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On Clairaut-type equations

By SHYUICHI IZUMIYA (Sapporo)

Abstract. We study implicit first order ordinary differential equations with complete integral. We establish the principle of duality among these equations. We also give a characterization of first order differential equation with smooth complete solutions which we call Clairaut type equations.

0. Introduction

About 260 years ago Alex Claude Clairaut [4] studied the following equation which is called the Clairaut equation now: $y = x \cdot \dfrac{dy}{dx} + f\left(\dfrac{dy}{dx}\right)$. It is usually taught in the first or second year course of calculus in the university and treated as one of the typical examples of non-linear equations that are easily solved. Moreover it has a quite beautiful geometric structure as follows: There exists a "general solution" that consists of lines; $y = t \cdot x + f(t)$, where t is a parameter and the singular solution is the envelope of such a family.

In this article we shall discuss equations with the same geometric structure as the Clairaut equation. Here, we give another example as follows: $y - \left(\dfrac{dy}{dx}\right)^2 = 0$. We can easily solve this equation: the "general solution" is given by $y = \frac{1}{4}(x+t)^2$, where t is a parameter. Here, the "singular solution" is given by $y = 0$ that is the envelope of the family of graphs of the "general solution". The "general solution" of this equation does not consist of lines. However, the "singular solution" is the envelope of the graphs of the "general solution" like as the Clairaut equation. We will refer such an equation as a Clairaut-type equation.

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In ([5], [8]) it has been proved that Clairaut type equations are not generic in all implicit differential equations. However, these are generic in the set of implicit differential equations with complete solution (cf. [6]). In this paper we give a characterization of Clairaut type equations and establish the principle of duality among implicit differential equations with complete integral. We assert that Clairaut type equations play the principal role of such duality (cf. Corollary 4.2).

Al map germs considered here are differentiable of class C^∞, unless stated otherwise.

1. Basic notions

We stick to first order ordinary differential equations of the form $F\left(x, y, \dfrac{dy}{dx}\right) = 0$. We assume that $F : (\mathbb{R}^3, (x_0, y_0, p_0)) \to (\mathbb{R}, 0)$ is a function germ such that $\operatorname{grad} F \neq 0$. Then $S = F^{-1}(0)$ is a smooth surface in $(\mathbb{R}^3, (x_0, y_0, p_0))$.

We now define the notion of solutions. A *smooth solution* of $F = 0$ is a function germ $y = f(x)$ at the origin such that $(0, f(0), f'(0)) = (x_0, y_0, p_0)$ and $F(x, f(x), f'(x)) = 0$. This is the classical notion of solutions of the equation $F = 0$. The following is the geometric generalization of the notion of solution due to Lie. A *geometric solution* of $F = 0$ is a smooth immersion germ $\gamma : (\mathbb{R}, 0) \to F^{-1}(0)$ such that $y'(t) = p(t)x'(t)$ where $\gamma(t) = (x(t), y(t), p(t))$ in the canonical coordinate system of $\mathbb{R}^3$. In the terminology of contact geometry, the above curve is called a Legendrian curve (see [1]). The proof of the following lemma is just an exercise for readers.

Lemma 1.1. *Let $\gamma : (\mathbb{R}, 0) \to F^{-1}(0)$ be a geometric solution. Suppose that $x'(0) \neq 0$. Then there exist a diffeomorphism germ $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and a function germ f such that $\gamma \circ \phi(x) = (x, f(x), f'(x))$.*

According to the above property, we may define the notion of singular point of solutions. We say that t_0 is a *geometric singular point* of the solution γ if $x'(t_0) = 0$. Thus γ is multivalued around the geometric singular point. It is clear that t_0 is a geometric singular point of γ if and only if $(x'(t_0), y'(t_0)) = (0, 0)$.

On the other hand, there exists a notion of the Legendrian transformation by which a dual relationship can be set up between one equation and another. We adopt another coordinate system (X, Y, P) of $\mathbb{R}^3$ by $X = p$, $Y = x \cdot p - y$, $P = x$. We refer to a smooth mapping $*L : \mathbb{R}^3 \to \mathbb{R}^3$; $*L(x, y, p) = (p, x \cdot p - y, x)$ as a *Legendre transformation*. By the definition, we have $*L^{-1}(X, Y, P) = (P, X \cdot P - Y, X)$. If we apply the Legendre transformation to our equation, we obtain a new equation

$$F^*(X, Y, P) = F(P, X \cdot P - Y, X) = 0$$

in the new coordinate system (X, Y, P).

If we calculate partial derivatives at the point (X_0, Y_0, P_0) corresponding to (x_0, y_0, p_0), we can show the following:

$$F^*_P(X_0, Y_0, P_0) = (F_x + p \cdot F_y)(x_0, y_0, p_0)$$
$$F^*_Y(X_0, Y_0, P_0) = -F_y(x_0, y_0, p_0)$$
$$F^*_X(X_0, Y_0, P_0) = (F_p + x \cdot F_y)(x_0, y_0, p_0).$$

The following lemma is quite simple but important in the later sections.

Lemma 1.2. (1) *Let $\gamma : (\mathbb{R}, 0) \to F^{-1}(0)$ be a geometric solution of $F = 0$. Then $*L \circ \gamma : (\mathbb{R}, 0) \to F^{*-1}(0)$ is a geometric solution of $F^* = 0$.*

(2) *If t_0 is a geometric singular point of γ, then t_0 is a geometric non-singular point of $*L \circ \gamma$.*

We now call (x_0, y_0, p_0) a *π-singular point* of $F = 0$ if $F = F_p = 0$ at (x_0, y_0, p_0) and denote $\Sigma_\pi(F)$ as the set of π-singular points. We also call the set $D_F = \pi(\Sigma_\pi(F))$ a *discriminant set* of $F = 0$, where $\pi(x, y, p) = (x, y)$.

The following notion is a basis of our concerns. Let $\Gamma : (\mathbb{R} \times \mathbb{R}, 0) \to (F^{-1}(0), (x_0, y_0, p_0))$ be an one-parameter family of geometric solutions of $F = 0$. We say that Γ is *a complete solution at* (x_0, y_0, p_0) if

$$\operatorname{rank} \begin{pmatrix} x_t & y_t & p_t \\ x_c & y_c & p_c \end{pmatrix} = 2$$

and $\Gamma(0) = (x_0, y_0, p_0)$, where $\Gamma(t, c) = (x(t, c), y(t, c), p(t, c))$ and c is a parameter. In some classical textbooks (cf. [7]), the above term is used in a different sense. However, we adopt the above definition according to the terminology in the theory of first-order partial differential equations ([2], [3]). We say that an equation $F = 0$ is *completely integrable at* (x_0, y_0, p_0) if there exists a complete solution of $F = 0$ at (x_0, y_0, p_0). The uniqueness of the complete solution is dealt with in the following:

Proposition 1.3. *Let $\Gamma_i : (\mathbb{R} \times \mathbb{R}, 0) \to (F^{-1}(0), (x_0, y_0, p_0))$ $(i = 1, 2)$ be complete solutions of $F = 0$ at (x_0, y_0, p_0). Then there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}, 0)$ of the form $\Phi(t, c) = (\phi_1(t, c), \phi_2(c))$ such that $\Gamma_1 \circ \Phi = \Gamma_2$.*

Proof. Suppose that the assertion does not hold. Since the solution is an one-parameter family of curves in $F^{-1}(0)$, then there exists a point $(x_1, y_1, p_1) \in (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0)$ such that Γ_{1,c_1} and Γ_{2,c_2} are transversal near (x_1, y_1, p_1). Then we can construct an immersiv germ $\Gamma : (\mathbb{R} \times \mathbb{R}, (x_1, y_1, p_1)) \to F^{-1}(0)$ such that $\frac{\partial y}{\partial t} = p(t, c) \cdot \frac{\partial x}{\partial t}(t, c)$ and

$\frac{\partial y}{\partial c} = p(t,c)\cdot \frac{\partial x}{\partial c}(t,c)$, where $\Gamma(t,c) = (x(t,c), y(t,c), p(t,c))$. If we calculate second-order partial derivatives of both equalities, we get $\frac{\partial^2 y}{\partial t\,\partial c} = \frac{\partial p}{\partial c}\frac{\partial x}{\partial t} + p\cdot\frac{\partial^2 x}{\partial t\,\partial c}$ and $\frac{\partial^2 y}{\partial c\,\partial t} = \frac{\partial p}{\partial t}\frac{\partial x}{\partial c} + p\cdot\frac{\partial^2 x}{\partial c\,\partial t}$. This contradicts fact that γ is an immersion.

We now give some examples of completely integrable equations.

Example 1.4. 1) The dual of the Clairaut equation. Consider the equation: $y = f(x)$. This equation is given by the Legendre transform of the Clairaut equation. The geometric conplete solution is given by $\{(c, f(c), x) \mid (c,x) \in \mathbb{R}\times\mathbb{R}\}$.

2) Consider the following equation: $y - 2p^3 = 0$. We have the complete solution $\Gamma(t,c) = (3t^2 + c, 2t^3, t)$.

2. The Clairaut type equation

In this section we give a characterization of equations with smooth complete solution. By the definition of smoothness of the solution and a parameterized version of Lemma 1.1, a *smooth complete solution* of $F = 0$ is given by one-parameter family of smooth function germs $y = f(x,c)$ such that $F(x, f(x,c), \frac{\partial f}{\partial x}(x,c)) = 0$ and the mapping $j^1_* f : (\mathbb{R}\times\mathbb{R}, 0) \to F^{-1}(0)$ defined by $j^1_* f(x,c) = (x, f(x,c), \frac{\partial f}{\partial x}(x,c))$ is an immersion germ. We remark that $j^1_* f$ is an immersion gern if and only if $\left(\frac{\partial f}{\partial c}, \frac{\partial^2 f}{\partial x\,\partial c}\right) \neq (0,0)$.

The following definition is due to Dara [5]. We say that an equation $F = 0$ is *Clairaut type* at (x_0, y_0, p_0) if there exist smooth function germs $A(x,y,p)$, $B(x,y,p)$ at (x_0,y_0,p_0) such that $F_x + p\cdot F_y = A\cdot F + B\cdot F_p$. We now give some examples of Clairaut type equations.

Example 2.1. 1) Of course, one of the example is the Clairaut equation. In this case we can easily show that $F_x + p\cdot F_y = 0$. Then we may choose $A = B = 0$.

2) Consider the following equation: $y - p^2 = 0$. Then we have $F_x + p\cdot F_y = p$ and $F_p = -2p$, so that we may choose $A = 0$ and $B = -\frac{1}{2}$. Here, we can get the smooth complete solution as follows: $y = \frac{1}{4}(x+c)^2$.

Moreover, D_F is the envelope of the family graphs of the smooth complete solution.

3) "Free particle" on the line. Consider the following equation: $y^2 + p^2 - 1 = 0$. We can calculate that $F_x + p\cdot F_y = 2y\cdot p$ and $F_p = 2p$. Then we have $A = 0$ and $B = y$. The smooth complete solution around $(0, \pm 1, 0)$ is given by $y = \dfrac{\pm 1}{(c^2+2c+2)^{\frac{1}{2}}}\cdot\cos(t + ct)$, where (t,c) is a point near $(0,-1)$. In this case we can also show that D_F is the envelope of the family of graphs of the smooth complete solution.

The following theorem gives a characterization of equations with smooth complete solution.

Theorem 2.2. *For an equation $F = 0$, the followings are equivalent.*

(1) $F = 0$ *is Clairaut type equation at* (x_0, y_0, p_0).

(2) $F = 0$ *has a smooth complete solution at* (x_0, y_0, p_0).

Moreover, in this case, if $\Sigma_\pi(F) \neq 0$, then D_F is the envelope of the family of the graphs of the complete solution.

Proof. (1) $\Longrightarrow$ (2). By the assumption, there exist function germs A, B at (x_0, y_0, p_0) such that $F_z + p \cdot F_y = A \cdot F + B \cdot F_p$. We now consider a vector field germ $V = \frac{\partial}{\partial x} + p \cdot \frac{\partial}{\partial y} - B \cdot \frac{\partial}{\partial p}$ at (x_0, y_0, p_0). Let $c(t)$ be an integral curve of V such that $c(0) \in F^{-1}(0)$. Then we can calculate that $\left.\frac{dF(c(t))}{dt}\right|_{t=0} = F_x + p \cdot F_y - B \cdot F_p = 0$. It follows that V is tangent to $F^{-1}(0)$. If we set $c(t) = (x(t), y(t), p(t))$, then we have $x'(t) = 1$, $y'(t) = p(c(t))$ and $p'(t) = B(c(t))$. These equalities guarantee that $c(t)$ is a smooth solution of $F = 0$. Then the flows of the vector field V gives the smooth complete solution of $F = 0$.

(2) $\Longrightarrow$ (1). Let $y = f(x, c)$ be the complete solution of $F = 0$. If we calculate the partial derivative of $F(x, f(x, c), f_x(x, c)) = 0$ with respect to x, then we have $F_x + f_x \cdot F_y + f_{xx} \cdot F_p = 0$ at $(x, f(x, c), f_x(x, c)) \in F^{-1}(0)$.

Since the map $j^1_* f$ is an immersion, then there exists a function germ $B(x, y, p)$ near (x_0, y_0, p_0) such that $B \circ j^1_* f(x, c) = f_{xx}(x, c)$. For any $(x, y, p) \in (F^{-1}(0), (x_0, y_0, p_0))$, there exists (x, c) such that $(x, f(x, c), f_x(x, c)) = (x, y, p)$. Then we have $F_x + p \cdot F_y = B \cdot F_p$. Since $\operatorname{grad} F \neq 0$, then the above equality means that there exists a function germ $A(x, y, p)$ at (x_0, y_0, p_0) such that $F_x + p \cdot F_y = B \cdot F_p + A \cdot F$. This completes the proof of the first part.

For the proof of the second part, we may assume that there exists a smooth complete solution $y = f(x, c)$ of $F = 0$ around (x_0, y_0, p_0). By definition, $j^1_* f(x, c) \in \Sigma_\pi(F)$ if and only if

$$\operatorname{rank}\begin{pmatrix} 1 & f_x \\ 0 & f_c \end{pmatrix} < 2$$

at (x, c). It is equivalent to the fact that $f_c = 0$. Then the set $\Sigma_\pi(F)$ is given by the equation $f_c = 0$ near (x_0, y_0, p_0). We now consider the family of graphs of the smooth complete solution which is defined by the equation $f(x, c) - y = 0$ on the (x, y)-plane. Then the set $\{(x, f(x, c)) \mid \text{There exists } c \text{ such that } f_c(x, c) = 0\}$ is the envelope of this family by the definition. This set is equal to the discriminant set D_F by the previous argument. This completes the proof.

We now consider Clairaut equations rather than Clairaut type equations. We have the following theorem.

Theorem 2.3. *For an equation $F = 0$ at (x_0, y_0, p_0), the followings are equivalent.*

(1) *There exists a function germ $A : (\mathbb{R}^3, (x_0, y_0, p_0)) \to \mathbb{R}$ such that*

$$F_x + p \cdot F_y = A \cdot F$$

and $(x_0, y_0, p_0) \in \Sigma_\pi(F)$.

(2) *There exists a function germ $f : (\mathbb{R}, p_0) \to \mathbb{R}$ such that*

$$F^{-1}(0) = \{(x, y, p) \mid y = x \cdot p + f(p)\}.$$

This theorem has been proved by Dara [5], however, we now give an elementary proof.

Proof. Suppose that $F = 0$ satisfies condition (1). If $F_y = 0$ at (x_0, y_0, p_0), then $F = F_x = F_p = 0$ at (x_0, y_0, p_0). This contradicts the fact that $\operatorname{grad} F \neq 0$. Then $F_y \neq 0$ at (x_0, y_0, p_0). By the implicit function theorem, there exist a function germ $h(x, p)$ and a non vanishing function germ $\lambda(x, y, p)$ such that $F(x, y, p) = \lambda(x, y, p) \cdot (h(x, p) - y)$. We now consider the Legendre transform F^* of F. Then $F^*(X, Y, P) = \Lambda(X, Y, P) \cdot (H(X, P) - Y)$, where $\Lambda(X, Y, P) = -\lambda(P, X \cdot P - Y, X)$ and $H(X, P) = X \cdot P - h(P, X)$. It follows that $F^*_P = \Lambda \cdot H_P$ on $F^{*-1}(0)$. Since $F^*_P = F_x + p \cdot F_y$ and $*L(\{(x, y, p) \mid y = h(x, p)\}) = \{(X, Y, P) \mid Y = H(X, P)\}$, we have $H_P \equiv 0$. Then we can put $f(X) = -H(X, P)$. Pulling back by the Legendre transformation, we have $F^{-1}(0) = \{(x, y, p) \mid y = x \cdot p + f(p)\}$. The converse of the proof is given by a direct calculation.

3. The principle of duality

Dulaity is one of the most beautiful properties in projective geometry. As we already mentioned in §1, the dual relationship among the equations is given by the Legendre transformation in the classical theory of the ordinary differential equations. However, situations are confused in the classical theory as usual. The following arguments may be considered as the principle of duality among completely integrable equatuions.

Let $\mathcal{C}_I(x_0, y_0, p_0)$ be the set of germs corresponding to completely integrable first order ordinary differntial equations at (x_0, y_0, p_0). For any $F \in \mathcal{C}_I(x_0, y_0, p_0)$, we have a unique complete solution $\Gamma_F : (\mathbb{R} \times \mathbb{R}, 0) \to F^{-1}(0)$ such that $\Gamma_F(0) = (x_0, y_0, p_0)$. We denote it by $\Gamma_F(t, c) = (x_F(t, c), y_F(t, c), p_F(t, c))$. We also define three subsets of $\mathcal{C}_I(x_0, y_0, p_0)$ as follows:

$$\mathcal{C}_{I_0}(x_0,y_0,p_0)=\Big\{F\in\mathcal{C}_I(x_0,y_0,p_0)\;\Big|\;\frac{dx_F}{dt}(x_0,y_0,p_0)\neq 0$$
$$\text{and }\frac{dp_F}{dt}(x_0,y_0,p_0)\neq 0\Big\},$$
$$\mathcal{C}_{I_1}(x_0,y_0,p_0)=\Big\{F\in\mathcal{C}_I(x_0,y_0,p_0)\;\Big|\;\frac{dp_F}{dt}(x_0,y_0,p_0)=0\Big\},$$
$$\mathcal{C}_{I_2}(x_0,y_0,p_0)=\Big\{F\in\mathcal{C}_I(x_0,y_0,p_0)\;\Big|\;\frac{dx_F}{dt}(x_0,y_0,p_0)=0\Big\}.$$

By the uniqueness of the complete solution of $F=0$, these subsets are well–defined. We denote by $\mathcal{C}^*_I(X_0,Y_0,P_0)$ the set of complete integrable first order ordinary differential equations at (X_0,Y_0,P_0) in the coordinate system (X,Y,P). We also define sets $\mathcal{C}^*_{I_0}(X_0,Y_0,P_0)$, $\mathcal{C}^*_{I_1}(X_0,Y_0,P_0)$ and $\mathcal{C}^*_{I_2}(X_0,Y_0,P_0)$ by exactly the same definition as those of the above. Then we have the following duality theorem.

Theorem 3.1. *We have an one-to-one correspondence*

$$D:\mathcal{C}_I(x_0,y_0,p_0)\to\mathcal{C}^*_I(X_0,Y_0,P_0)$$

defined by $\mathcal{D}(F)=F^*$.

Furthermore, we have relations:

$$\mathcal{D}(\mathcal{C}_{I_0}(x_0,y_0,p_0))=\mathcal{C}^*_{I_0}(X_0,Y_0,P_0) \tag{1}$$
$$\mathcal{D}(\mathcal{C}_{I_1}(x_0,y_0,p_0))=\mathcal{C}^*_{I_2}(X_0,Y_0,P_0) \tag{2}$$
$$\mathcal{D}(\mathcal{C}_{I_2}(x_0,y_0,p_0))=\mathcal{C}^*_{I_1}(X_0,Y_0,P_0). \tag{3}$$

Proof. By the definition, we have $\mathcal{D}(F)=F^*=F\circ(*L)^{-1}$, where $*L$ is the Legendre transformation. For any $F\in\mathcal{C}_I(x_0,y_0,p_0)$, $*L\circ\Gamma_F$ is the unique complete solution of F^* by Lemma 1.2, (1). Then $\mathcal{D}$ is a well-defined and one-to-one correspondence. Since $*L\circ\Gamma_F(t,c)=(p_F(t,c), x_F(t,c)\cdot p_F(t,c)-y_F(t,c),x_F(t,c))$, then we can easily show that the relations (1), (2) and (3).

We now give the following quite beautiful result as a corollary of the above theorem.

Corollay 4.2. *Let $F=0$ be an equation at (x_0,y_0,p_0). Then $F=0$ is completely integrable at (x_0,y_0,p_0) if and only if $F=0$ is of Clairaut type at (x_0,y_0,p_0) or $F^*=0$ is of Clairaut type at (X_0,Y_0,P_0).*

We already presented interesting examples of completely integrable equations (see Example 1.4). We can easily verify that the duals of these examples are Clairaut type.

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References

[1] V. I. Arnol'd, Contact geometry and wave propagation, *Monographie de L'Enseignement Mathematique* **34** (1989).
[2] C. Carathéodory, Calculus of Variations and Partial Differential Equations of First Order, Part I, Partial Differential Equations of the First Order, *Holden–Day*, 1965.
[3] R. Courant and D. Hilbert, Methods of mathematical physics I, II, *Wiley*, *New York*, 1962.
[4] 4 A. C. Clairaut, Solution de plusieurs problems, *Histoire de l'Academie Royale de Sciences, Paris* (1734), 196–215.
[5] L. Dara, Sindularités générique des équations differentielles multiformes, *Bol. Soc. Brasil Mat.* **6** (1975), 95–128.
[6] A. Hayakawa, G. Ishikawa, S. Izumiya and K. Yamaguchi, Classification of generic integral diagram and first order ordinary differential equations, (*to appear in International Jour. of Math*).
[7] I. G. Petrovski, Ordinary differential equations, *Prentice-Hall*, 1966.
[8] R. Thom, Sur les equations différentielles multiformes et leurs integrals singulieres, *Bol. da Soc. Brasil Mat.* **3** (1972), 1–11.

SHYUICHI IZUMIYA
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HOKKAIDO UNIVERSITY
SAPPORO 060
JAPAN
E-mail: izumiya@math.hokudai.ac.jp

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