Publ. Math. Debrecen
82/3-4 (2013), 697–708
DOI: 10.5486/PMD.2013.5461

On group algebras with unit groups of derived length three in characteristic three

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Abstract. Let K be a field of characteristic 3 and let G be a finite 3-group of class 2. Necessary and sufficient conditions are obtained for the group of units $U(KG)$ to be solvable of derived length 3.

1. Introduction

Let KG be the group algebra of a finite group G over a field K of characteristic p. Let $U = U(KG)$ be the group of units of the group algebra KG. First description of the solvability of the unit group U is given in [16, Chapter VI]. This problem has been discussed by many authors as can be seen in [3], [4], [5], [6], [10], [11]. Computation of the derived length of U and its connection with the order and nature of the commutator subgroup G' of G is an interesting problem. Shalev [17] has found necessary and sufficient conditions for U to be metabelian when $p \geq 3$. This work was completed by Coleman and Sandling [7] and independently by Kurdics [9] for $p = 2$. For $p \neq 2$, a complete description of group algebras KG with centrally metabelian unit groups is given in [13]. The group algebras with $\gamma_3(\delta^1(U)) = 1$ have been listed in [15]. Baginski [2] and Balogh and Li [1] have computed the derived length of U for finite p-groups and arbitrary groups with cyclic commutator subgroup of order p^n $(p > 2)$, respectively. Recently we have obtained the necessary and sufficient conditions for U to have derived length 3 when $p \neq 2, 3$, see [8]. As in [8], for $p = 3$, if $U''' = 1$, then

Mathematics Subject Classification: Primary: 16S34, 16U60.
Key words and phrases: group rings, unit groups.
The first author is very thankful to UGC for providing financial support in the form of SRF.

$G = P \rtimes H$ where P is a normal Sylow 3-subgroup of G and H is an abelian $3'$-subgroup of G. So in this paper, we continue our work on this problem when $p = 3$ and G is a finite 3-group. In addition, we assume that $\gamma_3(G) = 1$. We will use $(x, y) = x^{-1}y^{-1}xy$ for the group commutator of elements x and y of a group G and if $o(x) = n$ then $\widehat{x} = 1 + x + x^2 + \cdots + x^{n-1}$. The Lie commutators are denoted by $[x, y] = xy - yx$, $x, y \in KG$.

Our main result is as follows:

Theorem 1.1. *Let K be a field such that $\mathrm{Char}\, K = 3$ and let G be a finite 3-group of class 2. Then the following conditions are equivalent:*

(i) *$U''' = 1$;*

(ii) *G' is elementary abelian 3-subgroup such that $|G'| \leq 3^3$.*

2. Proof of the Theorem

Throughout this section, K is a field of characteristic 3.

Lemma 2.1. *Let G be a finite 3-group of class 2 such that $U''' = 1$. Then G' has exponent 3.*

Proof. Let $x, y \in G$, $z = (x, y)$, $o(z) = 3^n$, where $n \geq 2$. If $u = 1 + x - y$, then

$$u_1 = (u, y) = 1 + u^{-1}x(z - 1),$$

$$u_2 = (u, x) = 1 + u^{-1}y(z - 1)z^{-1}$$

$$\text{and} \quad u_3 = (u, y^{-1}) = 1 - u^{-1}x(z - 1)z^{-1}.$$

Clearly $(u_1, u_3) = 1$. Now

$$v = (u_1, u_2) = 1 + u_1^{-1}u_2^{-1}u^{-2}yxu^{-1}(z - 1)^3 z^{-1}$$

$$\text{and} \quad w = (u_3, u_2) = 1 - u_3^{-1}u_2^{-1}u^{-2}yxu^{-1}(z - 1)^3 z^{-2}.$$

Since $U''' = 1$, so $[v, w] = [u_1^{-1}\beta, u_3^{-1}\beta](z - 1)^6 z^{-3} = 0$ where $\beta = u_2^{-1}u^{-2}yxu^{-1}$. The annihilator A of $(z-1)^6$ in KG is a two sided ideal. The above equation implies that $\overline{\beta}^{-1}\overline{u_1}$ and $\overline{\beta}^{-1}\overline{u_3}$ commute in KG/A where $\overline{w}$ is the image of $w \in KG$ in KG/A. Hence

$$[\beta^{-1}u_1, \beta^{-1}u_3](z - 1)^6 = 0.$$

On simplifying we get

$$\{2uu_2 - x^{-1}yu^{-1}xy^{-1}u^2u_2 - ux^{-1}u^{-1}y^{-1}xu^{-1}xu^2u_2z^{-1} + ux^{-1}u^{-1}y^{-1}xu^2u_2z^{-1} + ux^{-1}y^{-1}u^{-1}yxuu_2 - (z-1)z^{-1}\}(z-1)^8 = 0.$$

As $u_2 \in 1 + (z-1)KG$, hence on multiplying this equation by $(z-1)^{3^n-9}$, we get $(z-1)^{3^n-1} = 0$, which is a contradiction. Hence $n = 1$ and exponent of G' is 3. $\square$

To prove the theorem we only need to show that $|G'| \leq 3^3$. Suppose that $G' \neq C_3$. Then, there exist $x, y, z \in G$ such that $(x, y) \neq 1$ and $(x, z) \notin \langle (x, y) \rangle$. If for all such triplets $x, y, z \in G$, $(y, z), (x, g) \in \langle (x, y), (x, z) \rangle$ for all $g \in G$, then $G' = C_3 \times C_3$, [14, Theorem 14, Step III]. So if $G' \neq C_3 \times C_3$, then there exists a triplet $x, y, z \in G$ such that either $(y, z) \notin \langle (x, y), (x, z) \rangle$ or $(x, g) \notin \langle (x, y), (x, z) \rangle$ for some $g \in G$. We first prove some preliminary results based on these two cases.

Lemma 2.2. *Let G be a finite 3-group of class 2 such that $U''' = 1$. Let $u, v, x, y, z \in G$ such that $a = (x, y) \neq 1$, $b = (x, z) \notin \langle a \rangle$ and $c = (y, z) \notin \langle a, b \rangle$. If (u, x), (u, y), $(u, z) \in \langle a, b, c \rangle$ and $(u, v) \notin \langle a, b, c \rangle$, then $(u, x) = (u, y) = (u, z) = 1$.*

Proof. If $t \in G'$, then $1 + g(t-1)$ is a unit for all $g \in G$. Let $\alpha = (u, v) - 1$ and $(u, x) = a^l b^m c^n$. Now

$$\begin{aligned} r_1 &= (1 + x\alpha, y) = 1 + x\alpha(a-1) - x^2(a-1)\alpha^2 \\ \text{and} \quad r_2 &= (1 + u^{-1}, v^{-1}) = 1 + (1+u)^{-1}\alpha. \end{aligned}$$

$$\begin{aligned} \text{Then} \quad u_1 &= (r_1, r_2) = 1 + r_1^{-1}r_2^{-1}[r_1 - 1, r_2 - 1] = 1 + [x, (1+u)^{-1}](a-1)\alpha^2 \\ &= 1 - (1+u)^{-1}ux(1+u)^{-1}(a-1)((x, u) - 1)\alpha^2 \in U''. \end{aligned}$$

Also if

$$\begin{aligned} r_3 &= (1 + y^{-1}, x^{-1}) = 1 + (1+y)^{-1}(a^{-1} - 1) \\ \text{and } r_4 &= (1 + z^{-1}, y^{-1}) = 1 + (1+z)^{-1}(c^{-1} - 1). \end{aligned}$$

$$\text{Then } u_2 = (r_3, r_4) = 1 - r_3^{-1}(1+y)^{-2}(1+z)^{-2}zy(a^{-1} - 1)\widehat{c} \in U''.$$

As $[u_1, u_2] = 0$, so we have

$$[(1+u)^{-1}ux(1+u)^{-1}, (1+y)^{-2}(1+z)^{-2}zy]\widehat{a}\widehat{c}((x, u) - 1)\alpha^2 = 0.$$

If $M = \langle a, b, c \rangle$, then $\Delta^7(M) = 0$. Since (u, x), (u, y), $(u, z) \in M$, hence we have

$$[(1+u)^{-2}ux, (1+y)^{-2}(1+z)^{-2}zy]\widehat{a}\widehat{c}((x, u) - 1)\alpha^2 = 0.$$

Thus

$$[(1+u)^{-2}ux, y^{-1}z^{-1}(1+z)^2(1+y)^2]\widehat{a}\widehat{c}((x,u)-1)\alpha^2 = 0.$$

Equivalently

$$\begin{aligned}\{(1+u)^{-2}u[x, y^{-1}z^{-1}(1+z)^2(1+y)^2]x^{-1} \\ + [(1+u)^{-2}u, y^{-1}z^{-1}(1+z)^2(1+y)^2]\}\widehat{a}\widehat{c}((x,u)-1)\alpha^2 = 0. \quad (2.1)\end{aligned}$$

On replacing x by x^{-1} in r_1 we get

$$r_1' = 1 + x^{-1}\alpha(a^{-1}-1) - x^{-2}\alpha^2(a^{-1}-1).$$

Then

$$u_1' = (r_1', r_2) = 1 - (1+u)^{-1}ux^{-1}(1+u)^{-1}((x,u)^{-1}-1)(a^{-1}-1)\alpha^2.$$

As $[u_1', u_2] = 0$, so we have

$$\begin{aligned}\{(1+u)^{-2}u[x^{-1}, y^{-1}z^{-1}(1+z)^2(1+y)^2]x \\ + [(1+u)^{-2}u, y^{-1}z^{-1}(1+z)^2(1+y)^2]\}\widehat{a}\widehat{c}((x,u)-1)\alpha^2 = 0. \quad (2.2)\end{aligned}$$

Subtracting (2.2) from (2.1) we get

$$0 = \{[x, z^{-1}+z]x^{-1} - [x^{-1}, z^{-1}+z]x\}\widehat{a}\widehat{c}((x,u)-1)\alpha^2 = 2m(z-z^{-1})\widehat{a}\widehat{b}\widehat{c}\alpha^2.$$

Thus $m = 0$.

Now take $r_5 = (1+x\alpha, z)$, $r_5' = (1+x^{-1}\alpha, z)$, $r_6 = (1+y^{-1}, z^{-1})$, $r_7 = (1+z^{-1}, x^{-1})$ and the commutators $u_3 = (r_5, r_2)$, $u_3' = (r_5', r_2)$, $u_4 = (r_6, r_7)$ we have

$$0 = [u_3, u_4] = [(1+u)^{-1}ux(1+u)^{-1}, (1+z)^{-2}(1+y)^{-2}yz]\widehat{b}\widehat{c}((x,u)-1)\alpha^2.$$

Equivalently

$$\begin{aligned}\{(1+u)^{-2}u[x, z^{-1}y^{-1}(1+y)^2(1+z)^2]x^{-1} \\ + [(1+u)^{-2}u, z^{-1}y^{-1}(1+y)^2(1+z)^2]\}\widehat{b}\widehat{c}((x,u)-1)\alpha^2 = 0. \quad (2.3)\end{aligned}$$

Also $[u_3', u_4] = 0$ implies

$$\begin{aligned}\{(1+u)^{-2}u[x^{-1}, z^{-1}y^{-1}(1+y)^2(1+z)^2]x \\ + [(1+u)^{-2}u, z^{-1}y^{-1}(1+y)^2(1+z)^2]\}\widehat{b}\widehat{c}((x,u)-1)\alpha^2 = 0. \quad (2.4)\end{aligned}$$

Subtracting (2.4) from (2.3) we get

$$0=\{[x,y^{-1}+y]x^{-1}-[x^{-1},y^{-1}+y]x\}\widehat{b}\widehat{c}((x,u)-1)\alpha^2=2l(y-y^{-1})\widehat{a}\widehat{b}\widehat{c}\alpha^2.$$

Thus $l=0$. Now

$$u_5=(r_3,r_7)$$
$$=1+r_3^{-1}r_7^{-1}(1+z)^{-1}(1+y)^{-1}zy(1+y)^{-1}(1+z)^{-1}(a^{-1}-1)(b^{-1}-1)(c-1).$$

Hence $[u_1,u_5]=0$ yields

$$0=[(1+u)^{-1}ux(1+u)^{-1},r_7^{-1}(1+z)^{-1}(1+y)^{-1}$$
$$\times zy(1+y)^{-1}(1+z)^{-1}]\widehat{a}(b-1)(c-1)((x,u)-1)\alpha^2.$$

Equivalently

$$\begin{aligned}&\{(1+u)^{-2}u[x,y^{-1}z^{-1}(1+z)^2(1+y)^2]x^{-1}\\&\quad+[(1+u)^{-2}u,y^{-1}z^{-1}(1+z)^2(1+y)^2]\}\\&\quad\times\widehat{a}(b-1)(c-1)((x,u)-1)\alpha^2=0.\end{aligned}\tag{2.5}$$

Also $[u_1',u_5]=0$ yields

$$\begin{aligned}&\{(1+u)^{-2}u[x^{-1},y^{-1}z^{-1}(1+z)^2(1+y)^2]x\\&\quad+[(1+u)^{-2}u,y^{-1}z^{-1}(1+z)^2(1+y)^2]\}\\&\quad\times\widehat{a}(b-1)(c-1)((x,u)-1)\alpha^2=0.\end{aligned}\tag{2.6}$$

Subtracting (2.6) from (2.5) we get

$$\begin{aligned}0&=\{[x,z^{-1}+z]x^{-1}-[x^{-1},z^{-1}+z]x\}\widehat{a}(b-1)(c-1)((x,u)-1)\alpha^2\\&=2n(z-z^{-1})\widehat{a}\widehat{b}\widehat{c}\alpha^2.\end{aligned}$$

Thus $n=0$ and $(u,x)=1$.

On interchanging x and y in the above proof we get $(u,y)=1$. Similarly, if we interchange x and z we get $(u,z)=1$. □

If for all triplets $x,y,z\in G$ such that $a=(x,y)\neq 1$, $b=(x,z)\notin\langle a\rangle$, we have $c=a^ib^j$, then on replacing z by $x^{i-1}z$ and y by $x^{1-j}y$, we can assume that $c=ab$.

Lemma 2.3. *Let G be a finite 3-group of class 2 such that $U''' = 1$. Let for all triplets x, y, z in G such that $a = (x, y) \neq 1$, $b = (x, z) \notin \langle a\rangle$, $c = (y, z) \in \langle a, b\rangle$. If $u, v, g \in G$ such that $d = (x, g) \notin \langle a, b\rangle$, $(u, v) \notin \langle a, b, d\rangle$ and (u, x), (u, y), $(u, z) \in \langle a, b, d\rangle$. Then $(u, x) = (u, y) = (u, z) = 1$.*

Proof. Let $\alpha = (u, v) - 1$ and $(u, x) = a^l b^m d^n$. Now

$$r_8 = (1 + x\alpha, g) = 1 + x\alpha(d - 1) - x^2(d - 1)\alpha^2,$$
$$r_8' = (1 + x^{-1}\alpha, g) = 1 + x^{-1}\alpha(d^{-1} - 1) - x^{-2}\alpha^2(d^{-1} - 1)$$

and

$$r_9 = (1 + x^{-1}, g^{-1}) = 1 + (1 + x)^{-1}(d - 1).$$

Then

$$u_6 = (r_8, r_2) = 1 - (1 + u)^{-1}ux(1 + u)^{-1}(d - 1)((x, u) - 1)\alpha^2,$$
$$u_6' = (r_8', r_2) = 1 - (1 + u)^{-1}ux^{-1}(1 + u)^{-1}(d^{-1} - 1)((x, u)^{-1} - 1)\alpha^2$$

and

$$u_7 = (r_9, r_6)$$
$$= 1 + r_9^{-1}r_6^{-1}(1 + x)^{-1}(1 + y)^{-1}yx(1 + y)^{-1}(1 + x)^{-1}(a - 1)(c - 1)(d - 1).$$

As $[u_6, u_7] = 0$, so we have

$$[(1 + u)^{-1}ux(1 + u)^{-1}, r_6^{-1}(1 + x)^{-1}(1 + y)^{-1}yx(1 + y)^{-1}(1 + x)^{-1}]$$
$$(a - 1)(c - 1)\widehat{d}((x, u) - 1)\alpha^2 = 0.$$

Let $N = \langle a, b, d\rangle$, then $\Delta^7(N) = 0$. Since (u, x), $(u, y) \in N$, hence we have

$$[(1 + u)^{-2}ux, (1 + x)^{-2}(1 + y)^{-2}yx](a - 1)(c - 1)\widehat{d}((x, u) - 1)\alpha^2 = 0.$$

Thus

$$[(1 + u)^{-2}ux, x^{-1}y^{-1}(1 + y)^2(1 + x)^2](a - 1)(c - 1)\widehat{d}((x, u) - 1)\alpha^2 = 0.$$

Equivalently

$$\begin{aligned}\{(1 + u)^{-2}u[x, x^{-1}y^{-1}(1 + y)^2(1 + x)^2]x^{-1} \\ + [(1 + u)^{-2}u, x^{-1}y^{-1}(1 + y)^2(1 + x)^2]\} \\ \times (a - 1)(c - 1)\widehat{d}((x, u) - 1)\alpha^2 = 0.\end{aligned} \tag{2.7}$$

Also $[u_6', u_7] = 0$, so we have

$$\begin{aligned}\{(1+u)^{-2}u[x^{-1}, x^{-1}y^{-1}(1+y)^2(1+x)^2]x \\ + [(1+u)^{-2}u, x^{-1}y^{-1}(1+y)^2(1+x)^2]\} \\ \times (a-1)(c-1)\widehat{d}((x,u)-1)\alpha^2 = 0.\end{aligned} \tag{2.8}$$

Subtracting (2.8) from (2.7) we get

$$\begin{aligned}0 &= \{[x, y^{-1}+y]x^{-1} - [x^{-1}, y^{-1}+y]x\}(a-1)(c-1)\widehat{d}((x,u)-1)\alpha^2 \\ &= 2(y-y^{-1})\widehat{a}(c-1)\widehat{d}((x,u)-1)\alpha^2 = 2m(y-y^{-1})\widehat{a}\widehat{b}\widehat{d}\alpha^2.\end{aligned}$$

Thus $m = 0$.

Now take $r_{10} = (1+x^{-1}, z^{-1})$, $r_{11} = (1+y^{-1}, g^{-1})$ and $u_8 = (r_{10}, r_{11})$. We have

$$\begin{aligned}0 = [u_3, u_8] = [(1+u)^{-1}ux(1+u)^{-1}, r_{11}^{-1}(1+x)^{-1}(1+y)^{-1} \\ \times yx(1+y)^{-1}(1+x)^{-1}](a-1)\widehat{b}((y,g)-1)((x,u)-1)\alpha^2.\end{aligned}$$

Equivalently

$$\begin{aligned}(1+u)^{-2}u[x, x^{-1}y^{-1}(1+y)^2(1+x)^2]x^{-1} \\ + [(1+u)^{-2}u, x^{-1}y^{-1}(1+y)^2(1+x)^2]\} \\ \times (a-1)\widehat{b}((y,g)-1)((x,u)-1)\alpha^2 = 0.\end{aligned} \tag{2.9}$$

Also $[u_3', u_8] = 0$ implies

$$\begin{aligned}\{(1+u)^{-2}u[x^{-1}, x^{-1}y^{-1}(1+y)^2(1+x)^2]x \\ + [(1+u)^{-2}u, x^{-1}y^{-1}(1+y)^2(1+x)^2]\} \\ \times (a-1)\widehat{b}((y,g)-1)((x,u)-1)\alpha^2 = 0.\end{aligned} \tag{2.10}$$

Subtracting (2.10) from (2.9) we get

$$\begin{aligned}0 &= \{[x, y^{-1}+y]x^{-1} - [x^{-1}, y^{-1}+y]x\}(a-1)\widehat{b}((y,g)-1)((x,u)-1)\alpha^2 \\ &= 2(y-y^{-1})\widehat{a}\widehat{b}((y,g)-1)((x,u)-1)\alpha^2.\end{aligned}$$

Let $(y,g) = a^r d^s$. On replacing y by xy, if needed, we can assume that $s \neq 0$. Hence

$$2n(y-y^{-1})\widehat{a}\widehat{b}\widehat{d}\alpha^2 = 0.$$

Thus $n = 0$. Now let $u_9 = (r_9, r_4)$. Then

$$0 = [u_6, u_9] = [(1+u)^{-1}ux(1+u)^{-1}, r_4^{-1}(1+x)^{-1}(1+z)^{-1} \times zx(1+z)^{-1}(1+x)^{-1}](b-1)(c-1)\widehat{d}((x,u)-1)\alpha^2.$$

Equivalently

$$\{(1+u)^{-2}u[x, x^{-1}z^{-1}(1+z)^2(1+x)^2]x^{-1} + [(1+u)^{-2}u, x^{-1}z^{-1}(1+z)^2(1+x)^2]\} \times (b-1)(c-1)\widehat{d}((x,u)-1)\alpha^2 = 0. \tag{2.11}$$

Also $[u_6', u_9] = 0$ implies

$$\{(1+u)^{-2}u[x^{-1}, x^{-1}z^{-1}(1+z)^2(1+x)^2]x + [(1+u)^{-2}u, x^{-1}z^{-1}(1+z)^2(1+x)^2]\} \times (b-1)(c-1)\widehat{d}((x,u)-1)\alpha^2 = 0. \tag{2.12}$$

Subtracting (2.12) from (2.11) we get

$$\begin{aligned} 0 &= \{[x, z^{-1}+z]x^{-1} - [x^{-1}, z^{-1}+z]x\}(b-1)\widehat{d}(c-1)((x,u)-1)\alpha^2 \\ &= 2(z - z^{-1})\widehat{b}(c-1)\widehat{d}((x,u)-1)\alpha^2 = 2l(z-z^{-1})\widehat{a}\widehat{b}\widehat{d}\alpha^2. \end{aligned}$$

Thus $l = 0$ and $(u, x) = 1$. On interchanging x and y in the above proof we get $(u, y) = 1$. Similarly interchanging x and z leads to $(u, z) = 1$. □

Theorem 2.1. *Let G be a group of class 2. Then $U''' = 1$ if and only if G' is an elementary abelian $3-$subgroup of G such that $|G'| \leq 3^3$.*

Proof. Let $U''' = 1$. Then by Lemma 2.1, G' is an elementary abelian 3-subgroup of G. Suppose that $G' \neq C_3$, then there exist $x, y, z \in G$ such that $(x, y) \neq 1$ and $(x, z) \notin \langle (x, y) \rangle$. If for all such triplets $x, y, z \in G$, $(y, z), (x, g) \in \langle (x, y), (x, z) \rangle$ for all $g \in G$, then $G' = C_3 \times C_3$ by [14]. Now we have two cases which we examine one by one:

Case (I): Let $x, y, z \in G$ such that $a = (x, y) \neq 1$, $b = (x, z) \notin \langle a \rangle$, $c = (y, z) \notin \langle a, b \rangle$. Then we show that $G' = \langle a, b, c \rangle = M$. Let, if possible, $u, v \in G$ such that $(u, v) \notin M$. For any $t \in G$, let $r_{12} = (1+x^{-1}, y^{-1})$, $r_{13} = (1+z((x,t)-1), x)$, $r_{14} = (1+x^{-1}, t^{-1})$,

$$u_{10} = (r_{12}, r_6) = 1 + r_6^{-1}(1+x)^{-2}(1+y)^{-2}yx\widehat{a}(c-1) \in U''$$

and

$$u_{11} = (r_{13}, r_{14}) = 1 - (1+x)^{-2} zx\widehat{b}\widehat{(x,t)} \in U''.$$

Thus $[u_{10}, u_{11}] = 0$ leads to

$$[r_6^{-1}(1+x)^{-2}(1+y)^{-2}yx, (1+x)^{-2}zx]\widehat{a}\widehat{b}(c-1)\widehat{(x,t)} = 0.$$

Equivalently

$$0 = [y^{-1}(1+y)^2, z]\widehat{a}\widehat{b}(c-1)\widehat{(x,t)} = [y+y^{-1}, z]\widehat{a}\widehat{b}(c-1)\widehat{(x,t)} = (1-y)\widehat{a}\widehat{b}\widehat{c}\widehat{(x,t)}.$$

Thus $(x,t) \in M$, for all $t \in G$. On interchanging x and y in the above proof, we get $(y,t) \in M$, for all $t \in G$. Similarly, interchanging x and z leads to $(z,t) \in M$, for all $t \in G$.

Now by Lemma 2.2, $(u,x) = 1 = (u,y)$. So $(zu,x) = b^{-1}$, $(zu,y) = c^{-1}$ and $a = (x,y) \notin \langle (zu,x), (zu,y)\rangle = \langle b, c\rangle$. This yields $(zu,v) = (z,v)(u,v) \in M$. Hence $(u,v) \in M$.

Case (II): For all $x,y,z \in G$ such that $(x,y) \neq 1$ and $(x,z) \notin \langle (x,y)\rangle$, let $(y,z) \in \langle (x,y), (x,z)\rangle$. Out of all such triplets, there is a triplet $x,y,z \in G$ such that $(x,g) \notin \langle (x,y), (x,z)\rangle$ for some $g \in G$. Let $a = (x,y)$, $b = (x,z)$, $c = (y,z)$ and $d = (x,g)$. Then we shall prove that $G' = \langle a,b,d\rangle = N$. As noted earlier, we can assume that $c = ab$. For all $t \in G$, let $r_{15} = (1+g^{-1}, x^{-1})$, $r_{16} = (1 + y((x,t)-1), z)$,

$$u_{12} = (r_{16}, r_{14}) = 1 - (1+x)^{-1}xy(1+x)^{-1}(a^{-1}-1)(c-1)\widehat{(x,t)} \in U''$$

and

$$u_{13} = (r_{15}, r_{10}) = 1 + r_{10}^{-1}(1+g)^{-2}(1+x)^{-2}gx(b-1)\widehat{d} \in U''.$$

Thus

$$\begin{aligned} &0 = [u_{12}, u_{13}] \\ &= [(1+x)^{-1}xy(1+x)^{-1}, r_{10}^{-1}(1+g)^{-2}(1+x)^{-2}gx](a^{-1}-1)(b-1)(c-1)\widehat{d}\widehat{(x,t)}. \end{aligned}$$

Equivalently

$$\begin{aligned} 0 &= [y, x^{-1}(1+x)^2 g^{-1}(1+g)^2](a-1)(b-1)(c-1)\widehat{d}\widehat{(x,t)} \\ &= \{[y, x^{-1}+x]g^{-1}(1+g)^2 + x^{-1}(1+x)^2[y, g^{-1}+g]\} \\ &\quad \times (a-1)(b-1)(c-1)\widehat{d}\widehat{(x,t)}. \end{aligned} \tag{2.13}$$

Since $d \notin \langle a\rangle$, we can write $(y,g) = a^r d^s$. On replacing g by $x^r g$, we get

$(y, x^r g) = d^s$ and the above equation yields

$$(x-1)\widehat{a}\widehat{b}\widehat{d}\widehat{(x,t)} = 0.$$

Thus $(x, t) \in N$. On interchanging x and y in (2.13) we get

$$\{[x, y^{-1} + y]g^{-1}(1+g)^2 + y^{-1}(1+y)^2[x, g^{-1} + g]\} (a-1)(b-1)(c-1)\widehat{(y,g)}\widehat{(y,t)} = 0.$$

As before, on replacing g by $x^r g$ we get

$$s^2(y-1)\widehat{a}\widehat{b}\widehat{d}\widehat{(y,t)} = 0.$$

So if $s \neq 0$, then $(y, t) \in N$. If $s = 0$, then replace y by xy to get

$$(xy-1)\widehat{a}\widehat{b}\widehat{d}\widehat{(xy,t)} = 0.$$

Since $(x, t) \in N$, we conclude $(y, t) \in N$. Similarly on interchanging x and z in (2.13) we get

$$\{[y, z^{-1}+z]g^{-1}(1+g)^2+z^{-1}(1+z)^2[y, g^{-1}+g]\}(a-1)(b-1)(c-1)\widehat{(z,g)}\widehat{(z,t)} = 0.$$

Now $d \notin \langle b \rangle$, so $(z, g) = b^l d^m$ and on replacing g by $x^l g$, we get $(z, x^l g) = d^m$ and $(y, x^l g) = a^{r-l} d^s$. Thus if $r = l$, we get

$$m^2(z-1)\widehat{a}\widehat{b}\widehat{d}\widehat{(z,t)} = 0$$

and $(z, t) \in N$, if $m \neq 0$. For $m = 0$, replacing z by $x^2 z$ leads to the same conclusion. If $r \neq l$, then replacing z by xz for $m \neq 2$ and by $x^2 z$ for $m = 2$, yields the same result.

Let $u, v \in G$ such that $(u, v) \notin N$. Then $(u, x) = 1 = (u, y)$, by Lemma 2.3. Thus $(x, uy) = a, (x, uz) = b$, $(x, ug) = d$, and hence $(uy, v) = (u, v)(y, v) \in N$. So $(u, v) \in N$.

Conversely, if G' is a central and elementary abelian 3-subgroup of G of order $\leq 3^3$ then by [12, Theorem 2.3], we have $\delta^{(3)}(KG) = 0$ and hence $U''' = 1$. $\square$

Finally, we give an example of a finite group G with $G' = C_3 \times C_3 \times C_3$ but non-central and show that for this group derived length of U is more than 3.

Example 2.1. Let $G = \langle a, b, c, d | a^3 = b^3 = c^3 = d^2 = 1, (a, b) = (a, c) = (b, c) = 1, (a, d) = a, (b, d) = b, (c, d) = c\rangle$. Then $G' = C_3 \times C_3 \times C_3 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ is not central in G and $U''' \neq 1$.

Let K be a field with $\operatorname{Char} K = 3$, then

$$\begin{aligned} u_1 &= (1 + d(a-1), d) = 1 + da(a-1) + \widehat{a}, \\ r &= (u_1, b) = 1 + da(a-1)(b^{-1}-1) + \widehat{a}(b^{-1}-1), \\ u_2 &= (1 + d(c-1), d) = 1 + dc(c-1) + \widehat{c} \\ \text{and} \quad s &= (u_2, b) = 1 + dc(c-1)(b^{-1}-1) + \widehat{c}(b^{-1}-1). \end{aligned}$$

Then

$$\begin{aligned} (r, s) &= 1 + r^{-1}s^{-1}[da(a-1)(b^{-1}-1) + \widehat{a}(b^{-1}-1), dc(c-1)(b^{-1}-1) + \widehat{c}(b^{-1}-1)] \\ &= 1 + r^{-1}s^{-1}d\widehat{b}(a-1)\{c(a-1) - a(c-1)\} = 1 + d\widehat{b}(a-1)(c-1)(a-c). \end{aligned}$$

This implies that $\delta^3(U(KG)) \neq 1$.

Acknowledgments: The authors wish to thank the referee for useful suggestions that led to a better presentation of the paper.

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(Received February 27, 2012; revised July 11, 2012)