Publ. Math. Debrecen
82/3-4 (2013), 709–725
DOI: 10.5486/PMD.2013.5492

The influence of $\mathfrak{F}_s$-quasinormality of subgroups on the structure of finite groups

By XIAOLONG YU (Hefei), XIAOYU CHEN (Hefei) and WENBIN GUO (Hefei)

Abstract. Let $\mathfrak{F}$ be a class of finite groups. A subgroup H of a finite group G is said to be $\mathfrak{F}_s$-quasinormal in G if there exists a normal subgroup T of G such that HT is s-permutable in G and $(H \cap T)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G. In this paper, we investigate further the influence of $\mathfrak{F}_s$-quasinormality of some subgroups on the structure of finite groups. New characterization of some classes of finite groups are obtained.

1. Introduction

Recall that a subgroup H of G is said to be s-quasinormal (or s-permutable) in G if H is permutable with every Sylow subgroup P of G (that is, $HP = PH$). The s-permutability of a subgroup of a finite group G often yields a wealth of information about the group G itself. In the past, it has been studied by many scholars (such as [1]–[2], [7]–[9], [13], [17]). Recently, Huang [10] introduced the following concept:

Definition 1.1. Let $\mathfrak{F}$ be a non-empty class of groups and H a subgroup of a group G. H is said to be $\mathfrak{F}_s$-quasinormal in G if there exists a normal subgroup T of G such that HT is s-permutable in G and $(H \cap T)H_G/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$, where H_G is the maximal normal subgroup of G contained in H.

Mathematics Subject Classification: 20D10, 20D15, 20D20.
Key words and phrases: $\mathfrak{F}_s$-quasinormal subgroup, Sylow subgroup, maximal subgroup, n-maximal subgroup.
Research is supported by NNSF of China (Grant #11071229) and Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant #20113402110036).
The third author is the corresponding one.

Note that, for a class $\mathfrak{F}$ of groups, a chief factor H/K of a group G is called $\mathfrak{F}$-central (see [16] or [4, Definition 2.4.3]) if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$. The symbol $Z_\infty^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathfrak{F}$-central. A subgroup H of G is said to be $\mathfrak{F}$-hypercenter in G if $H \leq Z_\infty^{\mathfrak{F}}(G)$.

By using this new concept, Huang [10] has given some conditions under which a finite group belongs to some formations. In this paper, we will go to further into the influence of $\mathfrak{F}_s$-quasinormal subgroups on the structure of finite groups. New characterizations of some classes of finite groups are obtained.

All groups considered in the paper are finite and G denotes a finite group. The notations and terminology in this paper are standard, as in [4] and [14].

2. Preliminaries

Let $\mathfrak{F}$ be a class of finite groups. Then $\mathfrak{F}$ is called a formation if it is closed under homomorphic image and every group G has a smallest normal subgroup (called $\mathfrak{F}$-residual and denoted by $G^{\mathfrak{F}}$) with quotient is in $\mathfrak{F}$. $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathfrak{F}$. $\mathfrak{F}$ is said to be S-closed (S_n-closed) if it contains all subgroups (all normal subgroups, respectively) of all its groups.

We use $\mathfrak{N}$, $\mathfrak{U}$, and $\mathfrak{S}$ to denote the formations of all nilpotent groups, supersoluble groups and soluble groups, respectively.

The following known results are useful in our proof.

Lemma 2.1 ([8, Lemma 2.2]). *Let G be a group and $H \leq K \leq G$.*

(1) *If H is s-permutable in G, then H is s-permutable in K;*

(2) *Suppose that H is normal in G. Then K/H is s-permutable in G/H if and only if K is s-permutable in G;*

(3) *If H is s-permutable in G, then H is subnormal in G;*

(4) *If H and F are s-permutable in G, the $H \cap F$ is s-permutable in G;*

(5) *If H is s-permutable in G and $M \leq G$, then $H \cap M$ is s-permutable in M.*

Lemma 2.2 ([10, Lemma 2.3]). *Let G be a group and $H \leq K \leq G$.*

(1) *H is $\mathfrak{F}_s$-quasinormal in G if and only if there exists a normal subgroup T of G such that HT is s-permutable in G, $H_G \leq T$ and $H/H_G \cap T/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$;*

(2) *Suppose that H is normal in G. Then K/H is $\mathfrak{F}_s$-quasinormal in G/H if and only if K is $\mathfrak{F}_s$-quasinormal in G;*

(3) *Suppose that H is normal in G. Then, for every $\mathfrak{F}_s$-quasinormal subgroup E of G satisfying $(|H|,|E|)=1$, HE/H is $\mathfrak{F}_s$-quasinormal in G/H;*

(4) *If H is $\mathfrak{F}_s$-quasinormal in G and $\mathfrak{F}$ is S-closed, then H is $\mathfrak{F}_s$-quasinormal in K;*

(5) *If H is $\mathfrak{F}_s$-quasinormal in G, K is normal in G and $\mathfrak{F}$ is S_n-closed, then H is $\mathfrak{F}_s$-quasinormal in K;*

(6) *If $G \in \mathfrak{F}$, then every subgroup of G is $\mathfrak{F}_s$-quasinormal in G.*

Lemma 2.3 ([6, Lemma 2.2]). *If H is a p-subgroup of G for some prime p and H is s-permutable in G, then:*

(1) $H \leq O_p(G)$;

(2) $O^p(G) \leq N_G(H)$.

Lemma 2.4 ([18]). *If A is a subnormal subgroup of a group G and A is a π-group, then $A \leq O_\pi(G)$.*

Lemma 2.5 ([15, II, Lemma 7.9]). *Let N be a nilpotent normal subgroup of G. If $N \neq 1$ and $N \cap \Phi(G) = 1$, then N is a direct product of some minimal normal subgroups of G.*

Lemma 2.6 ([5, Lemma 2.3]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

Recall that a subgroup H of G is said to be $\mathfrak{F}$-supplemented in G if there exists a subgroup T of G such that $G = HT$ and $T \in \mathfrak{F}$, where $\mathfrak{F}$ is some class of groups. The following Lemma is clear.

Lemma 2.7. *Let $\mathfrak{F}$ be a formation and H a subgroup of G. If H has an $\mathfrak{F}$-supplement in G, then:*

(1) *If $N \trianglelefteq G$, then HN/N has an $\mathfrak{F}$-supplement in G/N.*

(2) *If $H \leq K \leq G$, then H has an $\mathfrak{F}$-supplement in K.*

Lemma 2.8 ([10, Theorem 3.1]). *Let $\mathfrak{F}$ be an S-closed saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup E such that $G/E \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of E not having a supersoluble supplement in G is $\mathfrak{U}_s$-quasinormal in G.*

Lemma 2.9 ([10, Theorem 3.2]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G a group. Then $G \in \mathfrak{F}$ if and only if G has a soluble normal subgroup E*

such that $G/E \in \mathfrak{F}$ and every maximal subgroup of every non-cyclic Sylow subgroup of $F(E)$ not having a supersoluble supplement in G is $\mathfrak{U}_s$-quasinormal in G.

Lemma 2.10 ([3, Main Theorem])**.** *Suppose G has a Hall π-subgroup and $2 \notin \pi$. Then all the Hall π-subgroups are conjugate in G.*

Lemma 2.11 ([6, Lemma 2.5])**.** *Let G be a group and p a prime such that $p^{n+1} \nmid |G|$ for some integer $n \geq 1$. If $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$, then G is p-nilpotent.*

The generalized Fitting subgroup $F^*(G)$ of a group G is the product of all normal quasinilpotent subgroups of G. We also need in our proofs the following well-known facts about this subgroups (see [12, Chapter X]).

Lemma 2.12. *Let G be a group and N a subgroup of G.*

(1) *If N is normal in G, then $F^*(N) \leq F^*(G)$.*

(2) *If N is normal in G and $N \leq F^*(G)$, then $F^*(G)/N \leq F^*(G/N)$.*

(3) *$F(G) \leq F^*(G) = F^*(F^*(G))$. If $F^*(G)$ is soluble, then $F^*(G) = F(G)$.*

(4) *$C_G(F^*(G)) \leq F(G)$.*

(5) *$F^*(G) = F(G)E(G)$, $F(G) \cap E(G) = Z(E(G))$ and $E(G)/Z(E(G))$ is the direct product of simple non-abelian groups, where $E(G)$ is the layer of G.*

Lemma 2.13 ([8, Lemma 2.15–2.16])**.** (1) *If H is a normal soluble subgroup of a group G, then $F^*(G/\Phi(H)) = F^*(G)/\Phi(H)$.*

(2) *If K is a normal p-subgroup of a group G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

3. New characterization of supersoluble groups

Lemma 3.1. *Let p be the smallest prime dividing $|G|$ and P some Sylow p-subgroup of G. Then G is soluble if and only if every maximal subgroup of P is $\mathfrak{S}_s$-quasinormal in G.*

Proof. The necessity is obvious since $Z_\infty^{\mathfrak{S}}(G) = G$ whenever $G \in \mathfrak{S}$. Hence we only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. Then $p = 2$ by the well known Feit-Thompson Theorem of groups of odd order. We proceed the proof via the following steps:

(1) $O_2(G) = 1$.

Assume that $N = O_2(G) \neq 1$. Then P/N is a Sylow 2-subgroup of G/N. Let M/N be a maximal subgroup of P/N. Then M is a maximal subgroup of P. By the hypothesis and Lemma 2.2(2), M/N is $\mathfrak{S}_s$-quasinormal in G/N. The minimal choice of G implies that G/N is soluble. It follows that G is soluble, a contradiction. Hence (1) holds.

(2) $O_{2'}(G) = 1$.

Assume that $D = O_{2'}(G) \neq 1$. Then PD/D is a Sylow 2-subgroup of G/D. Suppose that M/D is a maximal subgroup of PD/D. Then there exists a maximal subgroup P_1 of P such that $M = P_1D$. By the hypothesis and Lemma 2.2(3), $M/D = P_1D/D$ is $\mathfrak{S}_s$-quasinormal in G/D. Hence G/D is soluble by the choice of G. It follows that G is soluble, a contradiction.

(3) Final contradiction.

Let P_1 be a maximal subgroup of P. By the hypothesis, there exists a normal subroup K of G such that P_1K is s-permutable in G and $(P_1 \cap K)(P_1)_G/(P_1)_G \leq Z_\infty^{\mathfrak{S}}(G/(P_1)_G)$. Note that $Z_\infty^{\mathfrak{S}}(G)$ is a soluble normal subgroup of G. By (1) and (2), we have $(P_1)_G = 1$ and $Z_\infty^{\mathfrak{S}}(G) = 1$. This induces that $P_1 \cap K = 1$. If $K = 1$, then P_1 is s-permutable in G and so $P_1 = 1$ by (1) (2) and Lemma 2.3(1). This means that $|P| = 2$. Then by [14, (10.1.9)], G is 2-nilpotent and so G is soluble, a contradiction. We may, therefore, assume that $K \neq 1$. If $2 \mid |K|$, then $|K_2| = 2$, where K_2 is a Sylow 2-subgroup of K. By [14, (10.1.9)] again, we see that K is 2-nilpotent, and so K has a normal 2-complement $K_{2'}$. Since $K_{2'}$ char $K \trianglelefteq G$, $K_{2'} \trianglelefteq G$. Hence by (2), $K_{2'} = 1$. Consequently $|K| = 2$, which contradicts (1). If $2 \nmid |K|$, then K is a $2'$-group. Hence by (2), $K \leq O_{2'}(G) = 1$, also a contradiction. This completes the proof. □

Theorem 3.2. *Let $G = AB$, where A is a subnormal subgroup of G, and B is a supersoluble Hall subgroup of G in which all Sylow subgroups are cyclic. If every maximal subgroup of every non-cyclic Sylow subgroup of A is $\mathfrak{U}_s$-quasinormal in G, then G is supersoluble.*

Proof. Suppose that the assertion is false and let G be a counterexample of minimal order. Then:

(1) *Each proper subgroup of G containing A is supersoluble.*

Let $A \leq M < G$. Then $M = M \cap AB = A(M \cap B)$. Obviously, $M \cap B$ is a Hall subgroup of M and every Sylow subgroup of $M \cap B$ is cyclic. By Lemma 2.2(4), every maximal subgroup of every non-cyclic Sylow subgroup of A is $\mathfrak{U}_s$-quasinormal in M. The minimal choice of G implies that M is supersoluble.

(2) *Let H be a non-trivial normal p-subgroup of G for some prime p. If H*

contains some Sylow p-subgroup of A or a Sylow p-subgroup of A is cyclic or $H \leq A$*, then* G/H *is supersoluble.*

If $A \leq H$, then $G/H = BH/H \cong B/(B \cap H)$ is supersoluble. Now we can assume that $A \nleq H$. Clearly, $G/H = (AH/H)(BH/H)$, where AH/H is subnormal in G/H and BH/H is supersoluble. Let Q/H be any non-cyclic Sylow q-subgroup of AH/H and Q_1/H a maximal subgroup of Q/H. Then there exists a non-cyclic Sylow q-subgroup A_q of A such that $Q = A_qH$ and a maximal subgroup A_1 of A_q such that $Q_1 = A_1H$. If $H \leq A$, then the assertion holds by the choice of G and Lemma 2.2(2). We may, therefore, assume that $H \nleq A$. Let P be a Sylow p-subgroup of A. Assume that P is cyclic or $P \leq H$. Then $p \neq q$. Clearly, $Q_1 \cap A_q = A_1$ is a maximal subgroup of A_q. By the hypothesis, A_1 is $\mathfrak{U}_s$-quasinormal in G. Therefore, $Q_1/H = A_1H/H$ is $\mathfrak{U}_s$-quasinormal in G/H by Lemma 2.2(3). This shows that the conditions of the theorem are true for G/H and so G/H is supersoluble by the minimal choice of G.

(3) *There exists at least one Sylow subgroup of A which is non-cyclic.*

It follows from the well known fact that a group G is supersoluble if all its Sylow subgroups are cyclic.

(4) *G is soluble.*

If $A \neq G$, then A is supersoluble by (1). Let p be the largest prime divisor of $|A|$. Then $A_p \trianglelefteq A$. By Lemma 2.4, $A_p \leq O_p(G)$. By (2), $G/O_p(G)$ is supersoluble. It follows that G is soluble.

We now only need to consider the case that $A = G$. If G is not soluble and let p be the minimal prime divisor of $|G|$. Then $p = 2$ by the well-known Feit-Thompson Theorem. Hence by Lemma 3.1, G is soluble.

(5) *G has a unique minimal normal subgroup N such that* $N = O_p(G) = C_G(N)$ *is a non-cyclic p-subgroup of G for some prime p and* $G = [N]M$*, where M is a supersoluble maximal subgroup of G.*

Let N be an arbitrary minimal normal subgroup of G. By (4), N is a p-group. If $p \in \pi(B)$, then the Sylow p-subgroups of G are cyclic and so the Sylow p-subgroups of A are cyclic. If $p \notin \pi(B)$, then clearly, $N \subseteq A$. Hence by (2), G/N is supersoluble. If N is cyclic, then by Lemma 2.6, G is supersoluble, a contradiction. Since the class of all supersoluble groups is a saturated formation, N is the only minimal normal subgroup N of G and $\Phi(G) = 1$. This implies that (5) holds.

(6) *N is not a Sylow subgroup of G and* $Z_\infty^{\mathfrak{U}}(G) = 1$.

By (5), clearly, $Z_\infty^{\mathfrak{U}}(G) = 1$. Assume that N is a Sylow p-subgroup of G. Let N_1 be a maximal subgroup of N. Then by hypothesis, N_1 is $\mathfrak{U}_s$-quasinormal in G.

Hence there exists a normal subgroup K of G such that N_1K is s-permutable in G and $N_1 \cap K \leq Z_\infty^{\mathfrak{U}}(G) = 1$ since $(N_1)_G = 1$. It follows that $N_1 \leq N_1 \cap N \leq N_1 \cap K = 1$. Hence $|N| = p$. This contradiction shows that N is not a Sylow p-subgroup of G.

(7) *A is supersoluble.*

If A is not supersoluble, then $G = A$ by (1). Let q be the largest prime divisor of $|G|$ and Q is a Sylow q-subgroup of G. Then QN/N is a Sylow q-subgroup of G/N. Since G/N is supersoluble, $QN/N \trianglelefteq G/N$. It follows that $QN \trianglelefteq G$. Let P be a non-cycli Sylow p-subgroup of $G = A$. If $p = q$, then $P = Q = QN \trianglelefteq G$. Therefore $N = O_p(G) = P$ is the Sylow p-subgroup of G, a contradiction. Assume that $q > p$. Then clearly $QP = QNP$ is a subgroup of G. Since $N \nleq \Phi(G)$, $N \nleq \Phi(P)$ by [11, III, Lemma 3.3(a)]. Let P_1 be a maximal subgroup of P such that $N \nleq P_1$. Then $(P_1)_G = 1$. By the hypothesis, P_1 is $\mathfrak{U}_s$-quasinormal in G. Hence, there exists a normal subgroup T of G such that P_1T is s-permutable in G and $P_1 \cap T \leq Z_\infty^{\mathfrak{U}}(G) = 1$. Obviously, $T \neq 1$ (In fact, if $T = 1$, then $P_1 \leq O_p(G) = N$ by Lemma 2.3(1). Hence $P_1 = N$ or $P = N$. This is impossible). Thus $N \leq T$, and so $P_1 \cap N \leq P_1 \cap T = 1$. This induces that $|N| = |P : P_1| = p$, which contradicts (5). Thus (7) holds.

(8) *The final contradiction.*

Let q be the largest prime divisor of $|A|$ and A_q a Sylow p-subgroup of A. Since A is supersoluble by (7), $A_q \trianglelefteq A$. Hence $A_q \leq O_q(G)$. If $q \mid |B|$, then $O_q(G) \leq G_q$, where G_q is a cyclic Sylow q-subgroup of B and so $O_q(G)$ is cyclic. In view of (2), $G/O_q(G)$ is supersoluble. It follows that G is supersoluble, a contradiction. Hence $q \nmid |B|$. Then, A_q is a Sylow q-subgroup of G and so $A_q = O_q(G) \neq 1$. This means that $q = p$ and so $N = A_p = G_p$, which contradicts (6). The final contradiction completes the proof. □

Theorem 3.3. *Let $\mathfrak{F}$ be an S-closed saturated formation containing $\mathfrak{U}$ and H a normal subgroup of G such that $G/H \in \mathfrak{F}$. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup of $F^*(H)$ having no supersoluble supplement in G is $\mathfrak{U}_s$-quasinormal in G. Then $G \in \mathfrak{F}$.*

Proof. We first prove that the theorem is true if $\mathfrak{F} = \mathfrak{U}$. Suppose that the assertion is false and consider a counterexample for which $|G||H|$ is minimal. Then:

(1) *$H = G$ and $F^*(G) = F(G)$.*

By Lemma 2.8, $F^*(H)$ is supersoluble. Hence $F^*(H) = F(H)$ by Lemma 2.12(3). Since (H, H) satisfies the hypothesis, the minimal choice of (G, H)

implies that H is supersoluble if $H < G$. Then $G \in \mathfrak{U}$ by Lemma 2.9, a contradiction.

(2) *Every proper normal subgroup N of G containing $F^*(G)$ is supersoluble.*

Let N be a proper normal subgroup of G containing $F^*(G)$. By Lemma 2.12, $F^*(G) = F^*(F^*(G)) \leq F^*(N) \leq F^*(G)$. Hence $F^*(N) = F^*(G)$. Let M be a maximal subgroup of any non-cyclic Sylow subgroup of $F^*(N)$. If there exists a supersoluble subgroup T such that $G = MT$, then $N = M(N \cap T)$ and $N \cap T \in \mathfrak{U}$. This means that M has a supersoluble supplement in N. Now assume that M has no supersoluble supplement in G. Then by hypothesis and Lemma 2.2(4), M is $\mathfrak{U}_s$-quasinormal in N. This shows that (N, N) satisfies the hypothesis. Hence N is supersoluble by the minimal choice of (G, H).

(3) *If $p \in \pi(F(G))$, then $\Phi(O_p(G)) = 1$ and so $O_p(G)$ is elementary abelian. In particular, $F^*(G) = F(G)$ is abelian and $C_G(F(G)) = F(G)$.*

Suppose that $\Phi(O_p(G)) \neq 1$ for some $p \in \pi(F(G))$. By Lemma 2.13(1), we have $F^*(G/\Phi(O_p(G))) = F^*(G)/\Phi(O_p(G))$. By using Lemma 2.2, we see that the pair $(G/\Phi(O_p\ (G)), F^*(G)/\Phi(O_p(G)))$ satisfies the hypothesis. The minimal choice of (G, H) implies $G/\Phi(O_p(G)) \in \mathfrak{U}$. Since $\mathfrak{U}$ is a saturated formation, we obtain that $G \in \mathfrak{U}$, a contradiction. This means that $\Phi(O_p(G)) = 1$ and so $O_p(G)$ is elementary abelian. Hence $F^*(G) = F(G)$ is abelian and $F(G) \leq C_G(F(G))$. Put $N = C_G(F(G))$. Then, clearly, $F(N) = F(G)$. If $N = G$, then $F(G) \leq Z(G)$. Let P_1 be a maximal subgroup of some Sylow p-subgroup of $F(G)$. Then $F(G/P_1) = F(G)/P_1$ by Lemma 2.13(2). Hence $(G/P_1, F(G)/P_1)$ satisfies the hypothesis and so $G/P_1 \in \mathfrak{F}$. Then since $P \leq Z(G)$, we obtain $G \in \mathfrak{F}$. This contradiction shows that $N < G$. Hence by (2), N is soluble and so $C_N(F(N)) \subseteq F(N)$. It follows that $N = C_G(F(G)) = F(G)$.

(4) *G has no normal subgroup of prime order contained in $F(G)$.*

Suppose that L is a normal subgroup of G contained in $F(G)$ and $|L| = p$. Put $C = C_G(L)$. Clearly, $F(G) \leq C \trianglelefteq G$. If $C < G$, then C is soluble by (2). Since G/C is cyclic, G is soluble. Then by the hypothesis and Lemma 2.9, $G \in \mathfrak{U}$, a contradiction. Hence $C = G$ and so $L \leq Z(G)$. By Lemma 2.13(2) $F^*(G/L) = F^*(G)/L = F(G)/L$. Hence G/L satisfies the hypothesis by Lemma 2.2. The minimal choice of (G, H) implies that $G/L \in \mathfrak{U}$ and consequently G is supersoluble, a contradiction.

(5) *For some $p \in \pi(F(G))$, $O_p(G)$ is a non-cyclic Sylow p-subgroup of $F(G)$.*

Clearly, $F(G) = O_{p_1}(G) \times O_{p_2}(G) \times \cdots \times O_{p_r}(G)$ for some primes p_i, $i = 1, 2, \ldots, r$. If all Sylow subgroups of $F(G)$ are cyclic, then $G/C_G(O_{p_i}(G))$ is abelian for any $i \in \{1 \cdots r\}$ and so $G/\cap_{i=1}^r C_G(O_{p_i}(G)) = G/C_G(F(G)) =$

$G/F(G)$ is abelian. Therefore G is soluble. It follows from Lemma 2.9 and the hypothesis that $G \in \mathfrak{U}$, a contradiction.

(6) *Every maximal subgroup of every non-cyclic Sylow subgroup of* $F(G)$ *has no supersoluble supplement in* G.

Let P be a non-cyclic Sylow subgroup of $F(G)$ and P_1 a maximal subgroup of P. Then $P = O_p(G)$ for some $p \in \pi(F(G))$. If P_1 has a supersoluble supplement in G, that is, there exists a supersoluble subgroup K of G such that $G = P_1K = O_p(G)K$, then $G/O_p(G) \simeq K/K \cap O_p(G)$ is supersoluble and so G is soluble. Hence as above, $G \in \mathfrak{U}$, a contradiction.

(7) $P \cap \Phi(G) \neq 1$*, for some non-cyclic Sylow subgroup* P *of* $F(G)$.

Assume that $P \cap \Phi(G) = 1$. Then $P = R_1 \times R_2 \times \cdots \times R_m$, where $R_i (i \in \{1, \cdots m\})$ is a minimal normal subgroup of G by Lemma 2.5. We claim that R_i are of order p for all $i \in \{1, \cdots m\}$. Assume that $|R_i| > p$, for some i. Without loss of generality, we let $|R_1| > p$. Let R_1^* be a maximal subgroup of R_1. Obviously, $R_1^* \neq 1$. Then $R_1^* \times R_2 \times \cdots \times R_m = P_1$ is a maximal subgroup of P. Put $T = R_2 \times \cdots \times R_m$, Clearly $(P_1)_G = T$. By (6) and the hyperthesis, P_1 is $\mathfrak{U}_s$-quasinormal in G. Hence by Lemma 2.2(1), there exists a normal subgroup N of G such that $(P_1)_G \leq N$, P_1N is s-permutable in G and $P_1/(P_1)_G \cap N/(P_1)_G \leq Z_\infty^{\mathfrak{U}}(G/(P_1)_G)$. Assume that $P_1/(P_1)_G \cap N/(P_1)_G \neq 1$. Let $Z_\infty^{\mathfrak{U}}(G/(P_1)_G) = V/(P_1)_G = V/T$. Then $P/T \cap V/T \trianglelefteq G/T$. Since $P \cap V \geq P_1 \cap N \cap V \geq P_1 \cap N > (P_1)_G = T$, we have $P/T \cap V/T \neq 1$. Because $P/T \simeq R_1$ and R_1 is a minimal normal subgroup of G, $P/T \subseteq V/T$. This implies that $|R_1| = |P/T| = p$. This contradiction shows that $P_1 \cap N = (P_1)_G = T$. Consequently $P_1N = R_1^*TN = R_1^*N$ and $R_1^* \cap N = 1$. Since $R_1 \cap N \trianglelefteq G$, $R_1 \cap N = 1$ or $R_1 \cap N = R_1$. But since $R_1^* \cap N = 1$, we have that $R_1 \cap N = 1$. Thus $R_1^* = R_1^*(R_1 \cap N) = R_1 \cap R_1^*N$ is s-permutable in G. It follows from Lemma 2.3(2) that $O^p(G) \leq N_G(R_1^*)$. Thus $|G : N_G(R_1^*)|$ is a power of p for every maximal subgroup R_1^* of R_1. This induces that p divides the number of all maximal subgroups of R_1. This contradicts [11, III, Theorem 8.5(d)]. Therefore $|R_i| = p$, which contradicts (4). Thus (7) holds.

(8) $F(G) = P$ *is a* p*-group,* P *contains a unique minimal normal subgroup* L *of* G *and* $L \subseteq \Phi(G)$.

Suppose that $1 \neq Q$ is a Sylow q-subgroup of $F(G)$ for some prime $q \neq p$ and let L be a minimal normal subgroup of G contained in $P \cap \Phi(G)$. By (3), Q is elementary abelian. By Lemma 2.12, $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G/L), E(G/L)] = 1$, where $E(G/L)$ is the layer of G/L. Since $L \leq \Phi(G)$, $F(G/L) = F(G)/L$. Now let $E/L = E(G/L)$. Since Q is normal in G and $[F(G)/L, E/L] = 1$, we have $[Q, E] \leq Q \cap L = 1$. It follows from (3) that

$F(G)E \leq C_G(Q) \trianglelefteq G$. If $C_G(Q) < G$, then $C_G(Q)$ is supersoluble by (1) and (2). Thus $E(G/L) = E/L$ is supersoluble and consequently $F^*(G/L) = F(G)/L$ by Lemma 2.12(5). Now, by Lemma 2.2, we see that $(G/L, F(G)/L)$ satisfies the hypothesis. The minimal choice of (G, H) implies that G/L is supersoluble and so is G. This contradiction shows that $C_G(Q) = G$, i.e. $Q \leq Z(G)$, which contradicts (4). Thus $F(G) = P$.

Let X be a minimal normal subgroup of G contained in P with $X \neq L$. Let $E/L = E(G/L)$ is the layer of G/L. As above, we see that $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G)/L, E/L] = 1$. Hence $[X, E] \leq X \cap L = 1$, i.e., $[X, E] = 1$. It follows from (3) that $F(G)E \leq C_G(X) \trianglelefteq G$. If $C_G(X) < G$, then $C_G(X)$ is supersoluble by (1) and (2). Thus $E(G/L) = E/L$ is supersoluble and consequently $F^*(G/L) = F(G)/L$. Obviously, G/L satisfies the hypothesis. By the choice of (G, H), we have that G/L is supersoluble and so is G, a contradiction. Hence $C_G(X) = G$, i.e. $X \leq Z(G)$, which also contradicts (4). Thus L is the unique minimal normal subgroup of G contained in P. Finally, $L \subseteq \Phi(G)$ by (7).

(9) $L < P$.

Suppose $L = P$. Let P_1 be a maximal subgroup of P such that P_1 is normal in some Sylow subgroup of G. Then $(P_1)_G = 1$. By the hypothesis and (8), P_1 is $\mathfrak{U}_s$-quasinormal in G. Hence there exists a normal subgroup K of G such that P_1K is s-permutable in G and $P_1 \cap K \leq Z_\infty^{\mathfrak{U}}(G)$. If $P_1 \cap K \neq 1$, then $1 < P_1 \cap K \leq P \cap Z_\infty^{\mathfrak{U}}(G)$, which implies that $P = P \cap Z_\infty^{\mathfrak{U}}(G)$ and $|P| = p$ since P is a minimal normal subgroup of G. This contradicts (4). So we may assume $P_1 \cap K = 1$. Since P is a minimal normal subgroup of G, $P \cap K = P$ or 1. If $P \cap K = P$, then $P \subseteq K$, and so $|P| = p$, which contradicts (4). If $P \cap K = 1$, then $P \cap P_1K = P_1(P \cap K) = P_1$. Hence P_1 is s-permutable in G. Then by Lemma 2.3(2), $O^p(G) \leq N_G(P_1)$. This induces that $P_1 \trianglelefteq G$. This means that $P_1 = (P_1)_G = 1$ and $|P| = p$, also a contradiction.

(10) *Final contradiction (for $\mathfrak{F} = \mathfrak{U}$).*

By (3) and (8), P is an elementary abelian group, and so L has a complement in P, T say. Let $P_1 = TL_1$, where L_1 is a maximal subgroup of L. Then $1 \neq P_1$ and clearly P_1 is a maximal subgroup of P such that P_1 is normal in some Sylow subgroup of G. Hence by (6), P_1 is $\mathfrak{U}_s$-quasinormal in G and $(P_1)_G = 1$ since L is the unique minimal normal subgroup of G contained in P. Hence there exists a normal subgroup S of G such that P_1S is s-permutable in G and $P_1 \cap S \leq Z_\infty^{\mathfrak{U}}(G)$. If $P_1 \cap S \neq 1$, then $1 < P_1 \cap S \leq P \cap Z_\infty^{\mathfrak{U}}(G)$ and so G has a minimal normal subgroup N of order p contained in P, which is contrary to (4). Hence $P_1 \cap S = 1$. If $P \cap S \neq 1$, then $L \leq P \cap S$ and so $L_1 \leq S$, which contradicts $P_1 \cap S = 1$.

If $P \cap S = 1$, then $P_1 = P_1(P \cap S) = P \cap P_1S$ is s-permutable in G. Hence $O^p(G) \leq N_G(P_1)$ by Lemma 2.3(2). It follows that $P_1 \trianglelefteq G$, which contradicts $(P_1)_G = 1$. The final contradiction shows that the theorem holds when $\mathfrak{F} = \mathfrak{U}$.

Now we prove that the theorem holds for $\mathfrak{F}$.

Since $H/H \in \mathfrak{U}$, by the assertion proved above and Lemma 2.2, we see that H is supersoluble. In particular, H is soluble and hence $F^*(H) = F(H)$. Now by using Lemma 2.9, we obtain that $G \in \mathfrak{F}$. This completes the proof of the theorem. □

4. New characterization of p-nilpotent groups

Lemma 4.1. *Let G be a group and p a prime divisor of $|G|$ with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geq 1$. Suppose P is a Sylow p-subgroup of G and every n-maximal subgroup of P(if exists) has a p-nilpotent supplement in G. Then G is p-nilpotent.*

Proof. Assume that $p^{n+1} \mid |G|$. Let P_{n1} be an n-maximal subgroup of P. By hypothesis, P_{n1} has a p-nilpotent supplement T_1 in G. Let K_1 be a normal Hall p'-subgroup of T_1. Obviously, K_1 is a Hall p'-subgroup of G. Hence $G = P_{n1}T_1 = P_{n1}N_G(K_1)$. We claim that $K_1 \trianglelefteq G$. Indeed, if $K_1 \ntrianglelefteq G$, then $N_P(K_1) = N_G(K_1) \cap P \neq P$ since $T_1 \subseteq N_G(K_1)$. Therefore, there exists a maximal subgroup P_2 of P such that $N_P(K_1) \leq P_2$. Let P_{n2} be an n-maximal subgroup of P contained in P_2. Since $P = P \cap G = P \cap P_{n1}N_G(K_1) = P_{n1}(P \cap N_G(K_1)) = P_{n1}N_P(K_1)$, we have $P_{n1} \neq P_{n2}$. By hypothesis, P_{n2} has a p-nilpotent supplement in G. With the same discussion as above, we can find a Hall p'-subgroup K_2 of G such that $G = P_{n2}N_G(K_2) = P_2N_G(K_2)$. If $p = 2$, then by Lemma 2.10, K_1 conjugates with K_2 in G. If $p > 2$, then G is soluble by Feit–Thompson theorem. Hence, K_1 also conjugates with K_2 in G. This means that there exists an element $g \in P_2$, such that $(K_2)^g = K_1$. Then $G = (P_2N_G(K_2))^g = P_2N_G(K_1)$. Hence, $P = P \cap G = P \cap P_2N_G(K_1) = P_2(P \cap N_G(K_1)) = P_2N_P(K_1) = P_2$. This contradiction shows that $p^{n+1} \nmid |G|$. Thus G is p-nilpotent by Lemma 2.11. □

Lemma 4.2. *Let G be a group and p a prime divisor of $|G|$ with $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geq 1$. Suppose that G has a Sylow p-subgroup P such that every n-maximal subgroup of P (if exists) either has a p-nilpotent supplement or is $\mathfrak{U}_s$-quasinormal in G, then G is p-nilpotent.*

Proof. Suppose the Lemma is false and let G be a counterexample of minimal order. By Lemma 2.11, we have $p^{n+1} \mid |G|$. Hence P has a non-trivial

n-maximal subgroup. We proceed via the following steps:

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$. Then we may choose a minimal normal subgroup N of G such that $N \leq O_{p'}(G)$. Clearly, $(|G/N|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ and PN/N is a Sylow p-subgroup of G/N. Assume that L/N is an n-maximal subgroup of PN/N. Then, clearly, $L/N = M_pN/N$, where M_p is an n-maximal subgroup of P. By hypothesis, M_p either has a p-nilpotent supplement or is $\mathfrak{U}_s$-quasinormal in G. By Lemma 2.7(1) and Lemma 2.2(3), we see that G/N (with respect to PN/N) satisfies the hypothesis. The minimal choice of G implies that G/N is p-nilpotent and consequently G is p-nilpotent, a contradiction.

(2) *P has a maximal subgroup P_1 such that P_1 has no p-nilpotent supplement in G* (This follows from Lemma 4.1).

(3) *G is soluble.*

Suppose that G is not soluble. Then $p = 2$ by the well known Feit-Thompson Theorem. Assume that $O_2(G) \neq 1$. By Lemma 2.7 and Lemma 2.2(2), $G/O_2(G)$ satisfies the hypothesis. Hence $G/O_2(G)$ is 2-nilpotent. It follow that G is soluble, a contradiction. Now assume that $O_2(G) = 1$. Then $(P_n)_G = 1$, where P_n is an n-maximal subgroup of P. Since P_n has no p-nilpotent supplement in G, P_n is $\mathfrak{U}_s$-quasinormal in G by the hypothesis. Hence there exists $K \trianglelefteq G$ such that P_nK is s-permutable in G and $P_n \cap K \leq Z_\infty^{\mathfrak{U}}(G)$. If $K = 1$, then $P_n \leq O_2(G) = 1$ by Lemma 2.3(1), a contradiction. Thus, $K \neq 1$. If $Z_\infty^{\mathfrak{U}}(G) \neq 1$, then there exists a minimal normal subgroup H of G contained in $Z_\infty^{\mathfrak{U}}(G)$. Hence H is of prime power order. This is impossible since $O_{2'}(G) = 1$ and $O_2(G) = 1$. Hence $P_n \cap K = 1$ and so $2^{n+1} \nmid |K|$. Then by Lemma 2.11, K has a normal Hall $2'$-subgroup T. Since T char $K \trianglelefteq G$, $T \trianglelefteq G$. It follows from (1) that $T = 1$. Consequently, $K \leq O_2(G) = 1$, a contradiction again. Hence (3) holds.

(4) *$N = O_p(G)$ is the only minimal normal subgroup of G and $G = [N]M$, where M is a maximal subgroup of G and M is p-nilpotent.*

Let N be a minimal normal subgroup of G. By (1) and (3), N is an elementary abelian p-group and $N \leq O_p(G)$. By Lemma 2.7(1) and Lemma 2.2(2), G/N satisfies the hypothesis and so G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$. Hence $O_p(G) = N = C_G(N)$, and consequently $G = [N]M$, where M is a p-nilpotent maximal subgroup of G. Thus (4) holds.

(5) *The final contradiction.*

Let P_n be an n-maximal subgroup of P such that $P_n \leq P_1$. Then P_n has also no p-nilpotent supplement in G. Hence there exists a normal subgroup K of G

such that P_nK is s-permutable in G and $(P_n \cap K)(P_n)_G/(P_n)_G \leq Z_\infty^{\mathfrak{F}}(G/(P_n)_G)$. We claim that $(P_n)_G = 1$. Indeed, if $(P_n)_G \neq 1$, then by (2), $O_p(G) = N = (P_n)_G$. Hence $G = NM = (P_n)_G M = P_n M$, which contradicts (2). Therefore, $P_n \cap K \leq Z_\infty^{\mathfrak{U}}(G)$. If $K = 1$, then P_n is s-permutable in G, and so $P_n \leq O_p(G) = N$ and $O^p(G) \leq N_G(P_n)$ by Lemma 2.3. Hence $1 \neq P_n \leq P_n^G = P_n^{O^p(G)P} = P_n^P = (P_n \cap N)^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N$. On the other hand, obviously, $N \leq P_n^G$. Thus $N = P_n^G = P_1 \cap N$. It follows that $N \leq P_1$, and so $G = NM = P_1M$. This means that P_1 has a p-nilpotent supplement in G. This contradiction shows that $K \neq 1$. If $P_n \cap K = 1$, then $p^{n+1} \nmid |K|$. By Lemma 2.11, K is p-nilpotent and $K_{p'} \leq O_{p'}(G) = 1$ by (1). Hence $K = N = O_p(G)$. It follows from Lemma 2.3(1) that $P_nK = K$ and so $P_n \cap K \neq 1$, a contradiction. Hence $P_n \cap K \neq 1$. This means that $Z_\infty^{\mathfrak{U}}(G) \neq 1$ and so $N \leq Z_\infty^{\mathfrak{U}}(G)$. Consequently, $|N| = |O_p(G)| = p$. Therefore, $G/N \cong G/C_G(N)$ is isomorphic with some subgroup of $Aut(N)$ of order $p-1$. Since $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$, $G/N = 1$. Consequently, $G = N$ is an elementary abelian p-group. The final contradiction completes the proof. □

Theorem 4.3. *Let p be a prime, $\mathfrak{F}$ a saturated formation containing all p-nilpotent groups and G a group. Suppose that $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$ for some integer $n \geq 1$. Then $G \in \mathfrak{F}$ if and only if G has a normal subgroup E such that G/E is p-nilpotent and every n-maximal subgroup of P (if exists) either has a p-nilpotent supplement or is $\mathfrak{U}_s$-quasinormal in G, where P is a Sylow p-subgroup of E.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose it is false and let G be a counterexample of minimal order. By Lemma 2.7(2) and Lemma 2.2(4), every n-maximal subgroup of P either has a p-nilpotent supplement or is $\mathfrak{U}_s$-quasinormal in E. Hence E is p-nilpotent by Lemma 4.2. Then, $E \neq G$. Let T be a normal Hall p'-subgroup of E. Clearly, $T \trianglelefteq G$. We proceed the proof via the following steps:

(1) *$T = 1$, and so $P = E \trianglelefteq G$.*

Suppose that $T \neq 1$. Since T is a normal Hall p'-subgroup of E and $E \trianglelefteq G$, then $T \trianglelefteq G$. We show that G/T (with respect to E/T) satisfies the hypothesis. Indeed, $(G/T)/(E/T) \simeq G/E$ is p-nilpotent and $E/T = PT/T$ is a p-group. Suppose that M_n/T is an n-maximal subgroup of PT/T and $P_n = M_n \cap P$. Then P_n is an n-maximal subgroup of P and $M_n = P_nT$. By the hypothesis, P_n either has a p-nilpotent supplement or is $\mathfrak{U}_s$-quasinormal in G. By Lemma 2.7(1) and Lemma 2.2(3), $M_n/T = P_nT/T$ either has a p-nilpotent supplement or is $\mathfrak{U}_s$-quasinormal in G/T. The minimal choice of G implies that G/T is p-nilpotent.

This implies that G is p-nilpotent. This contradiction shows $T = 1$. Hence $P = E \trianglelefteq G$.

(2) *Let Q be a Sylow q-subgroup of G, where q is a prime divisor of $|G|$ with $q \neq p$. Then $PQ = P \times Q$.*

By (1), $P = E \trianglelefteq G$, PQ is a subgroup of G. By Lemma 2.7(2) and Lemma 2.2(4), every n-maximal subgroup of P either has a p-nilpotent supplement or is $\mathfrak{U}_s$-quasinormal in PQ. By using Lemma 4.2, we have that PQ is p-nilpotent. Hence $Q \trianglelefteq PQ$ and thereby $PQ = P \times Q$.

(3) *The final contradiction.*

From (2), we have $O^p(G) \leq C_G(P)$. This induces that $E = P \leq Z_\infty(G) \leq Z_\infty^{\mathfrak{F}}(G)$. Therefore $G \in \mathfrak{F}$. The final contradiction completes the proof. □

Theorem 4.4. *Let G be a finite group and p a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if G has a soluble normal subgroup H of G such that G/H is p-nilpotent and every maximal subgroup of every Sylow subgroup of $F(H)$ is $\mathfrak{U}_s$-quasinormal in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that it is false and let G be a counterexample with $|G||H|$ is minimal. Let P be an arbitrary given Sylow p-subgroup of $F(H)$. Clearly, $P \trianglelefteq G$. We proceed the proof as follows.

(1) $\Phi(G) \cap P = 1$.

If not, then $1 \neq \Phi(G) \cap P \trianglelefteq G$. Let $R = \Phi(G) \cap P$. Clearly, $(G/R)/(H/R) \simeq G/H \in \mathfrak{F}$. By Gaschütz theorem (see [11, III, Theorem 3.5]), we have that $F(H/R) = F(H)/R$. Assume that P/R is a Sylow p-subgroup of $F(H/R)$ and P_1/R is a maximal subgroup of P/R. Then P is a Sylow p-subgroup of $F(G)$ and P_1 is a maximal subgroup of P. By Lemma 2.2(2) and the hypothesis, P_1/R is $\mathfrak{U}_s$-quasinormal in G/R. Now, let Q/R be a maximal subgroup of some Sylow q-subgroup of $F(H/R) = F(H)/R$, where $q \neq p$. Then $Q = Q_1R$, where Q_1 is a maximal subgroup of the Sylow q-subgroup of $F(H)$. By hypothesis, Q_1 is $\mathfrak{U}_s$-quasinormal in G. Hence $Q/R = Q_1R/R$ is $\mathfrak{U}_s$-quasinormal in G/R by Lemma 2.2(3). This shows that $(G/R, H/R)$ satisfies the hypothesis. The minimal choice of (G, H) implies that G/R is p-nilpotent. It follows that G is p-nilpotent, a contradiction. Hence (1) holds.

(2) *$P = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_m \rangle$, where every $\langle x_i \rangle$ $(i \in \{1 \cdots m\})$ is a normal subgroup of G with order p.*

By (1) and Lemma 2.5, $P = R_1 \times R_2 \times \cdots \times R_m$, where R_i $(i \in \{1 \cdots m\})$ is a minimal normal subgroup of G. We now prove that R_i is of order p, for $i \in \{1 \cdots m\}$.

Assume that $|R_i| > p$, for some i. Without loss of generality, we let $|R_1| > p$ and R_1^* be a maximal subgroup of R_1. Then, $R_1^* \neq 1$ and $R_1^* \times R_2 \times \cdots \times R_m = P_1$ is a maximal subgroup of P. Put $T = R_2 \times \cdots \times R_m$. Then, clearly, $(P_1)_G = T$. By hypothesis, P_1 is $\mathfrak{U}_s$-quasinormal in G. Hence by Lemma 2.2(1), there exists a normal subgroup N of G such that $(P_1)_G \leq N$, P_1N is s-permutable in G and $P_1/(P_1)_G \cap N/(P_1)_G \leq Z_\infty^{\mathfrak{U}}(G/(P_1)_G)$. Assume that $P_1/(P_1)_G \cap N/(P_1)_G \neq 1$. Let $Z_\infty^{\mathfrak{U}}(G/(P_1)_G) = V/(P_1)_G = V/T$. Then $P_1 \cap N \leq V$ and $P/T \cap V/T \trianglelefteq G/T$. Since $P \cap V \geq P_1 \cap N \cap V \geq P_1 \cap N > (P_1)_G = T$, $P/T \cap V/T \neq 1$. As $P/T \simeq R_1$ and R_1 is a minimal normal subgroup of G, we have $P/T \subseteq V/T$. This implies that $|R_1| = |P/T| = p$. This contradiction shows that $P_1 \cap N = (P_1)_G = T$. Consequently, $P_1N = R_1^*TN = R_1^*N$ and $R_1^* \cap N = 1$. Since $R_1 \cap N \trianglelefteq G$, $R_1 \cap N = 1$ or $R_1 \cap N = R_1$. If $R_1 \cap N = R_1$, then $R_1^* \subseteq R_1 \subseteq N$, which contradicts $R_1^* \cap N = 1$. Hence $R_1 \cap N = 1$. It follows that $R_1^* = R_1^*(R_1 \cap N) = R_1 \cap R_1^*N$ is s-permutable in G. Thus $O^p(G) \leq N_G(R_1^*)$ by Lemma 2.3(2). This induces that for every maximal subgroup R_1^* of R_1, we have that $|G : N_G(R_1^*)| = p^\alpha$, where α is an integer. Let $\{R_1^*, R_2^*, \cdots, R_t^*\}$ be the set of all maximal subgroups of R_1. Then p divides t. This contradicts to [11, III, Theorem 8.5(d)]. Thus (2) holds.

(3) *$G/F(H)$ is p-nilpotent.*

By (2), $F(H) = \langle y_1\rangle \times \langle y_2\rangle \times \cdots \times \langle y_n\rangle$, where $\langle y_i\rangle$ $(i \in \{1 \cdots n\})$ is a normal subgroup of G of order p. Since $G/C_G(\langle y_i\rangle)$ is isomorphic with some subgroup of $Aut(\langle y_i\rangle)$, $G/C_G(\langle y_i\rangle)$ is cyclic. Hence, $G/C_G(\langle y_i\rangle)$ is p-nilpotent for every i. It follows that $G/\cap_{i=1}^n C_G(\langle y_i\rangle)$ is p-nilpotent. Obviously, $C_G(F(G)) = \cap_{i=1}^n C_G(\langle y_i\rangle)$. Hence $G/C_G(F(G))$ is p-nilpotent. Consequently, $G/(H \cap C_G(F(G))) = G/C_H(F(H))$ is p-nilpotent. Since $F(H)$ is abelian, $F(H) \leq C_H(F(H))$. On the other hand, $C_H(F(H)) \leq F(H)$ since H is soluble. Thus $F(H) = C_H(F(H))$ and so $G/F(H)$ is p-nilpotent.

(4) *If K is a minimal normal subgroup of G contained in H, then $K \subseteq F(H)$ and G/K is p-nilpotent.*

Let K be an arbitrary minimal normal subgroup of G contained in H. Then K is an elementary abelian p-group for some prime p since H is soluble. Hence $K \leq F(H)$. By Lemma 2.2(2) and (3), we see that G/K (with respect to H/K) satisfies the hypothesis. The minimal choice of (G, H) implies that G/K is p-nilpotent.

(5) *The final contradiction.*

Since the class of all p-nilpotent groups is a saturated formation, by (2) and (4), we see that $K = F(H) = \langle x\rangle$ is the unique minimal normal subgroup of G contained in H, where $\langle x\rangle$ is a cyclic group of order p for some prime p. Since

G/K is p-nilpotent, it has a normal p-complement L/K. By Schur-Zassenhaus Theorem, $L = G_{p'}K$, where $G_{p'}$ is a Hall p'-subgroup of G. Since p is the prime divisor of $|G|$ with $(|G|, p-1) = 1$ and $N_L(K)/C_L(K) \simeq Aut(K)$ is a subgroup of a cyclic group of order $p-1$, we see that $N_L(K) = C_L(K)$. Then, by Burnside Theorem (see [14, (10.1.8)]), we have that L is p-nilpotent. Then $G_{p'}$ char $L \trianglelefteq G$, so $G_{p'} \trianglelefteq G$. Hence G is p-nilpotent. The final contradiction completes the proof. □

Acknowledgements. The authors are very grateful to the helpful suggestions of the referee.

References

[1] M. Assad and Piroska Csörgő, Characterization of finite groups with some S-quasinormal subgroups, *Monatsh. Math.* **146** (2005), 263–266.

[2] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, Sufficient conditions for supersolubility of finite groups, *J. Pure Appl. Algebra* **127** (1998), 113–118.

[3] F. Gross, Conjugacy of odd order Hall subgroups, *Bull. London Math. Soc.* **19** (1987), 311–319.

[4] W. Guo, The Theory of Classes of Groups, *Science Press/Kluwer, Academic Publishers, Beijing New York*, 2000.

[5] W. Guo, On $\mathfrak{F}$-supplemented subgroups of finite group, *Manuscripta Math.* **127** (2008), 139–150.

[6] W. Guo, K. P. Shum and F. Xie, Finite groups with some weakly S-supplemented subgroups, *Glasgow Math. J.* **53** (2011), 211–222.

[7] W. Guo, K. P. Shum and A. N. Skiba, X-semipermutable subgroup of finite groups, *J. Algebra* **315** (2007), 31–41.

[8] W. Guo and A. N. Skiba, Finite groups with given s-embedded and n-embedded subgroups, *J. Algebra* **321** (2009), 2843–2860.

[9] W. Guo and A. N. Skiba, New criterions of existence and conjugacy of Hall subgroups of finite groups, *Proc. Amer. Math. Soc.* **139** (2011), 2327–2336.

[10] J. Huang, On $\mathfrak{F}_s$-quasinormal subgroups of finite groups, *Comm. Algebra* **38** (2010), 4063–4076.

[11] B. Huppert, Endliche Gruppen I, *Springer-Verlag, Berlin*, 1967.

[12] B. Huppert and N. Blackburn, Finite Groups III, *Springer-Verlag, Berlin, New York*, 1982.

[13] O. Kegel, Sylow-gruppen and subnormalteiler endlicher gruppen, *Math. Z.* **78** (1962), 205–221.

[14] D. J. S. Robinson, A course in the Theory of Groups, *Springer, New York*, 1982.

[15] L. A. Shemetkov, Formations of Finite groups, *Nauka, Moscow*, 1978.

[16] L. A. Shemetkov and A. N. Skiba, Formations of Algebraic Systems, *Nauka, Moscow*, 1989.

[17] A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Algebra* **315** (2007), 192–209.

[18] H. Wielandt, Subnormal subgroups and permutation groups, Lectures given at the Ohio State University, *Columbus, Ohio*, 1971.

XIAOLONG YU
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE
AND TECHNOLOGY OF CHINA
HEFEI, 230026
P.R. CHINA

E-mail: yuxiaolong0710@sina.com

XIAOYU CHEN
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE
AND TECHNOLOGY OF CHINA
HEFEI, 230026
P.R. CHINA

E-mail: jelly@mail.ustc.edu.cn

WENBIN GUO
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE
AND TECHNOLOGY OF CHINA
HEFEI, 230026
P.R. CHINA

E-mail: wbguo@ustc.edu.cn

(Received March 19, 2012; revised August 24, 2012)