirm that there are no other solutions (n, k, a, ℓ) to equation (2) than those mentioned in Conjecture 1. Therefore, Theorem 1 is proved.

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