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Certain Riemannian invariants for Sasakian submanifolds

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Abstract. In [2], B. Y. Chen introduced a series of Riemannian invariants on Kaehler manifolds and proved sharp estimates of these invariants for Kaehler submanifolds in complex space forms in terms of the main extrinsic invariant, namely the squared mean curvature. In this article we define analogous Chen invariants for Sasakian manifolds and obtain inequalities involving these invariants for invariant submanifolds in Sasakian space forms.

1. Introduction

B. Y. Chen introduced in 1993 a series of Riemannian invariants, i.e., intrinsic characteristics of a Riemannian manifold, known as *Chen invariants*. In this way, he initiated a modern topic in Submanifold Theory, the *theory of Chen invariants*, looking for the answers of one of the most interesting problem: find relationships between the intrinsic and extrinsic invariants of submanifolds. The most known Chen invariant (called the *Chen first invariant*) is given by

$$\delta_M(p) = \tau(p) - (\inf K)(p),$$

where M is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$, and $\tau(p)$ is the scalar curvature

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at p. Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen first equality were studied in [5]. A recent survey of results involving this type of invariants can be read in [3].

In [2], B. Y. Chen introduced certain invariants on Kaehler manifolds. He obtained general inequalities involving those invariants for Kaehler submanifolds in complex space forms and determined such submanifolds satisfying the equality cases of the above inequalities.

In the present paper we define a series of Chen-like invariants for Sasakian manifolds. It is well-known that the Sasakian manifolds are the odd version of Kaehler manifolds and the geometry studying Sasakian manifolds, i.e., *contact geometry*, is an important field of Differential Geometry.

It is known that any invariant submanifold of a Sasakian manifold is Sasakian. In this respect, we consider that is interesting to study the behaviour of invariant submanifolds of Sasakian manifolds from this point of view, of Riemannian invariants and, more precisely, corresponding Chen-like invariants to those introduced by B. Y. Chen in [2].

In this study of such submanifolds (we must observe that the dimension of the submanifold should be ≥ 5) in Sasakian space forms we consider the notion of totally real plane section (similar to that defined by B. Y. Chen in Kaehler geometry); we need to impose the condition that the plane section must be orthogonal to the Reeb vector field ξ.

We estimate the sectional curvature of totally real plane sections of an invariant submanifold in terms of the ϕ-sectional curvature of the embedding Sasakian space form; the characterization of the equality case is given.

We define a series of Chen-like invariants δ_k^r on any Sasakian manifold. By using the above estimate of the sectional curvature of totally real plane sections we obtain sharp inequalities for these invariants for invariant submanifolds of a Sasakian space form.

Also, we derive characterizations of the equality cases in terms of the shape operators and give one example which shows that the inequality fails for $k \geq 4$.

2. Preliminaries

Let M^n be an n-dimensional Riemannian manifold. We denote by $K(\pi)$ the sectional curvature of M^n associated with a plane section $\pi \subset T_pM^n$, $p \in M^n$. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM^n, the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

Let $\widetilde{M}^{2m+1}$ be a $(2m+1)$-differentiable manifold. The triple (ϕ, ξ, η) on $\widetilde{M}^{2m+1}$ is called a (ϕ, ξ, η)-*structure* if it satisfies $\eta(\xi) = 1$ and $\phi^2 = -Id + \eta \otimes \xi$, where ϕ is an endomorphism of the tangent bundle, ξ is a vector field, η is a 1-form and Id is the identity tensor.

We recall that $\phi\xi = 0$ and $\eta \circ \phi = 0$.

If the manifold $\widetilde{M}^{2m+1}$ with a (ϕ, ξ, η)-*structure* admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y, $\widetilde{M}^{2m+1}$ has a (ϕ, ξ, η, g)-*almost contact metric structure*. For more details see [1], [7].

If, moreover, $d\eta(X, Y) = g(X, \phi Y)$, for all vector fields X, Y, then $\widetilde{M}^{2m+1}$ is a *contact metric* manifold.

A $(2m+1)$-dimensional Riemannian manifold $(\widetilde{M}^{2m+1}, g)$ is said to be a *Sasakian manifold* if it admits a normal contact metric structure, or equivalently, satisfying

$$(\widetilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad \widetilde{\nabla}_X \xi = \phi X,$$

for any vector fields X, Y on $T\widetilde{M}^{2m+1}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

A plane section π in $T_p\widetilde{M}^{2m+1}$ is called a ϕ-*section* if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ-*sectional curvature*.

A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\widetilde{M}^{2m+1}(c)$.

As examples of Sasakian space forms we have $\mathbf{R}^{2m+1}$, S^{2m+1} with standard Sasakian structures (see more details in [1]).

A plane section $\pi \subset T_p\widetilde{M}^{2m+1}$, orthogonal to ξ, where $\widetilde{M}^{2m+1}$ is a Sasakian manifold, is called *anti-invariant* or *totally real* if $\phi\pi$ is perpendicular to π.

Since π is orthogonal to ξ, we have $\phi^2 X = -X + g(X, \xi)\xi = -X$, $\forall X \in \pi$; then we can say that π is *totally real with respect to* ϕ.

For each real number k and $p \in \widetilde{M}^{2n+1}$, where $\widetilde{M}^{2m+1}$ is a Sasakian manifold, we define an invariant δ_k^r by

$$\delta_k^r(p) = \tau(p) - k \inf K^r(p),$$

where $\inf K^r(p) = \inf_{\pi^r}\{K(\pi^r)\}$ and π^r runs over all totally real plane sections in T_pM^n.

An n-dimensional submanifold M^n of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ tangent to ξ is called an *invariant submanifold* (or *Sasakian submanifold*) of $\widetilde{M}^{2m+1}(c)$ if $\phi(T_pM^n) \subseteq T_pM^n$.

We recall important results about invariant submanifolds in Sasakian manifolds [8].

Proposition 2.1. *Every invariant submanifold of a Sasakian manifold is a Sasakian manifold.*

Proposition 2.2. *Every invariant submanifold of a Sasakian manifold is minimal.*

Proposition 2.3. *If the second fundamental form of an invariant submanifold M^n of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ is parallel, then M^n is totally geodesic.*

Proposition 2.4. *Let M^n be an invariant submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ with ϕ-sectional curvature c. Then M^n is totally geodesic if and only if M^n has constant ϕ-sectional curvature c.*

We put $2q = 2m+1-n$ and choose $\{e_{n+1}, \ldots, e_{n+q}, e_{n+q+1} = \phi e_{n+1}, \ldots, e_{2m+1} = \phi e_{n+q}\}$ an orthonormal normal frame. Then the shape operators $A_\alpha = A_{e_{n+\alpha}}$ and $A_{\alpha^*} = A_{e_{n+q+\alpha}}, \alpha, \alpha^* = \overline{1,q}$, of an invariant submanifold M^n in a Sasakian manifold $\widetilde{M}^{2m+1}$ take the forms:

$$A_\alpha = \begin{pmatrix} A'_\alpha & A''_\alpha & 0 \\ A''_\alpha & -A'_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\alpha^*} = \begin{pmatrix} -A''_\alpha & A'_\alpha & 0 \\ A'_\alpha & A''_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

where A'_α and A''_α are $n \times n$ matrices.

We recall now two important examples of invariant submanifolds.

Example 2.1. Let S^{2m+1} be a unit sphere with standard Sasakian structure. An odd-dimensional unit sphere $S^{2n+1}(n < m)$ with induced structure is a totally geodesic Sasakian submanifold of S^{2m+1}. Obviously the Sasakian space form $\mathbf{R}^{2n+1}(-3)$ in $\mathbf{R}^{2m+1}(-3)$ is a totally geodesic Sasakian submanifold.

Example 2.2. The circle bundle (Q^n, S^1) over a hyperquadric in $\mathbf{C}P^{n+1}$ is a Sasakian submanifold of S^{2n+3} which is a η-Einstein manifold.

Chen inequalities for other classes of submanifolds (C-totally real and contact slant submanifolds, respectively) were obtained in [4] and [6].

3. Invariants of a Sasakian submanifold

In this section we prove the main results of the article.

First theorem gives an inequality between the infimum of K^r (intrinsic invariant) of an invariant submanifold and the ϕ-sectional curvature of the Sasakian space form (extrinsic invariant), i.e. the embedding space; the characterization of the equality case is given.

In the second theorem we obtain an inequality for δ_k^r of an invariant submanifold of a Sasakian space form and characterize the equality case for $k < 4$ (the submanifold is then totally geodesic) and $k = 4$ (in terms of the shape operator). For $k > 4$ the inequality fails.

Theorem 3.1. *For any invariant submanifold M^n in a Sasakian space form $\widetilde{M}^{2m+1}(c)$, we have:*

$$\inf K^r \leq \frac{c+3}{4}. \tag{3.1}$$

The equality case holds if and only if M^n is a totally geodesic submanifold.

Proof. By a ϕ-sectional curvature $H(X)$ of M^n with respect to a unit tangent vector X orthogonal to ξ, we mean the sectional curvature $K(X, \phi X)$ spanned by the vectors X and ϕX. Let $K(X,Y)$ be the sectional curvature determined by orthonormal vectors X and Y, with X,Y orthogonal to ξ, $g(X, \phi Y) = 0$. Then we have (see [1], p. 111):

$$K(X,Y) + K(X,\phi Y) = \frac{1}{4}[(H(X+\phi Y) + H(X-\phi Y) \\ + H(X+Y) + H(X-Y) - H(X) - H(Y) + 6].$$

Let T^1M^n denote the unit sphere bundle of M^n consisting of all unit tangent vectors on M^n. For each $x \in M^n$, we put

$$W_x = \{(X,Y);\ X,Y \in T_x^1 M^n,\ g(X,\xi) = g(Y,\xi) = g(X,Y) = g(X,\phi Y) = 0\}.$$

Then W_x is a closed subset of $T_x^1M^n \times T_x^1M^n$ and it is easy to verify that if $\{X,Y\}$ spans a totally real plane section, then both $\{X+\phi Y, X-\phi Y\}$ and $\{X+Y, X-Y\}$ also span totally real plane sections. We define a function $\hat{H} : W_x \to \mathbf{R}$ by

$$\hat{H}(X,Y) = H(X) + H(Y), \quad (X,Y) \in W_x.$$

Suppose that (X_m, Y_m) is a point in W_x such that $\hat{H}$ attains an absolute maximum value, say m_x, at (X_m, Y_m). It follows that

$$K(X_m, Y_m) + K(X_m, \phi Y_m) \leq \frac{1}{4}[\hat{H}(X_m, Y_m) + 6].$$

On the other hand, it is known that $H(X) \leq c$ (as in the Kaehler case, see [3]). Thus, from the previous relation, we obtain

$$K(X_m, Y_m) + K(X_m, \phi Y_m) \leq \frac{c+3}{2},$$

which implies the inequality (3.1).

For the equality case the proof is similar to the proof of Theorem 1 from [2]. □

Remark. It is well known that on a K-contact manifold, in particular Sasakian, the sectional curvature of a plane section which contains the vector ξ is equal to 1, i.e., $K(X, \xi) = 1$; thus we have considered only the case when X and Y are both orthogonal to ξ.

Theorem 3.2. *For any invariant submanifold M^n in a Sasakian space form $\widetilde{M}^{2m+1}(c)$, the following statements hold.*

(1) *For each $k \in (-\infty, 4]$, δ^r_k satisfies:*

$$\delta^r_k \leq [n(n-1) - 2k]\,\frac{c+3}{8} + (n-1)\frac{c-1}{8}. \tag{3.2}$$

(2) *Inequality (3.2) fails for every $k > 4$.*

(3) *$\delta^r_k = [n(n-1) - 2k]\,\frac{c+3}{8} + (n-1)\frac{c-1}{8}$ holds for some $k \in (-\infty, 4)$ if and only if M^n is a totally geodesic submanifold of $\widetilde{M}^{2m+1}(c)$.*

(4) *The invariant submanifold M^n satisfies*

$$\delta^r_4 = [n(n-1) - 8]\,\frac{c+3}{8} + (n-1)\frac{c-1}{8}$$

at a point $p \in M^n$ if and only if there exists an orthonormal basis

$$\{e_1, e_2 = \phi e_1, e_3, e_4 = \phi e_3, \ldots, e_{2k-1}, e_{2k} = \phi e_{2k-1}, e_{2k+1} = \xi, e_{n+1}, \ldots, e_{2m+1}\}$$

of $\widetilde{M}^{2m+1}(c)$ such that, with respect to this basis, the shape operator of M^n takes the forms (2.1), with

$$A'_\alpha = \begin{pmatrix} a_\alpha & b_\alpha & 0 \\ b_\alpha & -a_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A''_\alpha = \begin{pmatrix} a^*_\alpha & b^*_\alpha & 0 \\ b^*_\alpha & -a^*_\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $a_\alpha, b_\alpha, a_{\alpha^*}, b_{\alpha^*}$ are real numbers.

Proof. Let M^n be an invariant submanifold in a Sasakian space form $\widetilde{M}^{2m+1}(c)$. Then Gauss formula for the submanifold M^n is

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),$$

for any $X,Y,Z,W \in T_pM^n$.

Since $\widetilde{M}^{2m+1}(c)$ is a Sasakian space form, we have

$$\begin{aligned} 4\widetilde{R}(X,Y,Z,W) = {} & (c+3)\{-g(Y,Z)g(X,W) + g(X,Z)g(Y,W)\} \\ & + (c-1)\{-\eta(X)\eta(Z)g(Y,W) + \eta(Y)\eta(Z)g(X,W) \\ & - g(X,Z)\eta(Y)g(\xi,W) + g(Y,Z)\eta(X)g(\xi,W) \\ & - g(\phi Y,Z)g(\phi X,W) + g(\phi X,Z)g(\phi Y,W) \\ & + 2g(\phi X,Y)g(\phi Z,W)\}, \quad \forall X,Y,Z,W \in T_pM^n. \end{aligned} \tag{3.3}$$

We choose an orthonormal basis $\{e_1,\dots,e_n=\xi\} \subset T_pM^n$ and an orthonormal basis $\{e_{n+1},\dots,e_{2m+1}\} \subset T_p^{\perp}M^n$. For $X=Z=e_i$ and $Y=W=e_j$, and summing by $i,j=1,\dots,n$, in (3.3) we obtain:

$$\widetilde{R}(e_i,e_j,e_i,e_j) = (c+3)(-n+n^2) + (c-1)\Big[-2(n-1) + 3\sum_{i,j=1}^{n} g^2(\phi e_i,e_j)\Big].$$

In particular, we may choose an orthonormal frame

$$\{e_1, e_2=\phi e_1, e_3, e_4=\phi e_3, \dots, e_{2k-1}, e_{2k}=\phi e_{2k-1}, e_{2k+1}=\xi\},$$

$n=2k+1$. Then $\sum_{j=1}^{n} g^2(\phi e_i,e_j)=1$; we obtain

$$\widetilde{R}(e_i,e_j,e_i,e_j) = (n^2-n)(c+3) + (n-1)(c-1),$$

which implies that

$$2\tau = n(n-1)\frac{c+3}{4} + (n-1)\frac{c-1}{4} + n^2\|H\|^2 - \|h\|^2,$$

where $\|H\|^2$ and $\|h\|^2$ are the squared norm of the mean curvature vector

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i),$$

and the squared norm of the second fundamental form, respectively.

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

From Proposition 2.2 we have $H = 0$. Then:

$$2\tau = n(n-1)\frac{c+3}{4} + (n-1)\frac{c-1}{4} - \|h\|^2 . \tag{3.4}$$

From (3.4) we obtain

$$\tau \leq n(n-1)\frac{c+3}{8} + (n-1)\frac{c-1}{8}, \tag{3.5}$$

with equality if and only if M^n is totally geodesic, i.e., $h = 0$.

Next, let $\pi = sp(e_1, e_2)$ be a totally real plane section in T_pM^n. We compute $K(\pi)$ and consider in the Gauss equation $X = Z = e_1$ and $Y = W = e_2$.

Using (3.3), (3.4) and (2.1), we have

$$\begin{aligned} n(n-1)\frac{c+3}{4} + (n-1)\frac{c-1}{4} - 2\tau &= 4\sum_{\alpha=1}^{q}\{\|A'_\alpha\|^2 + \|A''_\alpha\|^2\} \\ \geq 4\sum_{\alpha=1}^{q}\{(h_{11}^{\alpha})^2 + (h_{22}^{\alpha})^2 &+ 2(h_{12}^{\alpha})^2 + (h_{11}^{\alpha^*})^2 + (h_{22}^{\alpha^*})^2 + 2(h_{12}^{\alpha^*})^2\} \\ \geq -8\sum_{\alpha=1}^{q}\{h_{11}^{\alpha}h_{22}^{\alpha} - (h_{12}^{\alpha})^2 &+ h_{11}^{\alpha^*}h_{22}^{\alpha^*} - (h_{12}^{\alpha^*})^2\} = -8K(\pi) + 2(c+3), \end{aligned}$$

where $h_{ij}^{r} = g(h(e_i, e_j), e_r)$.

Thus we have obtained

$$-2\tau + 8K(\pi) \geq 2(c+3) - (n-1)\frac{c-1}{4} - n(n-1)\frac{c+3}{4}. \tag{3.6}$$

Since inequality (3.6) holds for any totally real plane section, we get

$$\tau - 4\inf K^r \leq (n^2 - n - 8)\frac{c+3}{8} + (n-1)\frac{c-1}{8}. \tag{3.7}$$

For any positive number s, if we sum (3.5) multiplied by s and (3.7) we obtain:

$$(s+1)\tau - 4\inf K^r \leq [(s+1)n(n-1) - 8]\frac{c+3}{8} + (s+1)(n-1)\frac{c-1}{8}. \tag{3.8}$$

Since $s > 0$ and $\frac{4}{s+1} < 4$, we have

$$\tau - \frac{4}{s+1} \inf K^r \leq \left[n(n-1) - \frac{8}{s+1} \right] \frac{c+3}{8} + (n-1)\frac{c-1}{8}. \tag{3.9}$$

If we put $k = \frac{4}{s+1}$ and we use the definition of δ_k^r , we obtain

$$\delta_k^r \leq [n(n-1) - 2k] \frac{c+3}{8} + (n-1)\frac{c-1}{4}, \tag{3.10}$$

for any $k \in (0, 4)$. Combining (3.5), (3.7), (3.10) we get the inequality (3.2) for $k \in [0, 4]$.

The inequality for $k \leq 0$ follows from Theorem 3.1.

To prove statement (2) we get the next example: Let $\pi : S^7 \longrightarrow \mathbf{C}P^3$ be the Hopf fibration. The complex quadric Q_2 in $\mathbf{C}P^3$ is defined by:

$$Q_2 = \{(z_0, z_1, z_2, z_3) \in \mathbf{C}P^3 : z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0\}.$$

We put $M^5 = \pi^{-1}(Q_2)$. Then M^5 is an invariant submanifolds of S^7. We have the commutative diagram.

$$\begin{array}{ccc} M^5 & \rightarrow & S^7 \\ \downarrow \pi & & \downarrow \pi \\ Q_2 & \rightarrow & \mathbf{C}P^3 \end{array}.$$

If we assume that M^5 satisfies the equality case of the inequality (3.2), then Q_2 in $\mathbf{C}P^3$ satisfies the equality $\delta_k^r = 12 - k$. According to [2], one has $\delta_k^r = 8$. Hence this leads to a contradiction for any $k > 4$.

In order to prove statement (3) we consider 3 cases:

I) If $\delta_0^r = n(n-1)\frac{c+3}{8} + (n-1)\frac{c-1}{8}$, then (3.4) implies that M^n is totally geodesic.

II) If $\delta_k^r = [n(n-1) - 2k]\frac{c+3}{8} + (n-1)\frac{c-1}{8}$, for some $k \in (0, 4)$, then we can write :

$$\begin{aligned} [n(n-1) - 2k] \frac{c+3}{8} + (n-1)\frac{c-1}{8} &= (1 - \frac{k}{4})\delta_0^r + \frac{k}{4}\delta_4^r \\ &\leq [n(n-1) - 2k] \frac{c+3}{8} + (n-1)\frac{c-1}{8}, \end{aligned}$$

which implies that M^n is totally geodesic.

III) If $\delta_k^r = [n(n-1)-2k]\,\frac{c+3}{8} + (n-1)\frac{c-1}{8}$ for some $k \in (-\infty, 0)$, then we can write:

$$[n(n-1)-2k]\,\frac{c+3}{8} + (n-1)\frac{c-1}{8} = \tau - k \inf K^r$$
$$\leq [n(n-1)-2k]\,\frac{c+3}{8} + (n-1)\frac{c-1}{8},$$

which implies that M^n is totally geodesic.

For proving statement (4), we use the proof of statement (1) and the fact that the second fundamental form of M^n satisfies the conditions:

$$h_{11}^r + h_{22}^r = 0, \quad h_{1j}^r = h_{2j}^r = h_{jk}^r = 0, \quad j, k = 3, \ldots, n, r \in \{\alpha, \alpha^* | \alpha = \overline{1, q}\}.$$

Then, the shape operator of M^n takes the forms (2.1), with respect to the orthonormal basis

$$\{e_1, e_2 = \phi e_1, e_3, e_4 = \phi e_3, \ldots, e_{2k-1}, e_{2k} = \phi e_{2k-1}, e_{2k+1} = \xi, e_{n+1}, \ldots, e_{2m+1}\}.$$

Conversely, we suppose that the shape operator at a point $p \in M^n$ takes the form (2.1), with A'_α and A''_α given above, with respect to a suitable orthonormal basis. From the equation of Gauss we get $\inf K^r = K(e_1, e_2)$.

Also we have

$$n(n-1)\frac{c+3}{8} + (n-1) - \frac{c-1}{8} - 2\tau = -8K(e_1, e_2) + 2(c+3).$$

Thus, we get

$$\delta_4^r = [n(n-1)-8]\,\frac{c+3}{8} + (n-1)\frac{c-1}{8}. \qquad \square$$

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