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# Mixed-type reverse order laws for generalized inverses in rings with involution

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**Abstract.** We investigate mixed-type reverse order laws for the Moore–Penrose inverse in rings with involution. We extend some well-known results to more general settings, and also prove some new results.

## 1. Introduction

Many authors have studied the equivalent conditions for the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ to hold in setting of matrices, operators, $C^*$-algebras or rings [2], [9], [3], [5], [8], [10], [12], [16], [17]. This formula cannot trivially be extended to the other generalized inverses of the product $ab$. Since the reverse order law $(ab)^\dagger = b^\dagger a^\dagger$ does not always holds, it is not easy to simplify various expressions that involve the Moore–Penrose inverse of a product. In addition to $(ab)^\dagger = b^\dagger a^\dagger$, $(ab)^\dagger$ may be expressed as $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$, $(ab)^\dagger = b^\dagger a^\dagger - b^\dagger[(1-bb^\dagger)(1-a^\dagger a)]^\dagger a^\dagger$, etc. These equalities are called mixed-type reverse order laws for the Moore–Penrose inverse of a product and some of them are in fact equivalent (see [4], [12], [14]). In this paper we study necessary and sufficient conditions for mixed-type reverse order laws of the form: $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger$, $(ab)^\dagger = b^\dagger(abb^\dagger)^\dagger$, $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $(ab)^\dagger = (a^*ab)^\dagger a^*$, $(ab)^\dagger = b^*(abb^*)^\dagger$ and $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$ in rings with involution.

Let $\mathcal{R}$ be an associative ring with the unit 1. An involution $a \mapsto a^*$ in a ring

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$\mathcal{R}$ is an anti-isomorphism of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*.$$

An element $a \in \mathcal{R}$ is selfadjoint if $a^* = a$.

The *Moore–Penrose inverse* (or *MP-inverse*) of $a \in \mathcal{R}$ is the element $b \in \mathcal{R}$, such that the following equations hold [13]:

$$(1)\ aba = a, \quad (2)\ bab = b, \quad (3)\ (ab)^* = ab, \quad (4)\ (ba)^* = ba.$$

There is at most one $b$ such that above conditions hold (see [13]), and such $b$ is denoted by $a^\dagger$. The set of all Moore–Penrose invertible elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^\dagger$. If $a$ is invertible, then $a^\dagger$ coincides with the ordinary inverse of $a$.

If $\delta \subset \{1,2,3,4\}$ and $b$ satisfies the equations $(i)$ for all $i \in \delta$, then $b$ is an $\delta$-inverse of $a$. The set of all $\delta$-inverse of $a$ is denote by $a\{\delta\}$. Notice that $a\{1,2,3,4\} = \{a^\dagger\}$. If $a\{1\} \neq \emptyset$, then $a$ is regular.

Now, we state the following useful result.

**Theorem 1.1** ([6], [11]). *For any $a \in \mathcal{R}^\dagger$, the following is satisfied:*

(a) $(a^\dagger)^\dagger = a$;

(b) $(a^*)^\dagger = (a^\dagger)^*$;

(c) $(a^*a)^\dagger = a^\dagger (a^\dagger)^*$;

(d) $(aa^*)^\dagger = (a^\dagger)^* a^\dagger$;

(e) $a^* = a^\dagger a a^* = a^* a a^\dagger$;

(f) $a^\dagger = (a^*a)^\dagger a^* = a^* (aa^*)^\dagger$;

(g) $(a^*)^\dagger = a(a^*a)^\dagger = (aa^*)^\dagger a$.

The following result is well-known for complex matrices [1] and linear bounded Hilbert space operators [18], and it is equally true in rings with involution.

**Lemma 1.1.** *If $a, b \in \mathcal{R}$ such that $a$ is regular, then*

(a) $b \in a\{1,3\} \iff a^*ab = a^*$;

(b) $b \in a\{1,4\} \iff baa^* = a^*$.

Proof. (a) Let $b \in a\{1,3\}$, then we get $a^*ab = a^*(ab)^* = (aba)^* = a^*$.

Conversely, the equality $a^*ab = a^*$ implies

$$(ab)^* = b^*a^* = b^*a^*ab = (ab)^*ab \text{is selfadjoint}$$

and

$$aba = (ab)^*a = (a^*ab)^* = (a^*)^* = a.$$

Hence, $b \in a\{1,3\}$.

Similarly, we can verify the second statement. $\square$

The reverse-order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ was first studied by Galperin and Waksman [7]. A Hilbert space version of their result was given by Isumino [9]. Many results concerning the reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ for complex matrices appeared in Tian's papers [14] and [15], where the author used mostly properties of the rank of a complex matrices. In [12], a set of equivalent conditions for this reverse order rule for the Moore–Penrose inverse in the setting of $C^*$-algebra is studied.

Xiong and Qin [18] investigated the following mixed-type reverse order laws for the Moore–Penrose inverse of a product of Hilbert space operators: $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger$, $(ab)^\dagger = b^\dagger(abb^\dagger)^\dagger$, $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$. They used the technique of block operator matrices. We extend results from [18] to more general settings.

This paper is organized as follows. In Section 2, we extend the results from [18] to settings of rings with involution without the hypothesis corresponding to $R(A^*AB) \subseteq R(B)$. In Section 3, we consider the following mixed-type reverse order laws for the Moore–Penrose inverse in rings with involution: $(ab)^\dagger = (a^*ab)^\dagger a^*$, $(ab)^\dagger = b^*(abb^*)^\dagger$ and $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$. In this paper we apply a purely algebraic technique.

## 2. Reverse order laws $(a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger$, $b^\dagger(abb^\dagger)^\dagger = (ab)^\dagger$ and $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (ab)^\dagger$

In this section, we consider necessary and sufficient conditions for reverse order laws $(a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger$, $b^\dagger(abb^\dagger)^\dagger = (ab)^\dagger$ and $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (ab)^\dagger$ to be satisfied in rings with involution. The results in [18] for linear bounded Hilbert space operators are generalized, since we do not use any e hypothesis corresponding to the condition $R(A^*AB) \subseteq R(B)$ from [18].

**Theorem 2.1.** *If $a, b, a^\dagger ab \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$;

(2) $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{1,3\}$;

(3) $(a^\dagger ab)^\dagger a^\dagger = (ab)^\dagger$;

(4) $(a^\dagger ab)\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$.

Proof. (2) $\Longrightarrow$ (1): Since $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{1,3\}$, then $ab = ab(a^\dagger ab)^\dagger a^\dagger ab$ and

$$ab(a^\dagger ab)^\dagger a^\dagger = (ab(a^\dagger ab)^\dagger a^\dagger)^* = (aa^\dagger ab(a^\dagger ab)^\dagger a^\dagger)^* = (a^\dagger)^* a^\dagger ab(a^\dagger ab)^\dagger a^*,$$

which gives

$$\begin{aligned} a^*ab &= a^*(ab(a^\dagger ab)^\dagger a^\dagger)ab = a^*(a^\dagger)^* a^\dagger ab(a^\dagger ab)^\dagger a^* ab \\ &= a^\dagger a a^\dagger ab(a^\dagger ab)^\dagger a^* ab = a^\dagger ab(a^\dagger ab)^\dagger a^* ab. \end{aligned}$$

Therefore, $a^*ab\mathcal{R} = a^\dagger ab(a^\dagger ab)^\dagger a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$.

(1) $\Longrightarrow$ (4): The assumption $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$ implies that $a^*ab = a^\dagger abx$, for some $x \in \mathcal{R}$. Now, for any $(a^\dagger ab)^{(1,3)} \in (a^\dagger ab)\{1,3\}$ and $a^{(1,3)} \in a\{1,3\}$,

$$a^*ab = a^\dagger abx = a^\dagger ab(a^\dagger ab)^{(1,3)}(a^\dagger abx) = a^\dagger ab(a^\dagger ab)^{(1,3)}a^*ab. \tag{1}$$

Applying the involution to (1), we obtain

$$b^*a^*a = b^*a^*aa^\dagger ab(a^\dagger ab)^{(1,3)} = b^*a^*ab(a^\dagger ab)^{(1,3)}. \tag{2}$$

Multiplying the equality (2) by $a^{(1,3)}$ from the right side, we get

$$b^*a^* = b^*a^*ab(a^\dagger ab)^{(1,3)}a^{(1,3)}, \tag{3}$$

by $a^*aa^{(1,3)} = a^*(aa^{(1,3)})^* = (aa^{(1,3)}a)^* = a^*$. From the equality (3) and Lemma 1.1, we deduce that $(a^\dagger ab)^{(1,3)}a^{(1,3)} \in (ab)\{1,3\}$, for any $(a^\dagger ab)^{(1,3)} \in (a^\dagger ab)\{1,3\}$ and $a^{(1,3)} \in a\{1,3\}$. So, $(a^\dagger ab)\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$.

(4) $\Longrightarrow$ (2): Obviously, because $(a^\dagger ab)^\dagger \in (a^\dagger ab)\{1,3\}$ and $a^\dagger \in a\{1,3\}$.

(2) $\Longleftrightarrow$ (3): It is easy to check this equivalence. □

Using Lemma 1.1(b), we can prove the following theorem in the same way as Theorem 2.1.

**Theorem 2.2.** *If $a, b, abb^\dagger \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$;

(2) $b^\dagger(abb^\dagger)^\dagger \in (ab)\{1,4\}$;

(3) $b^\dagger(abb^\dagger)^\dagger = (ab)^\dagger$;

(4) $b\{1,4\} \cdot (abb^\dagger)\{1,4\} \subseteq (ab)\{1,4\}$.

In the following result, we consider some equivalent conditions for mixed-type reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ to hold.

**Theorem 2.3.** *If $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$ *and* $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$;

(2) $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,3,4\}$;

(3) $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger = (ab)^\dagger$;

(4) $b\{1,3\} \cdot (a^\dagger abb^\dagger)\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$ *and* $b\{1,4\} \cdot (a^\dagger abb^\dagger)\{1,4\} \cdot a\{1,4\} \subseteq (ab)\{1,4\}$.

Proof. (2) $\Longrightarrow$ (1): The condition $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{3\}$ gives

$$\begin{aligned} abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger &= (abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* = (aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)^* \\ &= (a^\dagger)^* a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*. \end{aligned}$$

Using this equality and the hypothesis $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1\}$, we have

$$\begin{aligned} a^*ab &= a^*(abb^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger)ab = a^*(a^\dagger)^* a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*ab \\ &= a^\dagger aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*ab = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^\dagger a^*ab, \end{aligned}$$

which yields $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$.

Similarly, we can prove that $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,4\}$ implies $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$.

(1) $\Longrightarrow$ (4): From $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$, by $b\mathcal{R} = bb^\dagger\mathcal{R}$, we get $a^*abb^\dagger\mathcal{R} \subseteq a^\dagger abb^\dagger\mathcal{R}$. Thus, $a^*abb^\dagger = a^\dagger abb^\dagger x$, for some $x \in \mathcal{R}$. Then, for any $(a^\dagger abb^\dagger)^{(1,3)} \in (a^\dagger abb^\dagger)\{1,3\}$, $a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$, we obtain

$$a^*abb^\dagger = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^{(1,3)}(a^\dagger abb^\dagger x) = a^\dagger abb^\dagger(a^\dagger abb^\dagger)^{(1,3)}a^*abb^\dagger. \tag{4}$$

If we apply the involution to (4), we see that

$$bb^\dagger a^*a = bb^\dagger a^*aa^\dagger abb^\dagger(a^\dagger abb^\dagger)^{(1,3)} = bb^\dagger a^*abb^\dagger(a^\dagger abb^\dagger)^{(1,3)}. \tag{5}$$

Multiplying the equality (5) from the left side by $b^*$ and from the right side by $a^{(1,3)}$, it follows

$$b^*a^* = b^*a^*abb^\dagger(a^\dagger abb^\dagger)^{(1,3)}a^{(1,3)}.$$

Notice that this equality and

$$bb^{(1,3)} = (bb^{(1,3)})^* = (bb^\dagger bb^{(1,3)})^* = bb^{(1,3)}bb^\dagger = bb^\dagger \tag{6}$$

imply

$$b^*a^* = b^*a^*abb^{(1,3)}(a^\dagger abb^\dagger)^{(1,3)}a^{(1,3)}. \tag{7}$$

By (7) and Lemma 1.1, we observe that $b^{(1,3)}(a^\dagger abb^\dagger)^{(1,3)}a^{(1,3)} \in (ab)\{1,3\}$, for any $(a^\dagger abb^\dagger)^{(1,3)} \in (a^\dagger abb^\dagger)\{1,3\}$, $a^{(1,3)} \in a\{1,3\}$ and $b^{(1,3)} \in b\{1,3\}$. Hence, $b\{1,3\} \cdot (a^\dagger abb^\dagger)\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$.

In the similar way, we can show that $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$ gives $b^*a^* = b^{(1,4)}(a^\dagger abb^\dagger)^{(1,4)}a^{(1,4)}abb^*a^*$, for any $(a^\dagger abb^\dagger)^{(1,4)} \in (a^\dagger abb^\dagger)\{1,4\}$, $a^{(1,4)} \in a\{1,4\}$ and $b^{(1,4)} \in b\{1,4\}$, i.e. $b\{1,4\} \cdot (a^\dagger abb^\dagger)\{1,4\} \cdot a\{1,4\} \subseteq (ab)\{1,4\}$.

(4) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3): Obviously. $\square$

## 3. Reverse order laws $(a^*ab)^\dagger a^* = (ab)^\dagger$, $b^*(abb^*)^\dagger = (ab)^\dagger$ and $b^*(a^*abb^*)^\dagger a^* = (ab)^\dagger$

In this section, we give the equivalent conditions related to reverse order laws $(a^*ab)^\dagger a^* = (ab)^\dagger$, $b^*(abb^*)^\dagger = (ab)^\dagger$ and $b^*(a^*abb^*)^\dagger a^* = (ab)^\dagger$ in settings of rings with involution.

**Theorem 3.1.** *If $a, b, a^*ab \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$;

(2) $(a^*ab)^\dagger a^* \in (ab)\{1,3\}$;

(3) $(a^*ab)^\dagger a^* = (ab)^\dagger$;

(4) $(a^*ab)\{1,3\} \cdot (a^\dagger)^*\{1,3\} \subseteq (ab)\{1,3\}$.

Proof. (2) $\Longrightarrow$ (1): Using the assumption $(a^*ab)^\dagger a^* \in (ab)\{1,3\}$, we have

$$\begin{aligned} ab(a^*ab)^\dagger a^* &= (ab(a^*ab)^\dagger a^*)^* = (aa^\dagger ab(a^*ab)^\dagger a^*)^* \\ &= ((a^\dagger)^* a^*ab(a^*ab)^\dagger a^*)^* = aa^*ab(a^*ab)^\dagger a^\dagger, \end{aligned}$$

and

$$a^\dagger ab = a^\dagger(ab(a^*ab)^\dagger a^*)ab = a^\dagger aa^*ab(a^*ab)^\dagger a^\dagger ab = a^*ab(a^*ab)^\dagger a^\dagger ab.$$

Thus, the condition (1) is satisfied.

(1) $\Longrightarrow$ (4): First, by the inclusion $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$, we conclude that $a^\dagger ab = a^*aby$, for some $y \in \mathcal{R}$. Further, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$ and $a' \in (a^\dagger)^*\{1,3\}$, we get

$$a^\dagger ab = a^*aby = a^*ab(a^*ab)^{(1,3)}(a^*aby) = a^*ab(a^*ab)^{(1,3)}a^\dagger ab. \tag{8}$$

When we apply the involution to (8), we observe that

$$b^*a^\dagger a = b^*a^\dagger aa^*ab(a^*ab)^{(1,3)} = b^*a^*ab(a^*ab)^{(1,3)}. \tag{9}$$

Since $a' \in (a^\dagger)^*\{1,3\}$, by the equality (6) and Theorem 1.1,

$$a^\dagger aa' = a^*[(a^\dagger)^*a'] = a^*(a^\dagger)^*[(a^\dagger)^*]^\dagger = a^\dagger aa^* = a^*. \tag{10}$$

If we multiply the equality (9) from the right side by $a'$ and use (10), we obtain

$$b^*a^* = b^*a^*ab(a^*ab)^{(1,3)}a',$$

which implies, by Lemma 1.1, $(a^*ab)^{(1,3)}a' \in (ab)\{1,3\}$, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$ and $a' \in (a^\dagger)^*\{1,3\}$, that is, the condition (4) holds.

(4) $\Longrightarrow$ (2): By Theorem 1.1, $a^* = [((a^\dagger)^\dagger]^* = [((a^\dagger)^*]^\dagger \in (a^\dagger)^*\{1,3\}$ and this implication follows.

(2) $\Longleftrightarrow$ (3): Obviously. $\square$

In the same manner as in the proof of Theorem 3.1, we can verify the following results.

**Theorem 3.2.** *If $a, b, abb^* \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$;

(2) $b^*(abb^*)^\dagger \in (ab)\{1,4\}$;

(3) $b^*(abb^*)^\dagger = (ab)^\dagger$;

(4) $(b^\dagger)^*\{1,4\} \cdot (abb^*)\{1,4\} \subseteq (ab)\{1,4\}$.

Necessary and sufficient conditions related to the reverse order law $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$ are studied in the next result.

**Theorem 3.3.** *If $a, b, a^*abb^* \in \mathcal{R}^\dagger$, then the following statements are equivalent:*

(1) $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$ *and* $bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$;

(2) $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1,3,4\}$;

(3) $b^*(a^*abb^*)^\dagger a^* = (ab)^\dagger$;

(4) $(b^\dagger)^*\{1,3\} \cdot (a^*abb^*)\{1,3\} \cdot (a^\dagger)^*\{1,3\} \subseteq (ab)\{1,3\}$ *and*
$(b^\dagger)^*\{1,4\} \cdot (a^*abb^*)\{1,4\} \cdot (a^\dagger)^*\{1,4\} \subseteq (ab)\{1,4\}$.

Proof. (2) $\Longrightarrow$ (1): From $b^*(a^*abb^*)^\dagger a^* \in (ab)\{3\}$,

$$\begin{aligned} abb^*(a^*abb^*)^\dagger a^* &= (abb^*(a^*abb^*)^\dagger a^*)^* = ((a^\dagger)^*a^*abb^*(a^*abb^*)^\dagger a^*)^* \\ &= aa^*abb^*(a^*abb^*)^\dagger a^\dagger. \end{aligned}$$

Now, by $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1\}$,

$$\begin{aligned} a^\dagger ab &= a^\dagger(abb^*(a^*abb^*)^\dagger a^*)ab = a^\dagger aa^*abb^*(a^*abb^*)^\dagger a^\dagger ab \\ &= a^*abb^*(a^*abb^*)^\dagger a^\dagger ab \end{aligned}$$

implying $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$.

Analogously, we can prove the implication $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1,4\} \Longrightarrow bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$.

(1) $\Longrightarrow$ (4): If $a^\dagger ab\mathcal{R} \subseteq a^*ab\mathcal{R}$, by $b\mathcal{R} = bb^*\mathcal{R}$, we see $a^\dagger abb^*\mathcal{R} \subseteq a^*abb^*\mathcal{R}$ and $a^\dagger abb^* = a^*abb^*y$, for some $y \in \mathcal{R}$. For any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$, $a' \in (a^\dagger)^*\{1,3\}$ and $b' \in (b^\dagger)^*\{1,3\}$, then

$$a^\dagger abb^* = a^*abb^*(a^*abb^*)^{(1,3)}(a^*abb^*y) = a^*abb^*(a^*abb^*)^{(1,3)}a^\dagger abb^*. \tag{11}$$

Applying the involution to (11), it follows

$$bb^*a^\dagger a = bb^*a^\dagger aa^*abb^*(a^*abb^*)^{(1,3)} = bb^*a^*abb^*(a^*abb^*)^{(1,3)}. \tag{12}$$

From the condition $b' \in (b^\dagger)^*\{1,3\}$ and the equality (10), we obtain

$$bb' = b(b^\dagger bb') = bb^*.$$

Now, multiplying (12) from the left side by $b^\dagger$ and from the right side by $a'$, we get, by (10) and the last equality,

$$b^*a^* = b^*a^*abb'(a^*abb^*)^{(1,3)}a'.$$

Thus, by Lemma 1.1, $b'(a^*abb^*)^{(1,3)}a' \in (ab)\{1,3\}$, for any $(a^*ab)^{(1,3)} \in (a^*ab)\{1,3\}$, $a' \in (a^\dagger)^*\{1,3\}$ and $b' \in (b^\dagger)^*\{1,3\}$, which is equivalent to $(b^\dagger)^*\{1,3\} \cdot (a^*abb^*)\{1,3\} \cdot (a^\dagger)^*\{1,3\} \subseteq (ab)\{1,3\}$.

Similarly, we show that $bb^\dagger a^*\mathcal{R} \subseteq bb^*a^*\mathcal{R}$ gives $(b^\dagger)^*\{1,4\} \cdot (a^*abb^*)\{1,4\} \cdot (a^\dagger)^*\{1,4\} \subseteq (ab)\{1,4\}$.

(4) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3): These parts can be check easy. $\square$

If we state in the proved results the elements $a^*$, $(a^\dagger)^*$, $a^\dagger$, $b^*$, $(b^\dagger)^*$ or $b^\dagger$ instead $a$ or $b$, we obtain various mixed-type reverse order laws for the Moore–Penrose inverses in rings with involution.

By the results presenting in Section 2 and Section 3, we can get the following consequence.

**Corollary 3.1.** *If $a, b, ab, a^\dagger ab, abb^\dagger, a^\dagger abb^\dagger, a^*ab, abb^*, a^*abb^* \in \mathcal{R}^\dagger$. Then the following statements are equivalent:*

(1) $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$;

(2) $(ab)^\dagger = (a^\dagger ab)^\dagger a^\dagger = b^\dagger(abb^\dagger)^\dagger$;

(3) $(ab)^\dagger = b^*(a^*abb^*)^\dagger a^*$;

(4) $(ab)^\dagger = (a^*ab)^\dagger a^* = b^*(abb^*)^\dagger$;

(5) $a^*ab\mathcal{R} \subseteq a^\dagger ab\mathcal{R}$ *and* $bb^*a^*\mathcal{R} \subseteq bb^\dagger a^*\mathcal{R}$;

(6) $b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger \in (ab)\{1,3,4\}$;

(7) $b\{1,3\} \cdot (a^\dagger abb^\dagger)\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$ *and* $b\{1,4\} \cdot (a^\dagger abb^\dagger)\{1,4\} \cdot a\{1,4\} \subseteq (ab)\{1,4\}$;

(8) $(a^\dagger ab)^\dagger a^\dagger \in (ab)\{1,3\}$ *and* $b^\dagger (abb^\dagger)^\dagger \in (ab)\{1,4\}$;

(9) $(a^\dagger ab)\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$ *and* $b\{1,4\} \cdot (abb^\dagger)\{1,4\} \subseteq (ab)\{1,4\}$;

(10) $a^\dagger ab\mathcal{R} \subseteq a^* ab\mathcal{R}$ *and* $bb^\dagger a^*\mathcal{R} \subseteq bb^* a^*\mathcal{R}$;

(11) $b^*(a^*abb^*)^\dagger a^* \in (ab)\{1,3,4\}$;

(12) $(b^\dagger)^*\{1,3\} \cdot (a^*abb^*)\{1,3\} \cdot (a^\dagger)^*\{1,3\} \subseteq (ab)\{1,3\}$ *and* $(b^\dagger)^*\{1,4\} \cdot (a^*abb^*)\{1,4\} \cdot (a^\dagger)^*\{1,4\} \subseteq (ab)\{1,4\}$;

(13) $(a^*ab)^\dagger a^* \in (ab)\{1,3\}$ *and* $b^*(abb^*)^\dagger \in (ab)\{1,4\}$;

(14) $(a^*ab)\{1,3\} \cdot (a^\dagger)^*\{1,3\} \subseteq (ab)\{1,3\}$ *and* $(b^\dagger)^*\{1,4\} \cdot (abb^*)\{1,4\} \subseteq (ab)\{1,4\}$.

PROOF. The equivalences of conditions (1)–(4) follow as in [12, Theorem 2.6] for elements of $C^*$-algebras. The rest follows from these equivalences and theorems in Section 2 and Section 3. □

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